2. Effective calculability.
Abbreviation of treatment. A
function is said to be ‘effectively calcu
lable’ if its values can be found by some purely
mechanical process. Although it is fairly easy to get
an intuitive grasp of this idea it is nevertheless desirable
to have some definite, mathematically expressible definition.
Such a definition was first given by Gödel at Princeton in 1934
(Gödel [2], 26) following in part an unpublished suggestion of Herb
rand, and has since been developed by Kleene (Kleene [2]). We shall
not be concerned much here with this particular definition. Another defini
tion of effective calculability has been given by Church (Church [3], 356–358)
who identifies it with λ-definability. The author has recently suggested a definition
corresponding more closely to the intuitive idea (Turing [1], see also Post [1]). It was
said above “a function is effectively calculable if its values can be found by some pure
ly mechanical process.” We may take this statement literally, understanding by a purely
mechanical process one which could be carried out by a machine. It is possible to give a
mathematical description, in a certain normal form, of the structures of these machines. T
he development of the idea leads to the author’s definition of a computable function, and
an identification of computability[3] with effective calculability. ([3]We shall use the expressio
n ‘computable function’ to mean a function calculable by a machine, and let ‘effectively c
alculable’ refer to the intuitive idea without particular identification with any one of these
definitions. We do not restrict the values taken by a computable function to be natural num
bers; we may for instance have computable propositional functions.) It is not difficult thou
gh somewhat laborious, to prove these three definitions equivalent (Kleene [3], Turing [2]
). In the present paper we shall make considerable use of Church’s identification of effec
tive calculability with λ-definability, or, what comes to the same, of the identification w
ith computability and one of the equivalence theorems. In most cases where we have to deal
with an effectively calculable function we shall introduce the corresponding W. F. F. with so
me such phrase as “the function *f* is effectively calculable, let *F* be a formula λ-defining it” or
“let *F* be a formula such that $F(\underline{n})$ is convertible to...whenever $\underline{n}$ represents a positive integer”.
In such cases there is no difficulty in seeing how a machine could in principle be designed to calcu
late the values of the function concerned, and assuming this done the equivalence theorem can be
applied. A statement as to what the formula *F* actually is may be omitted. We may introduce imm
ediately on this basis a W. F. F. ω with the property that $\omega(\underline{m}, \underline{n})$ conv $\underline{r}$ if r is the greatest positive
integer for which m^r divides n, if any, and is 1 if there is none. We also introduce Dt with the pr
operties: $\mathrm{Dt}(\underline{n}, \underline{n})$ conv 3; $\mathrm{Dt}(\underline{n} + \underline{m}, \underline{n})$ conv 2; $\mathrm{Dt}(\underline{n}, \underline{n} + \underline{m})$ conv 1. There is another point to
be made clear in connection with the point of view we are adopting. It is intended that all pr
oofs that are given should be regarded no more critically than proofs in classical analysis.
The subject matter, roughly speaking, is constructive systems of logic, but as the purp
ose is directed towards choosing a particular constructive system of logic for pract
ical use; an attempt at this stage to put our theorems into constructive form w
ould be putting the cart before the horse. Those computable functions whic
h take only the values 0 and 1 are of particular importance since they dete
rmine and are determined by computable properties, as may be seen by r
eplacing ‘0’ and ‘1’ by ‘true’ and ‘false’. But besides this type of proper
ty we may have to consider a different type, which is roughly speaking,
less constructive than the computable properties, but more so than the
general predicates of classical mathematics. Suppose we have a com
putable function of the natural members taking natural numbers as values,
then corresponding to this function there is the property of being a value o
f the function. Such a property we shall describe as ‘axiomatic’; the reason
for using this term is that it is possible to define such a property by giving a
set of axioms, the property to hold for a given argument if and only if it is p
ossible to deduce that it holds from the axioms. Axiomatic properties may
also be characterized in this way. A property ψ of positive integers is axioma
tic if and only if there is a computable property ϕ of two positive integers such th
at $\psi(x)$ is true if and only if there is a positive integer y such that $\phi(x, y)$ is true.
Or again ψ is axiomatic if and only if there is a W. F. F. *F* such that $\psi(n)$ is true if
and only if $F(\underline{n})$ conv 2. 3. Number theoretic theorems. By a numb er theoretic theorems[4]
([4]I belie ve there is no generally accepted meaning for this term, but it should be noticed that we are using
it in a rather restricted sense. The most generally accepted meaning is probably this: suppose we take an arbitrary
formula of the function calculus of first order and replace the function variables by primitive recursive relations. The resulting formula
represents a typical number theoretic theorem in this [more general] sense.) we shall mean a theorem of the form ‘$\theta(x)$ vanishes for infin
itely many natural numbers x’, where $\theta(x)$ is a primitive recursive[5] function. ([5]Primitive recursive functions of natural numbers are
defined inductively as follows.* The class of primitive recursive function is more restricted than the computable functions, but has the advantage
that there is a process whereby one can tell of a set of equations whether it defines a primitive recursive function in the manner described above.
If $\phi(x_1, \dots, x_n)$ is primitive recursive than $\phi(x_1, \dots, x_n) = 0$ is described as a primitive recursive between $x_1, \dots x_n$.) We shall say that a problem
is number theoretic if it has been shown that any solution of the problem may be put in the form of a proof of one or more number theoretic
theorems. More accurately we may say that a class of problems is number theoretic if the solution of any one of them can be transformed (by a
uniform process) into the form of proofs of number theoretic theorems. I shall now draw a few consequences from the definitions of ‘number
theoretic theorems’, and in section 5 will try to justify confining our considerations to this type of problem. An alternative form for number
theoretical theorems is ‘for each natural number x there exists a natural number y such that $\phi(x, y$ vanishes’, where $\phi(x, y)$ is primitive recursive
and conversely. In other words, there is a rule whereby given the function $\theta(x)$ we can find a functions $\phi(x, y)$, or given $\psi(x, y)$ we can find a
function $\theta(x)$, so that ‘$\theta(x)$ vanishes infinitely often’ is a necessary and sufficient condition for ‘for each x there is y so that $\phi(x, y) = 0$’. In fact given
$\theta(x)$ we define $\phi(x, y) = \theta(y) + x(x, y)$ where $x(x, y)$ is the (primitive recursive) function with the properties $\alpha(x, y = 1\ (y \leq x); = 0\ (y > x)$. If on the
other hand we are given $\phi(x, y)$ we define $\theta(x)$ by the equations $\theta_1(0) = 3$; $\theta_1(x + 1) = 3.\ 2/3\ (\theta_1(x))^5\ (\phi(\omega_3(\theta_1(x)) - 1, \omega_3(\theta_1(x)))$; $\theta(x) = \phi(\omega_3$
$(\theta_1(x)) - 1, \omega_2(\theta_1(x)))$ where $\omega_r(x)$ is to be defined so as to mean ‘the largest s for which r^s divides x ‘and $2/3x$ to be defined primitive recursively so
as to have its usual meaning if x is a multiple of 3. The function $\delta(x)$ is to be defined by the equations $\delta(0) = 0$, $\delta(x + 1) = 1$. It is easily verified
that the functions so defined have the desired properties. We shall now show that questions as to the truth of statements of form ‘does $f(x)$ vanish
identically’, where $f(x)$ is a computable function, can be reduced to questions as to the truth of number theoretical theorems. It is understood that
in each case the rule for the calculation of $f(x)$ is given and that one is satisfied that this rule is valid, i.e. that the machine which should calculate (x) is
circle free (Turing [1], 232). The function $f(s)$ being computable is general recursive in the Herbrand-Gödel sense, and therefore by a general theorem
due to Kleen[6] ([6]Kleen [3], 727. *Suppose $f(x_1, \dots, x_{n-1})$, $g(x_1, \dots, x_n)$, $h(x_1, \dots, x_{n+1})$ are primitive recursive then $\phi(x_1, \dots, x_n)$ is primitive recursive if it is defined
by one of the sets of equations (a) – (e). (a) $\phi(x_1, \dots, x_n) = h(x_1, \dots, x_{m-1}, g(x_1, \dots, x_n), x_{m-1}, \dots, x_{n-1}, x_n)$, $(1 \leq m \leq n)$; (b) $\phi(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1})$; (c) $\phi(x_1) = a$,
where $n = 1$ and a is some particular natural number; (d) $\phi(x_1) = x_1 + 1$ ($n = 1$); (e) $\phi(x_1, \dots, x_{n-1}, 0) = f(x_1, \dots, x_{n-1})$; $\phi(x_1, \dots, x_{n-1}, x_n + 1) = h(x_1, \dots, x_n, \phi(x_1, \dots, x_n))$. LMC

Universality and Computability

Fundamental questions:

Q. What is a general-purpose computer?

Q. Are there limits on the power of digital computers?

Q. Are there limits on the power of machines we can build?

Pioneering work in the 1930s.

- Princeton == center of universe.
- Automata, languages, computability, universality, complexity, logic

David Hilbert

Kurt Gödel

Alan Turing

Alonzo Church

John von Neumann

Context: Mathematics and Logic

Mathematics. Any formal system powerful enough to express arithmetic.

Principia Mathematics
Peano arithmetic
Zermelo-Fraenkel set theory

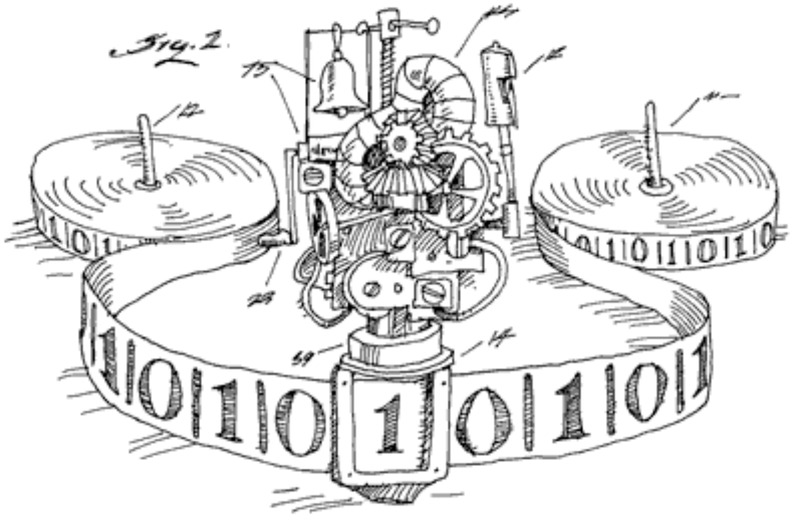
Complete. Can prove truth or falsity of any arithmetic statement.
Consistent. Can't prove contradictions like 2 + 2 = 5.
Decidable. Algorithm exists to determine truth of every statement.

Q. [Hilbert, 1900] Is mathematics complete and consistent?
A. [Gödel's Incompleteness Theorem, 1931] No!!!

Q. [Hilbert's Entscheidungsproblem] Is mathematics decidable?
A. [Church 1936, Turing 1936] No!

7.4 Turing Machines (revisited)

Alan Turing (1912-1954)



Turing Machine by Tom Dunne
American Scientist, March-April 2002

Turing Machine

Desiderata. Simple model of computation that is "as powerful" as conventional computers.

Intuition. Simulate how humans calculate.

Ex. Addition.

			1	2	3	4	5	6		
		+	3	1	4	1	5	9		

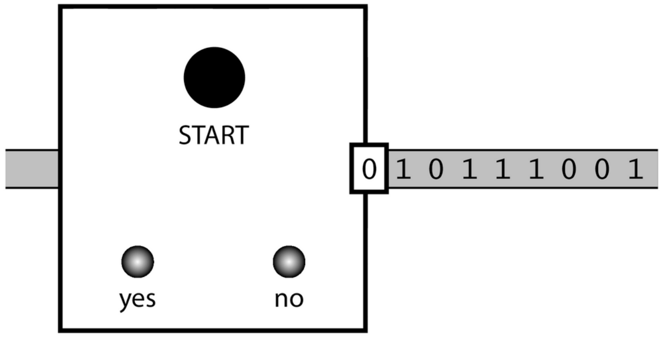
Last lecture: DFA

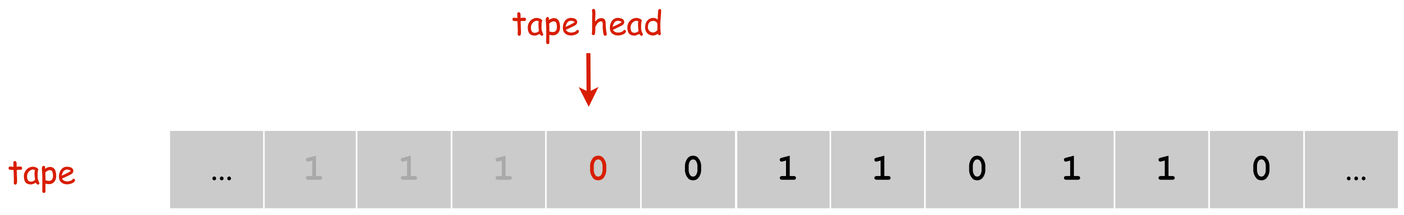
Tape.

- Stores input.
- One arbitrarily long strip, divided into cells.
- Finite alphabet of symbols.

Tape head.

- Points to one cell of tape.
- Reads a symbol from active cell.
- Moves right one cell at a time.





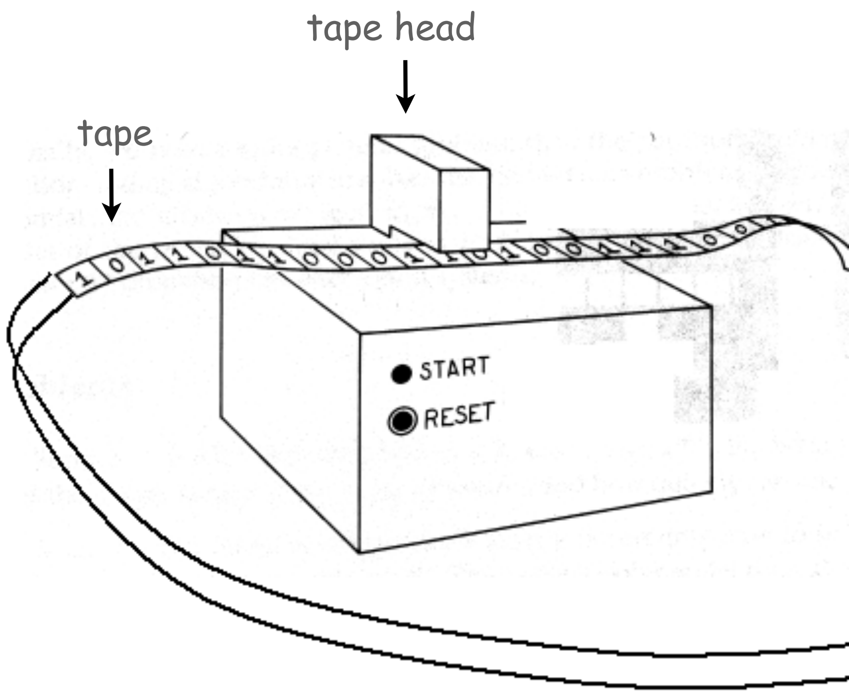
This lecture: Turing machine

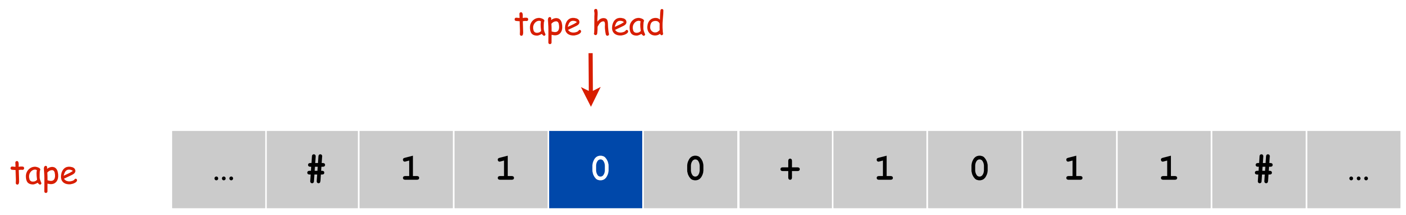
Tape.

- Stores input, output, and intermediate results.
- One arbitrarily long strip, divided into cells.
- Finite alphabet of symbols.

Tape head.

- Points to one cell of tape.
- Reads a symbol from active cell.
- Writes a symbol to active cell.
- Moves left or right one cell at a time.



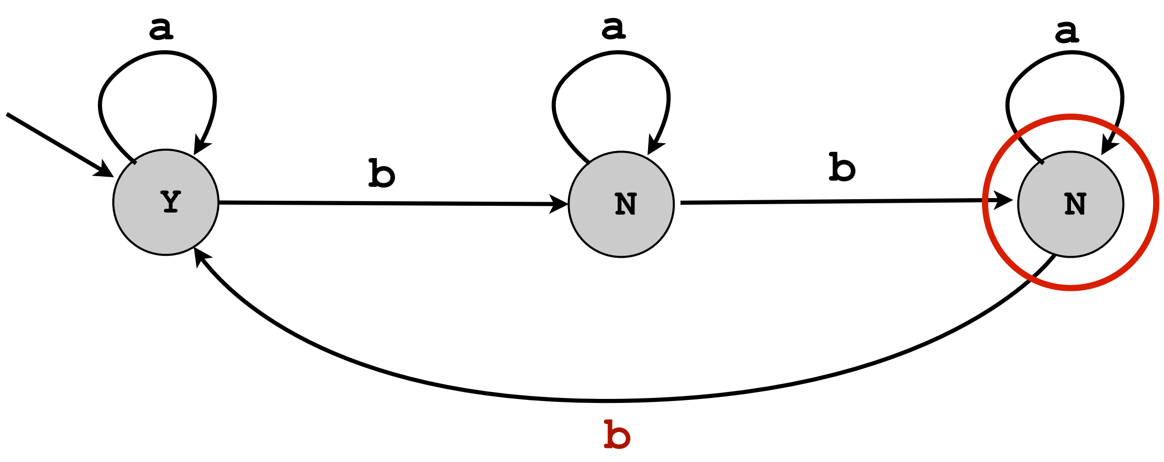


Last lecture: Deterministic Finite State Automaton (DFA)

Simple machine with N states.

- Begin in start state.
- Read first input symbol.
- Move to new state, depending on current state and input symbol.
- Repeat until last input symbol read.
- Accept input string if last state is labeled Y.

DFA



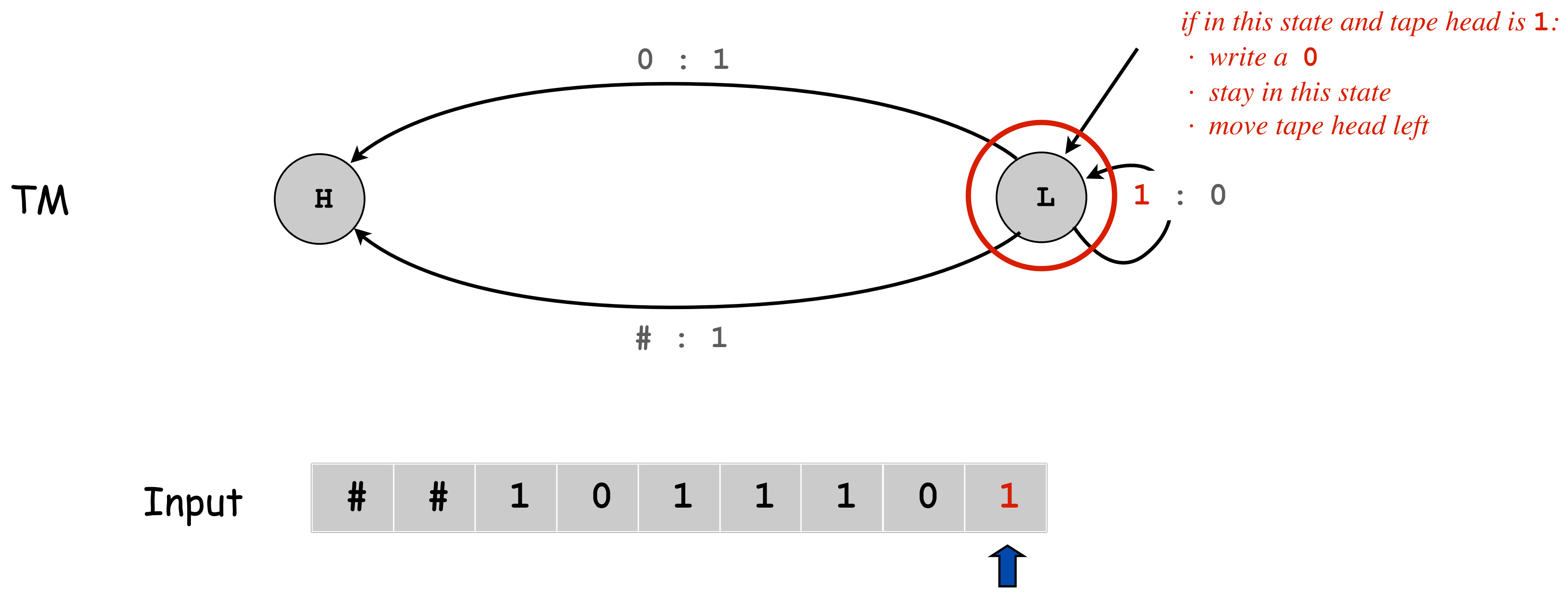
Input

b	b	a	a	b	b	a	b	b

This lecture: Turing Machine

Simple machine with N states.

- Begin in start state.
- Read first input symbol.
- Move to new state and write new symbol on tape, depending on current state and input symbol.
- Move tape head left if state is labeled L, right if state is labeled R.
- Repeat until entering a state labelled Y, N, or H.
- Accept input string if state is labeled Y, reject if N
  [or leave result of computation on tape].

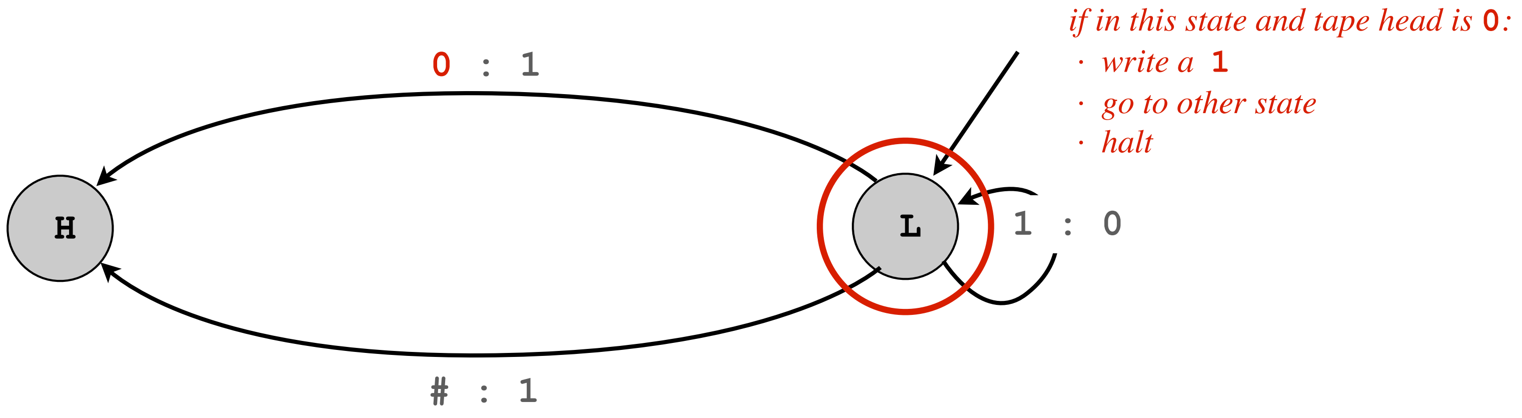


TM Example

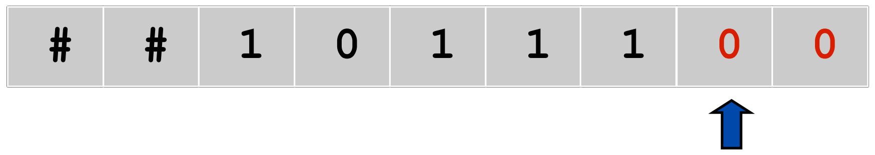
Simple machine with N states.

- Begin in start state.
- Read first input symbol.
- Move to new state and write new symbol on tape, depending on current state and input symbol.
- Move tape head left if state is labeled L, right if state is labeled R.
- Repeat until entering a state labelled Y, N, or H.
- Accept input string if state is labeled Y, reject if N
  [or leave result of computation on tape].

TM



Input

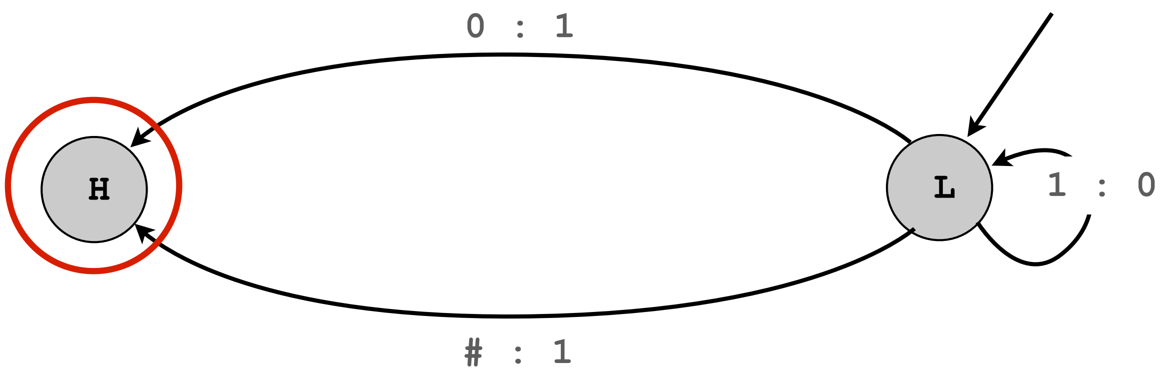


TM Example

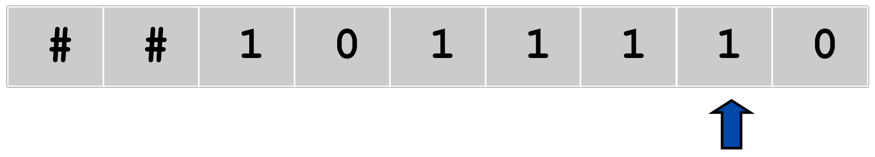
Simple machine with N states.

- Begin in start state.
- Read first input symbol.
- Move to new state and write new symbol on tape, depending on current state and input symbol.
- Move tape head left if state is labeled L, right if state is labeled R.
- Repeat until entering a state labelled Y, N, or H.
- Accept input string if state is labeled Y, reject if N
  [or leave result of computation on tape].

TM



Output



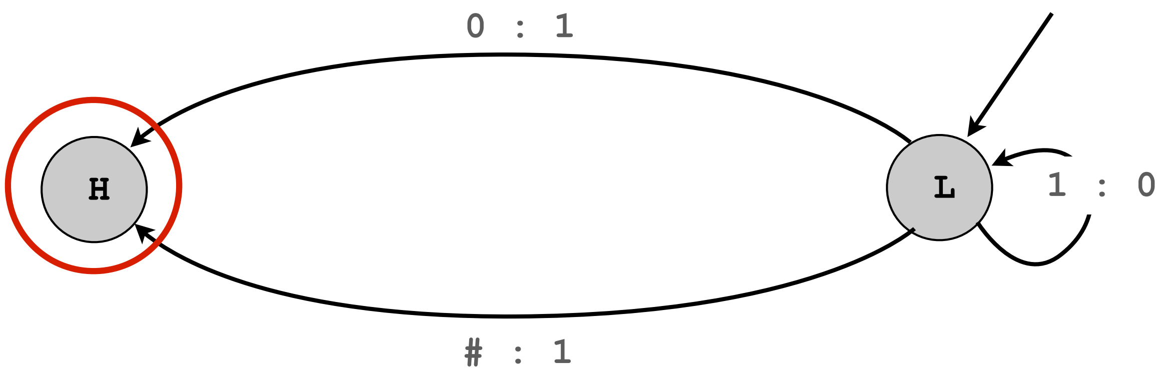
#	#	1	0	1	1	1	1	0

TM Example

Simple machine with N states.

- Begin in start state.
- Read first input symbol.
- Move to new state and write new symbol on tape, depending on current state and input symbol.
- Move tape head left if state is labeled L, right if state is labeled R.
- Repeat until entering a state labelled Y, N, or H.
- Accept input string if state is labeled Y, reject if N [or leave result of computation on tape].

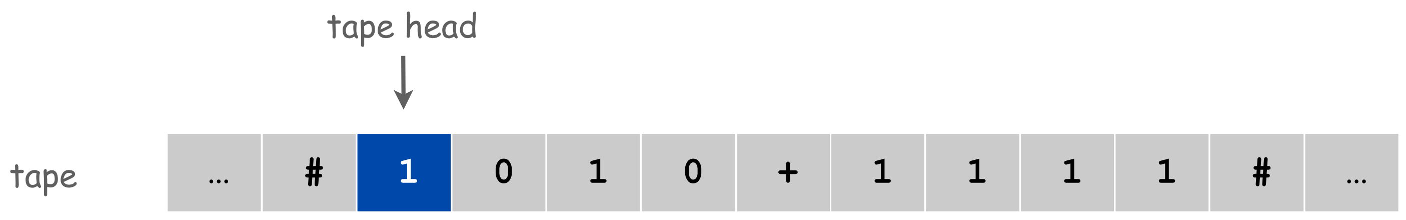
TM



Input	#	#	1	0	1	1	1	0	1
Output	#	#	1	0	1	1	1	1	0

Turing Machine: Initialization and Termination

Initialization. Set input on some portion of tape; set tape head.

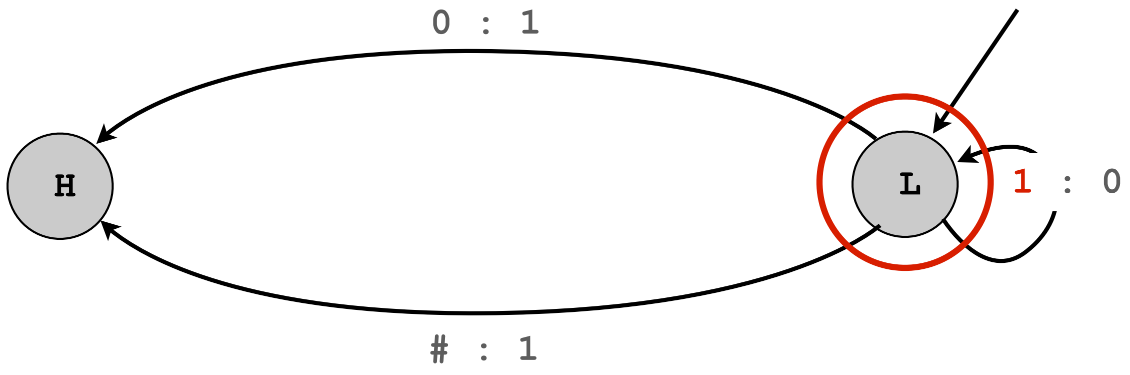


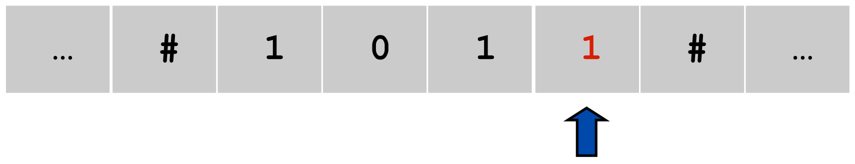
Termination. Stop if enter `yes`, `no`, or `halt` state.

Note: infinite loop possible

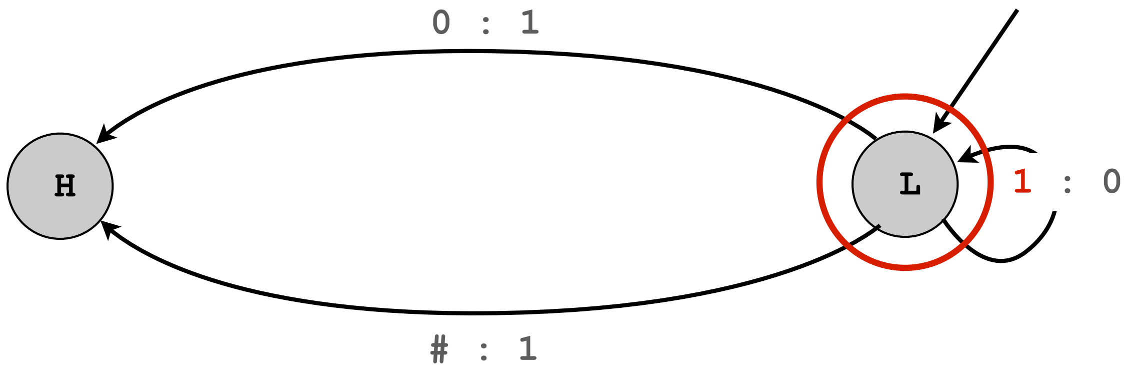
Output. Contents of tape.

TM Example 1: Binary Increment



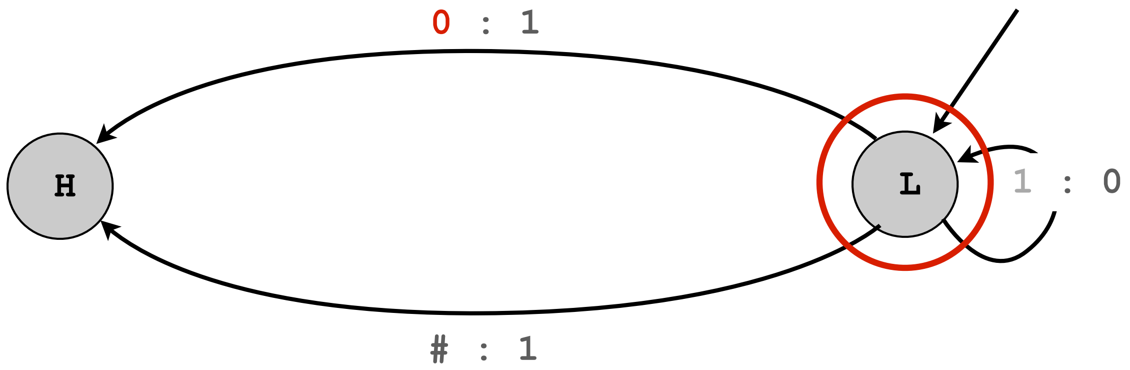


TM Example 1: Binary Increment



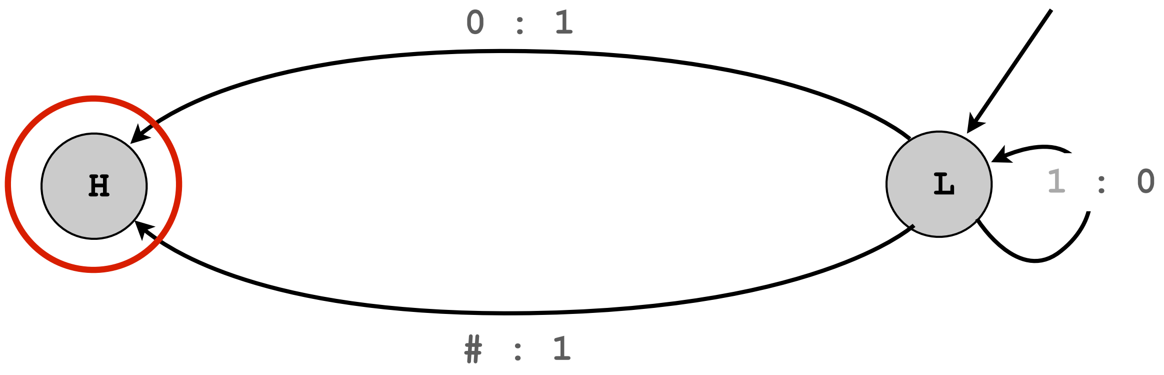
…	#	1	0	1	1	#	…
…	#	1	0	1	0	#	…

TM Example 1: Binary Increment

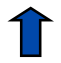


…	#	1	0	1	1	#	…
…	#	1	0	1	0	#	…
…	#	1	0	0	0	#	…

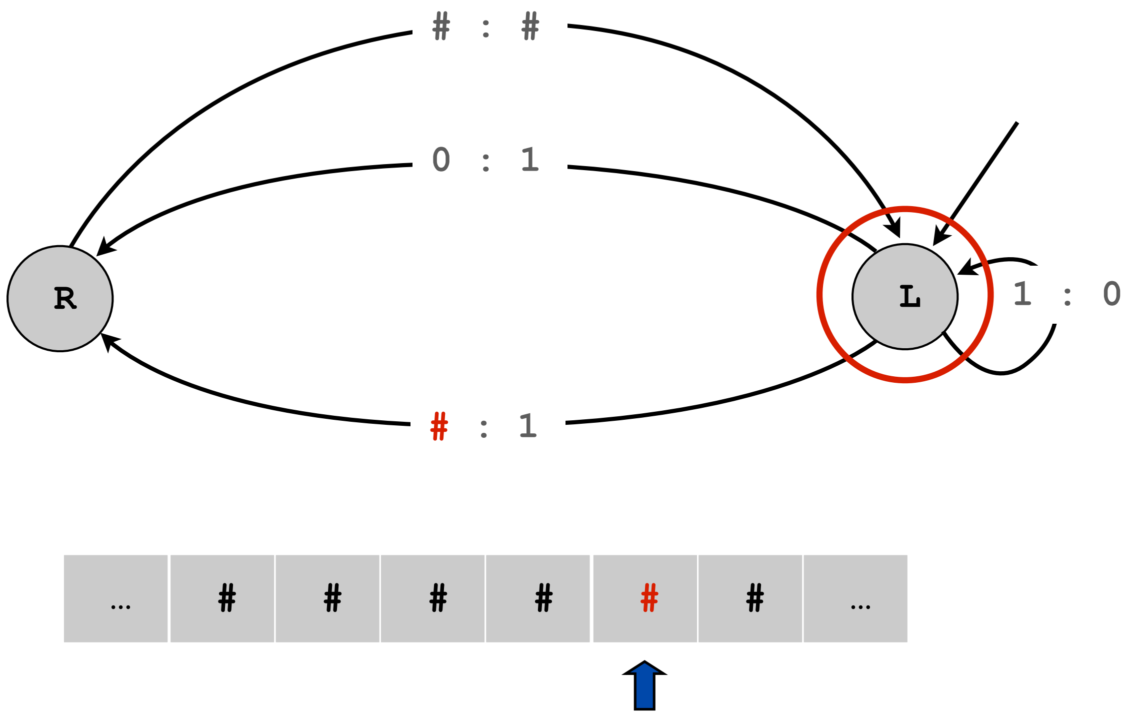
TM Example 1: Binary Increment



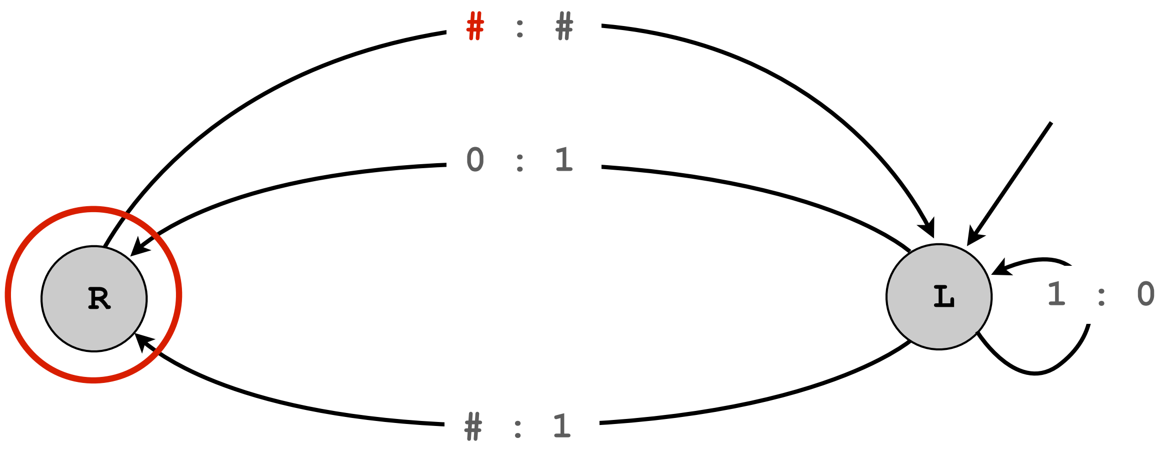
…	#	1	0	1	1	#	…
…	#	1	0	1	0	#	…
…	#	1	1	0	0	#	…



TM Example 2: Continuous Binary Counter



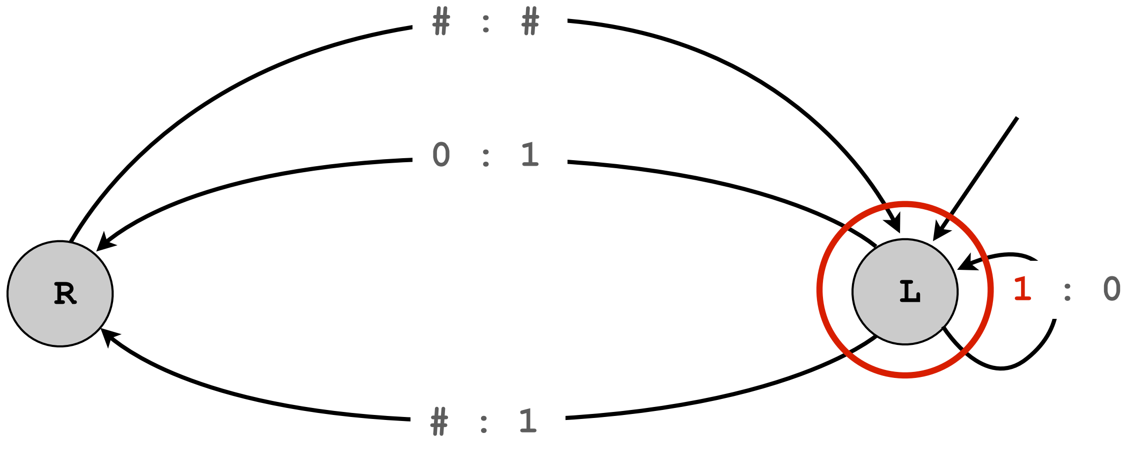
TM Example 2: Continuous Binary Counter



…	#	#	#	#	#	#	…

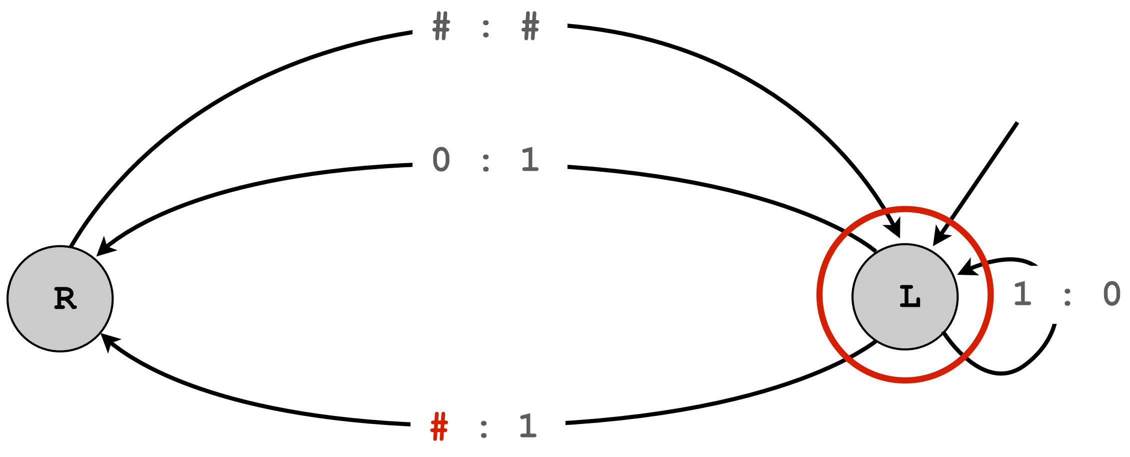
…	#	#	#	#	1	#	…

TM Example 2: Continuous Binary Counter



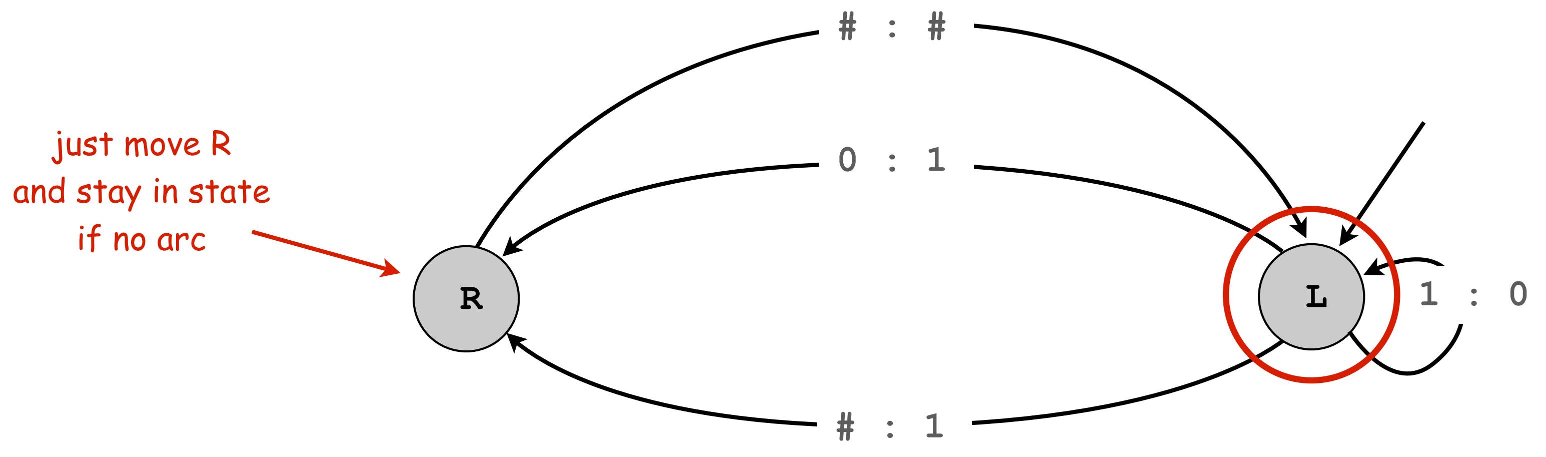
…	#	#	#	#	#	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	1	#	…

TM Example 2: Continuous Binary Counter



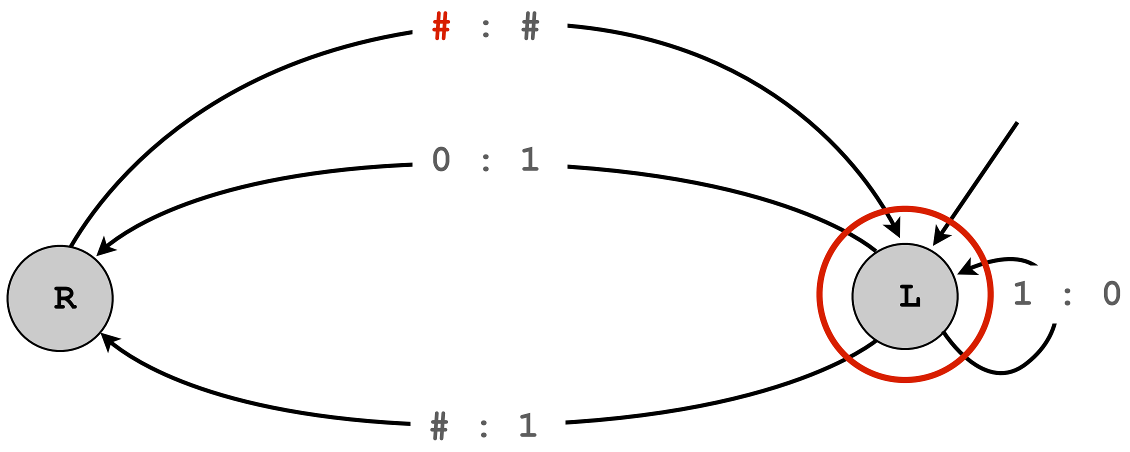
…	#	#	#	#	#	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	0	#	…

TM Example 2: Continuous Binary Counter



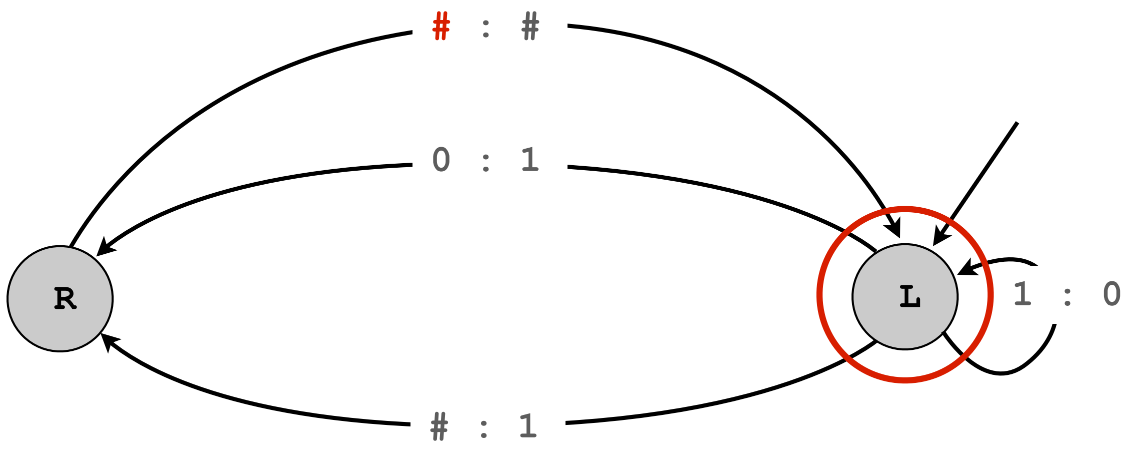
…	#	#	#	#	#	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	1	#	…
…	#	#	#	1	0	#	…

TM Example 2: Continuous Binary Counter



…	#	#	#	#	#	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	1	#	…
…	#	#	#	1	0	#	…

TM Example 2: Continuous Binary Counter

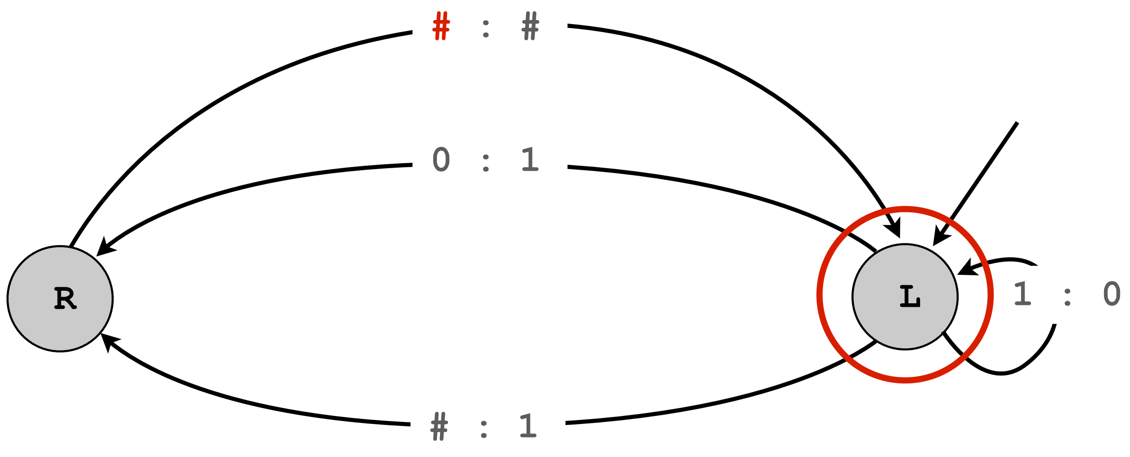


…	#	#	#	#	#	#	…
…	#	#	#	#	1	#	…
…	#	#	#	#	1	#	…
…	#	#	#	1	0	#	…

. . .

…	#	#	#	1	1	#	…

TM Example 2: Continuous Binary Counter



…	#	#	#	#	#	#	…

…	#	#	#	#	1	#	…

…	#	#	#	#	1	#	…

…	#	#	#	1	0	#	…

. . .

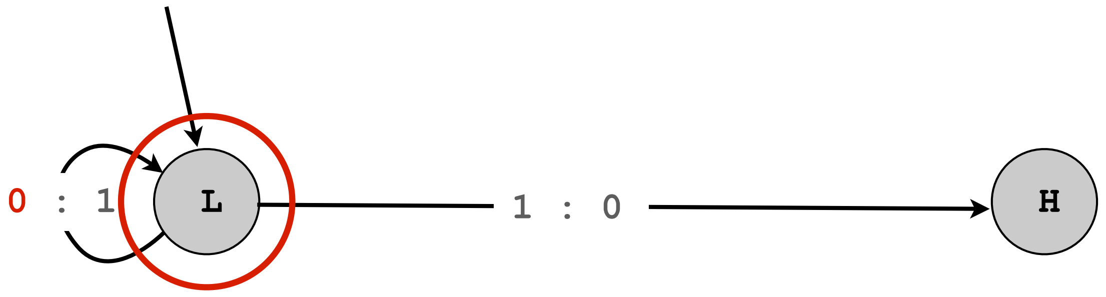
…	#	#	#	1	1	#	…

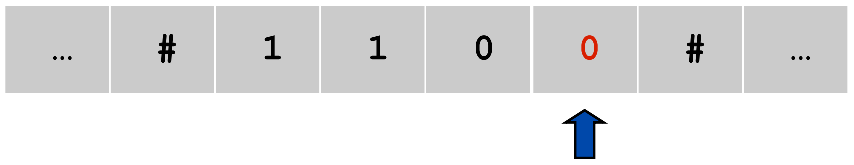
. . .

counts, never halts

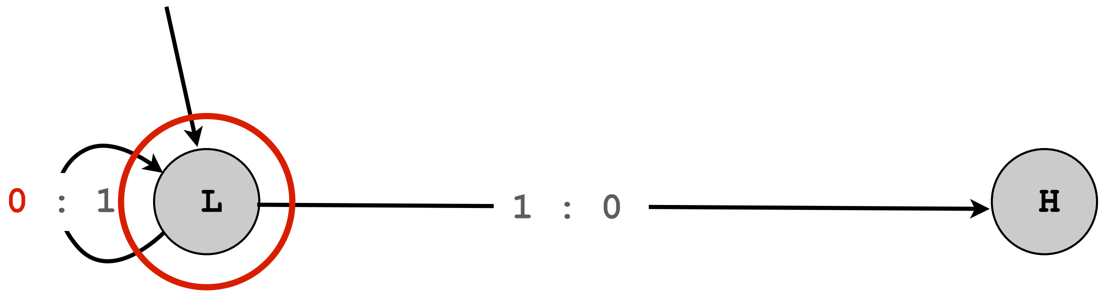
…	#	#	1	0	0	#	…

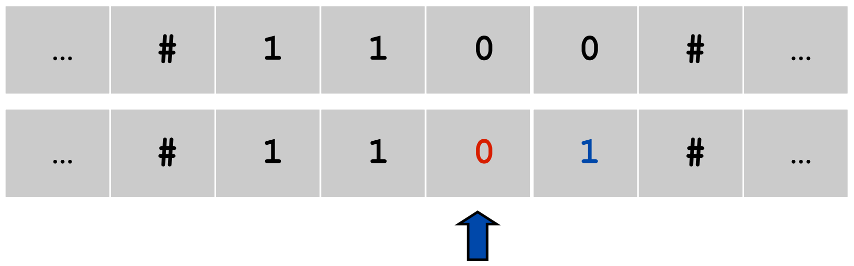
TM Example 3: Binary Decrement





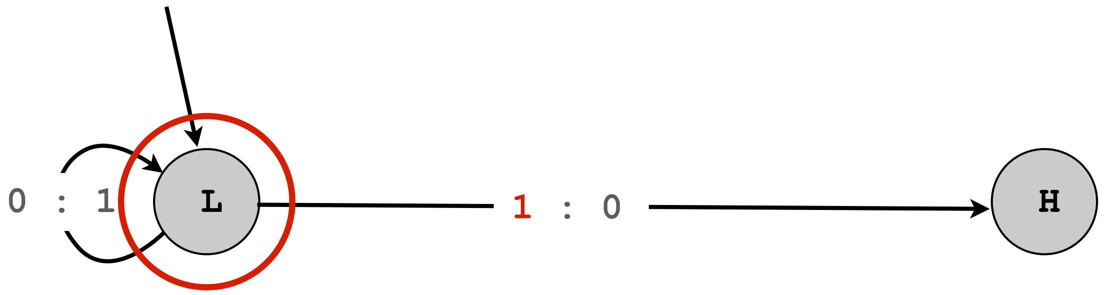
TM Example 3: Binary Decrement

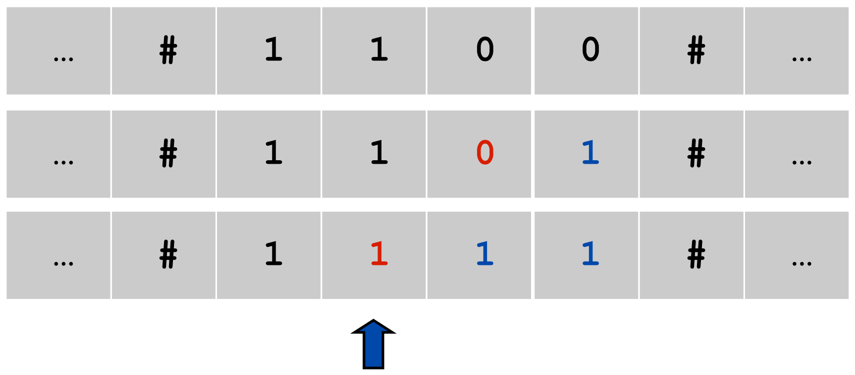




…	#	1	1	0	0	#	…
…	#	1	1	0	1	#	…

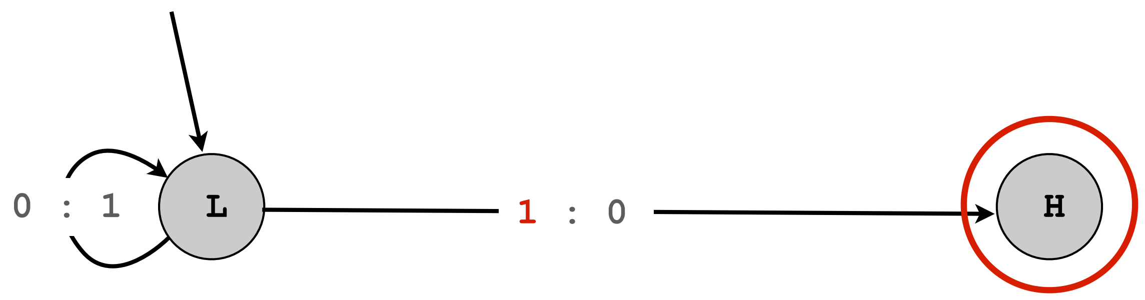
TM Example 3: Binary Decrement





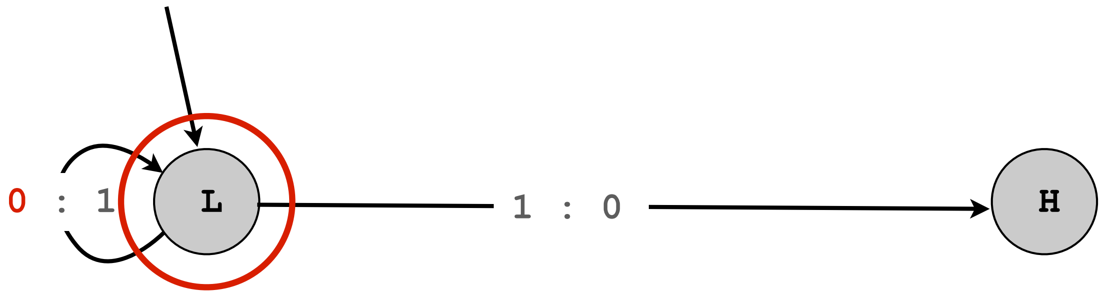
…	#	1	1	0	0	#	…
…	#	1	1	0	1	#	…
…	#	1	1	1	1	#	…

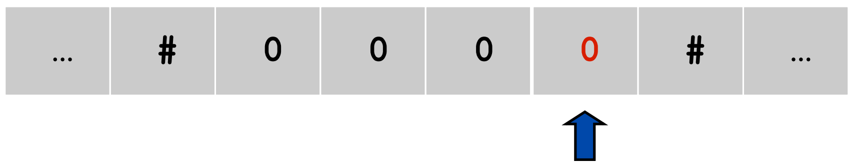
TM Example 3: Binary Decrement



...	#	1	1	0	0	#	...
...	#	1	1	0	1	#	...
...	#	1	0	1	1	#	...

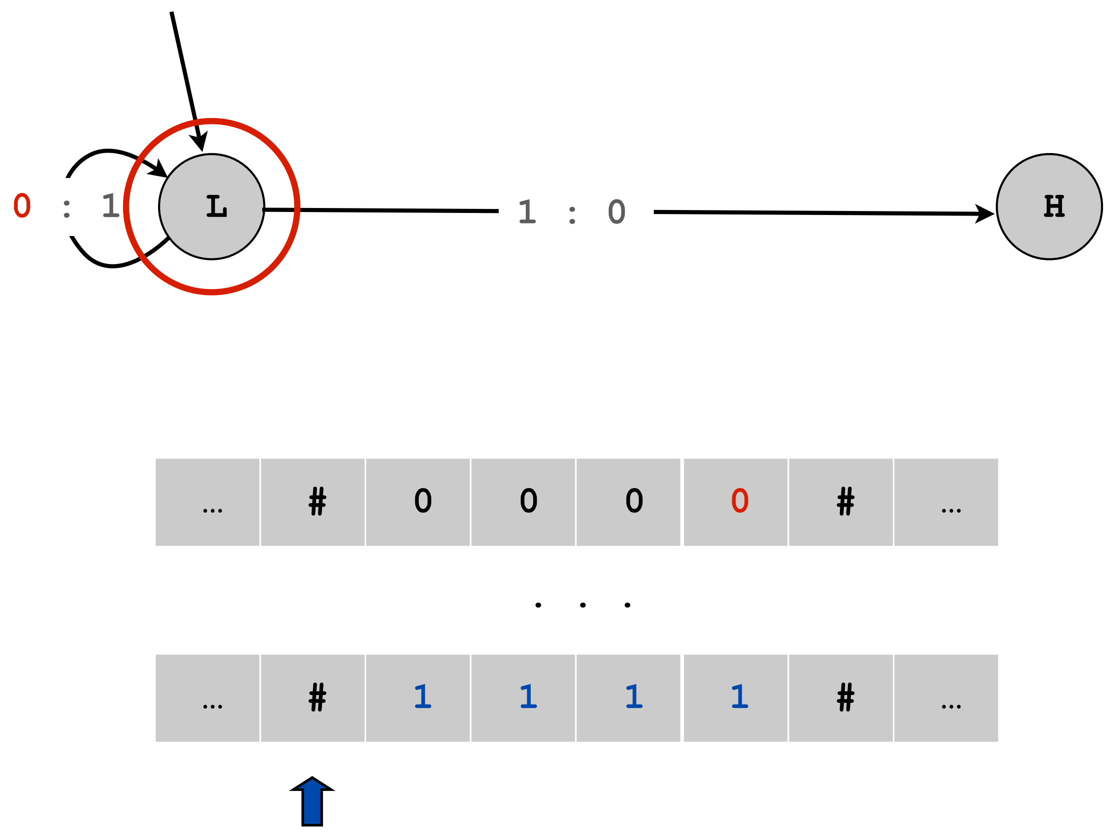
TM Example 3: Binary Decrement





Q. What happens if we try to decrement 0 ?

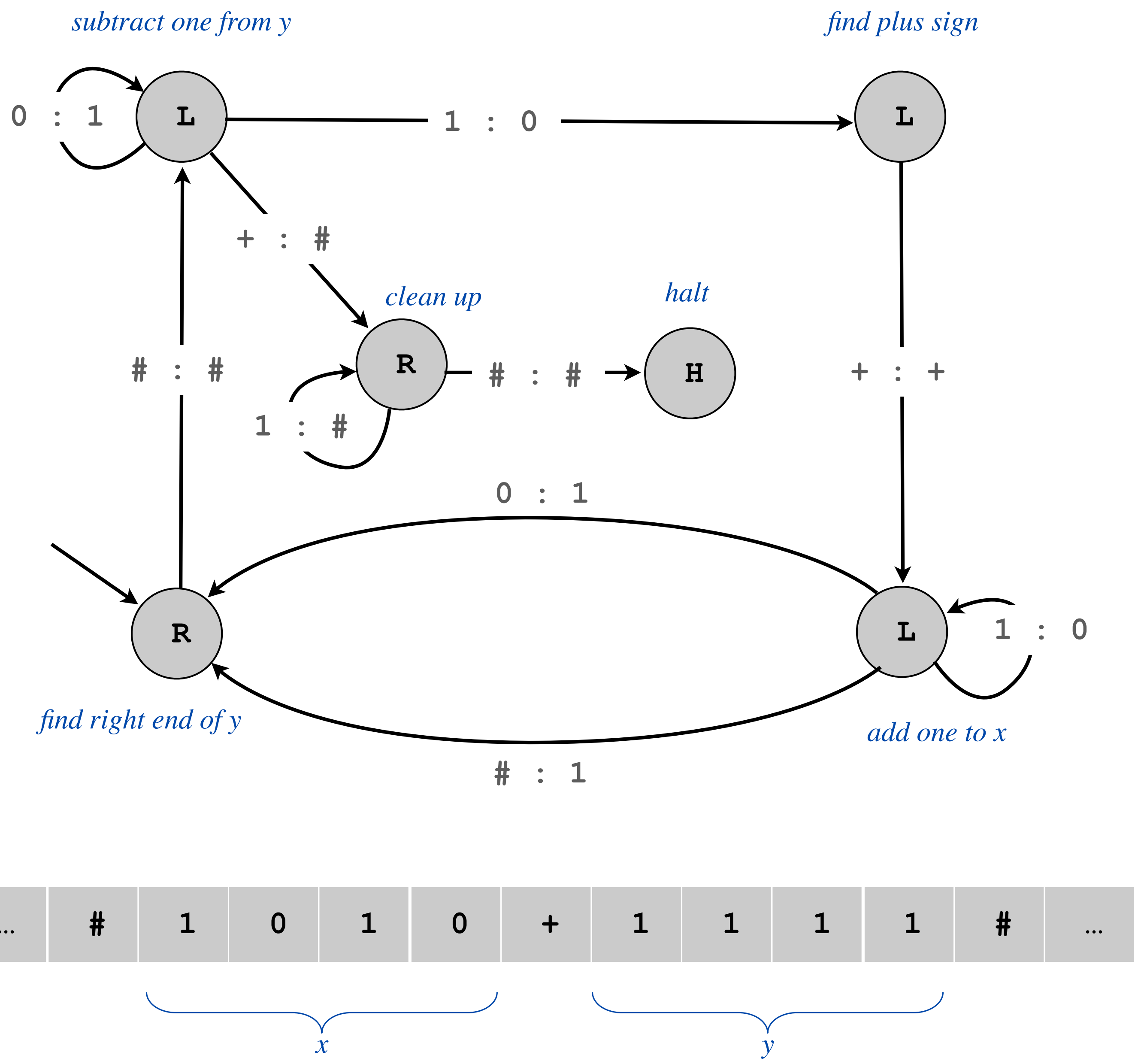
TM Example 3: Binary Decrement



Q. What happens if we try to decrement 0 ?

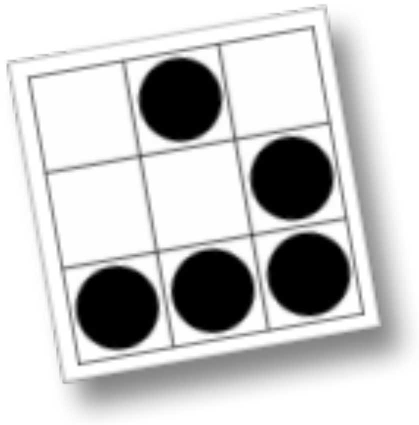
A. Doesn't halt! (TMs can have bugs, too.)

TM Example 4: Binary Adder



Ex. Use simulator to understand how this TM works.

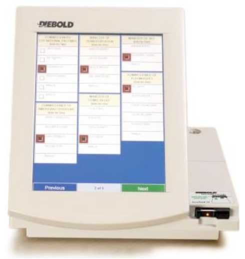
7.5 Universality



Universal Machines and Technologies

Dell PC

iMac



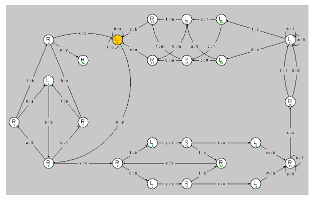
Diebold voting machine

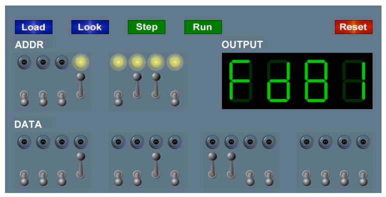
iPod

Printer

Xbox

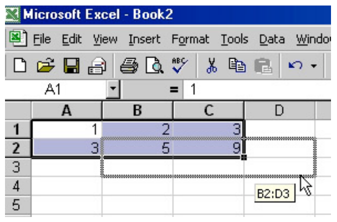
Tivo


Turing machine



TOY

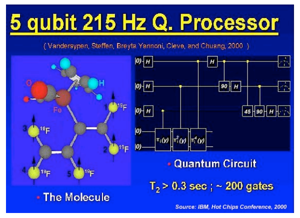
Java language

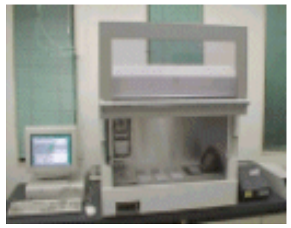


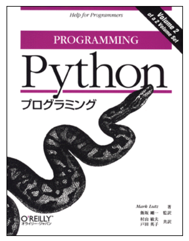
	A	B	C	D
1	1	2	3	
2	3	5	9	
3				
4				
5				

MS Excel

Blackberry


5 qubit 215 Hz Q. Processor
(Vandersypen, Steffen, Breyta, Yannoni, Cleve, and Chuang, 2000)
Quantum Circuit
T_2 > 0.3 sec ; ~ 200 gates
The Molecule
Source: IBM, Hot Chips Conference, 2000
Quantum computer


DNA computer



Python language

Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 1. A compiler is a program that takes a program in one language as input and outputs a program in another language.

Java

machine language

Your program

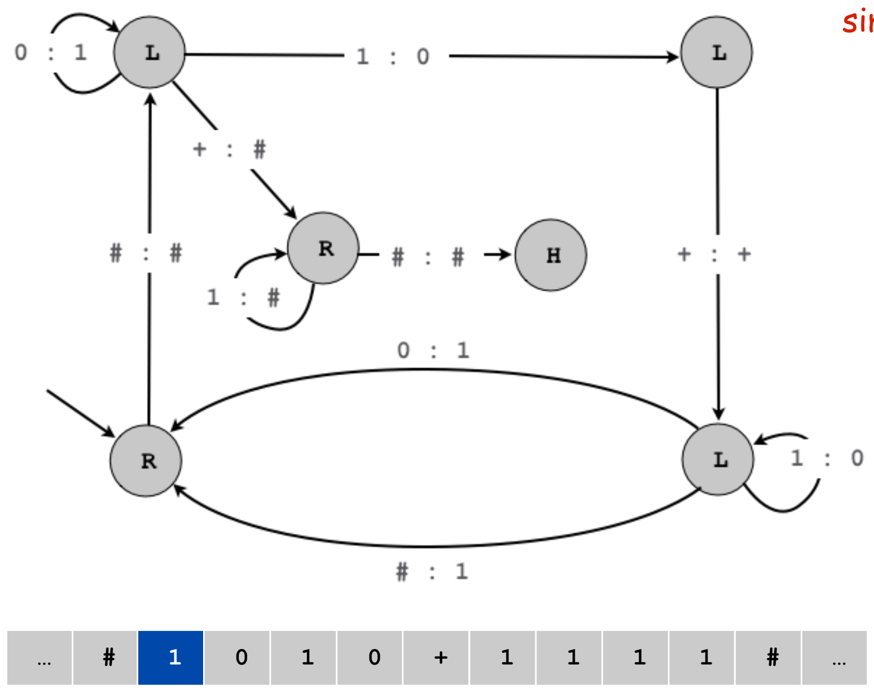
```
public class HelloWorld
{
    public static void main(String[] args)
    {
        System.out.println("Hello, World");
    }
}
```

is DATA to a compiler

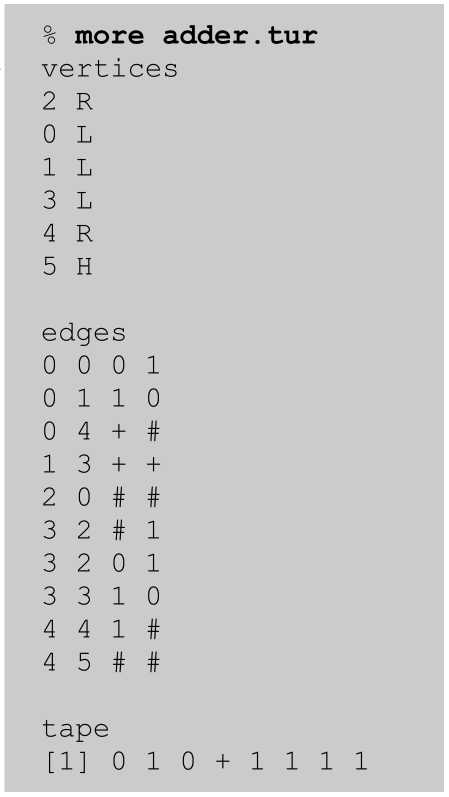
Program and Data

Data. Sequence of symbols (interpreted one way).
Program. Sequence of symbols (interpreted another way).

Ex 2. A simulator is a program that takes a program for one machine as input and simulates the operation of that program.



Data for simulator

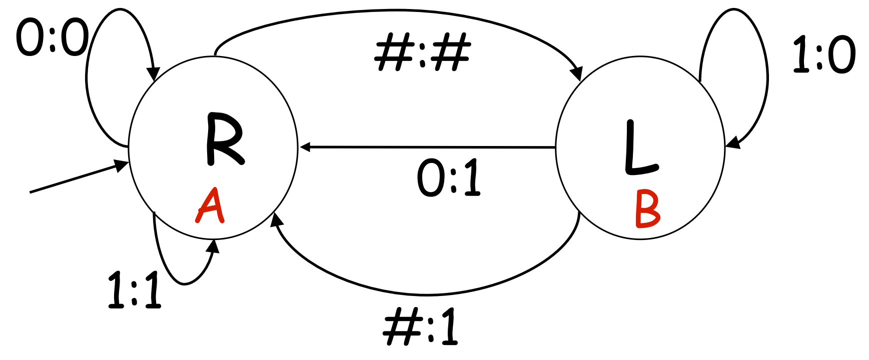


```
% more adder.tur
vertices
2 R
0 L
1 L
3 L
4 R
5 H

edges
0 0 0 1
0 1 1 0
0 4 + #
1 3 + +
2 0 # #
3 2 # 1
3 2 0 1
3 3 1 0
4 4 1 #
4 5 # #

tape
[1] 0 1 0 + 1 1 1 1
```

is a PROGRAM!

Representations of a Turing Machine

Graphical:



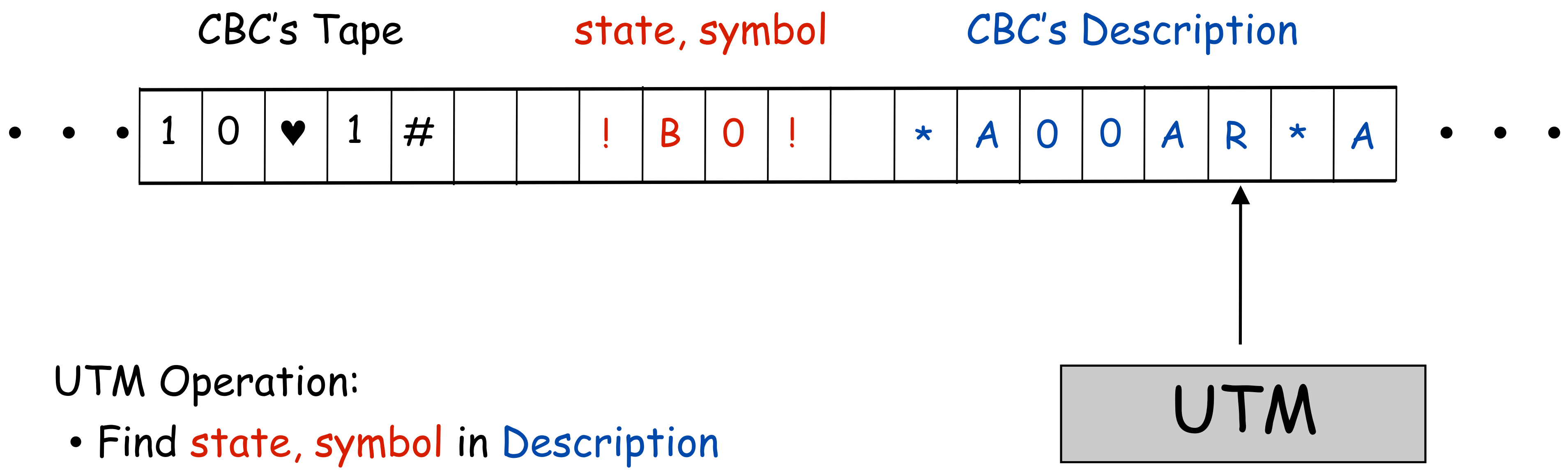
Continuous
Binary
Counter

Tabular:

Current state	Symbol read	Symbol to write	Next State	Direction
A	0	0	A	R
A	1	1	A	R
A	#	#	B	L
B	0	1	A	R
B	1	0	B	L
B	#	1	A	R

Linear: * A 0 0 A R * A 1 1 A R * A # # B L * B 0 1 A R * B 1 0 B L . . .

Universal Turing Machine

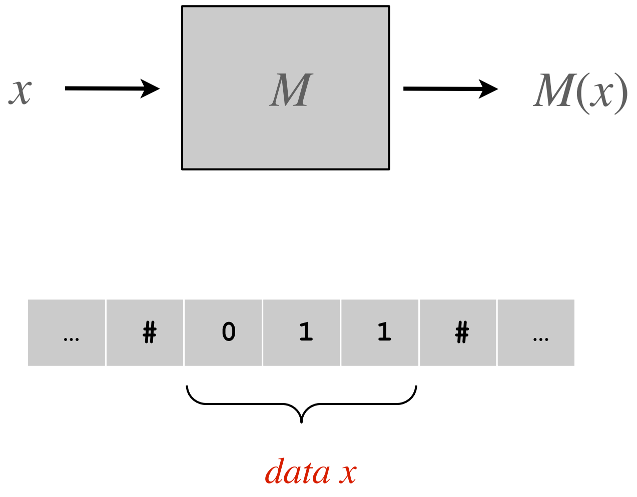


UTM Operation:

- Find state, symbol in Description
- Copy new symbol to CBI's tape
- Move ♥ L or R
- Update state, symbol
- Repeat

Universal Turing Machine

Turing machine M. Given input tape x, Turing machine M outputs $M(x)$.

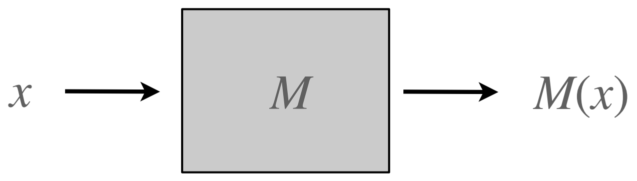


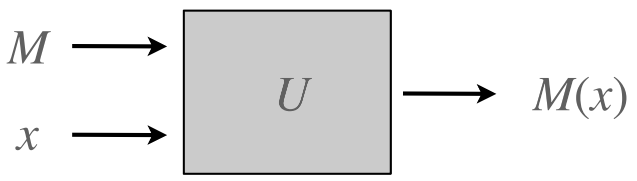
TM intuition. Software program that solves one particular problem.

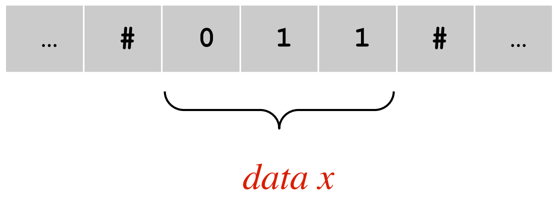
Universal Turing Machine

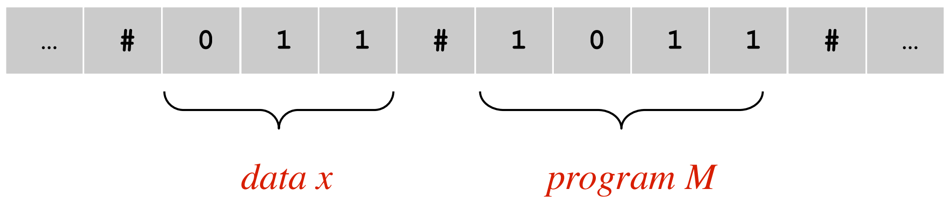
Turing machine M. Given input tape x, Turing machine M outputs $M(x)$.

Universal Turing machine U. Given input tape with x and M, universal Turing machine U outputs $M(x)$.









TM intuition. Software program that solves one particular problem.
UTM intuition. Hardware platform that can implement any algorithm.

Universal Turing Machine

Consequences. Your laptop (a UTM) can do any computational task.

↑ even tasks not yet contemplated when laptop was purchased

- Java programming.
- Pictures, music, movies, games.
- Email, browsing, downloading files, telephony.
- Word-processing, finance, scientific computing.
- . . .

" Again, it [the Analytical Engine] might act upon other things besides numbers… the engine might compose elaborate and scientific pieces of music of any degree of complexity or extent. " — Ada Lovelace

Church-Turing Thesis

Church Turing thesis (1936). Turing machines can do anything that can be described by any physically harnessable process of this universe.

Remark. "Thesis" and not a mathematical theorem because it's a statement about the physical world and not subject to proof.

but can be falsified

Use simulation to prove models equivalent.

- TOY simulator in Java
- Java compiler in TOY.

Implications.

- No need to seek more powerful machines or languages.
- Enables rigorous study of computation (in this universe).

Bottom line. Turing machine is a simple and universal model of computation.

Church-Turing Thesis: Evidence

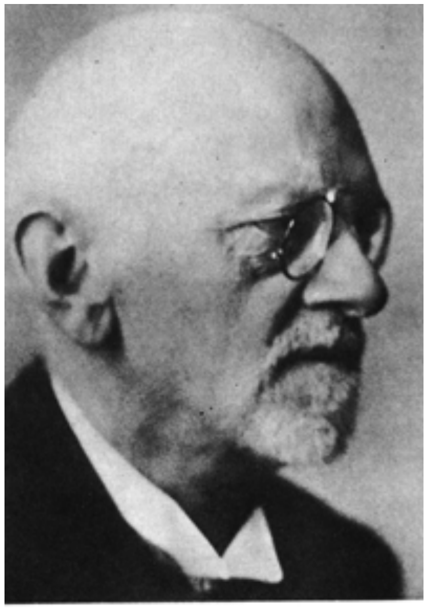
Evidence.

- 7 decades without a counterexample.
- Many, many models of computation that turned out to be equivalent.

"universal"

model of computation	description
enhanced Turing machines	multiple heads, multiple tapes, 2D tape, nondeterminism
untyped lambda calculus	method to define and manipulate functions
recursive functions	functions dealing with computation on integers
unrestricted grammars	iterative string replacement rules used by linguists
extended L-systems	parallel string replacement rules that model plant growth
programming languages	Java, C, C++, Perl, Python, PHP, Lisp, PostScript, Excel
random access machines	registers plus main memory, e.g., TOY, Pentium
cellular automata	cells which change state based on local interactions
quantum computer	compute using superposition of quantum states
DNA computer	compute using biological operations on DNA

7.6 Computability



Take any definite unsolved problem, such as the question as to the irrationality of the Euler-Mascheroni constant γ, or the existence of an infinite number of prime numbers of the form 2^n-1. However unapproachable these problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes.

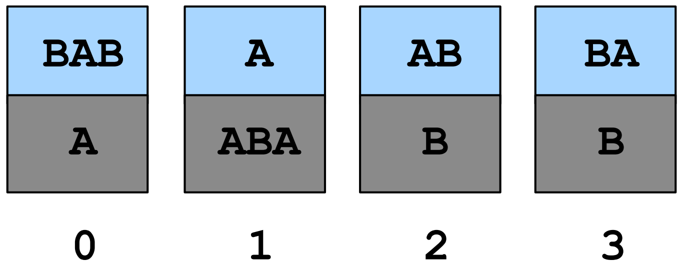
-David Hilbert, in his 1900 address to the International Congress of Mathematics

A Puzzle: Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
- Each card has a top string and bottom string.

Example 1:



N = 4

Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

A Puzzle: Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
- Each card has a top string and bottom string.

Example 1:

BAB	A	AB	BA
A	ABA	B	B
0	1	2	3

N = 4

Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

Solution 1.

Yes.

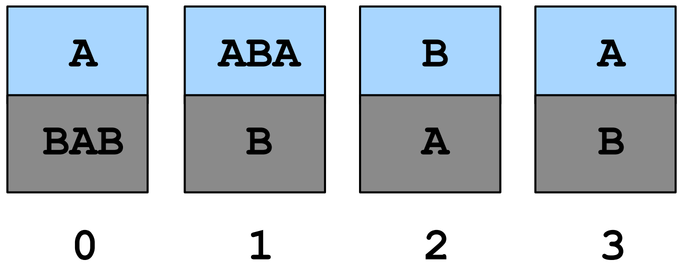
A	BA	BAB	AB	A
ABA	B	A	B	ABA
1	3	0	2	1

A Puzzle: Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
- Each card has a top string and bottom string.

Example 2:



N = 4

Puzzle:

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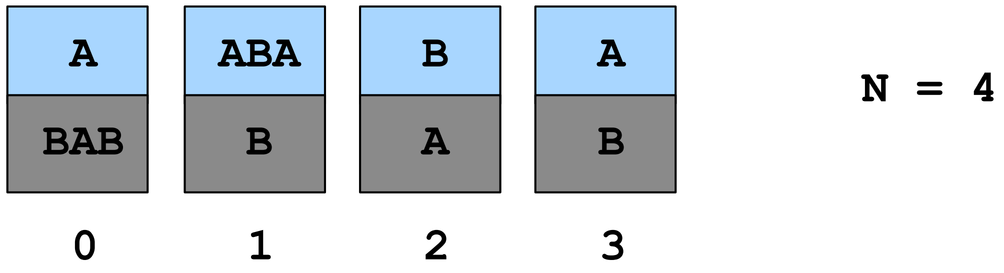
Example 2:



Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

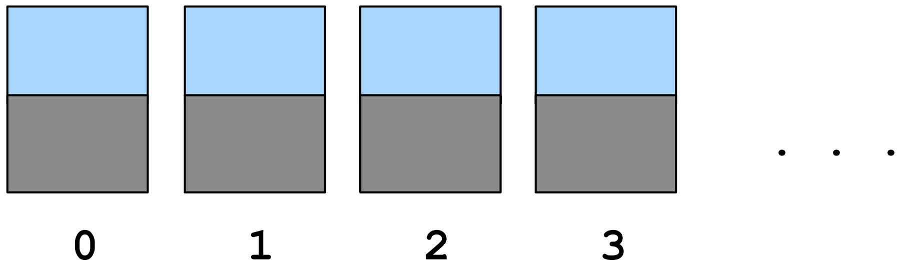
Solution 2.

No. First card in solution must contain same letter in leftmost position.

A Puzzle: Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
- Each card has a top string and bottom string.



Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

Challenge:

- Write a program to take cards as input and solve the puzzle.

A Puzzle: Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
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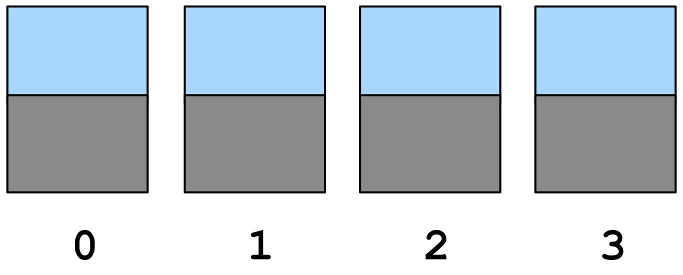

Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

Challenge:

- Write a program to take cards as input and solve the puzzle.

Surprising fact:

- It is NOT POSSIBLE to write such a program!

Halting Problem

Halting problem. Write a Java function that reads in a Java function **`f`** and its input **`x`**, and decides whether **`f(x)`** results in an infinite loop.

Easy for some functions, not so easy for others.

Ex. Does **`f(x)`** terminate?

```
public void f(int x)
{
   while (x != 1)
   {
      if   (x % 2 == 0)  x = x / 2;
      else               x = 3*x + 1;
   }
}
```

relates to famous open math conjecture

```
f(6):     6 3 10 5 16 8 4 2 1
f(27):    27 82 41 124 62 31 94 47 142 71 214 107 322 … 4 2 1
f(-17):   -17 -50 -25 -74 -37 -110 -55 -164 -82 -41 -122 …  -17 …
```

Undecidable Problem

A yes-no problem is undecidable if no Turing machine exists to solve it.

and (by universality) no Java program either

Theorem. [Turing 1937] The halting problem is undecidable.

Proof intuition: lying paradox.

- Divide all statements into two categories: truths and lies.
- How do we classify the statement: "I am lying" ?

Key element of lying paradox and halting proof: self-reference.

Halting Problem: Preliminaries

Some programs take other programs as input

- Java compiler, e.g.

Can a program take itself as input ??

Why not ?

- TextGenerator could take TextGenerator.java as input, produce a Markov model of itself, and generate Java-like text.

- GuitarHero could "play" the characters in GuitarHero.java.

- Almost always a peculiar thing to do, but we'll be interested only in whether the program halts, or goes into an infinite loop.

Halting Problem Proof

Assume the existence of **`halt(f,x)`**:

- Input: a function **`f`** and its input **`x`**.
- Output: **`true`** if **`f(x)`** halts, and **`false`** otherwise.

Note. **`halt(f,x)`** does not go into infinite loop.

We prove by contradiction that **`halt(f,x)`** does not exist.

- Reductio ad absurdum : if any logical argument based on an assumption leads to an absurd statement, then assumption is false.

encode f and x as strings

```
public boolean halt(String f, String x)
{
   if ( something terribly clever ) return true;
   else                             return false;
}
```

hypothetical halting function

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
- Output: `true` if `f(x)` halts, and `false` otherwise.

Construct function `strange(f)` as follows:

- If `halt(f,f)` returns `true`, then `strange(f)` goes into an infinite loop.
- If `halt(f,f)` returns `false`, then `strange(f)` halts.

f is a string so it is legal (if perverse) to use it for second argument

```
public void strange(String f)
{
   if (halt(f, f))
   {
      while (true) { } // an infinite loop
   }
}
```

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
- Output: `true` if `f(x)` halts, and `false` otherwise.

Construct function `strange(f)` as follows:

- If `halt(f,f)` returns `true`, then `strange(f)` goes into an infinite loop.
- If `halt(f,f)` returns `false`, then `strange(f)` halts.

In other words:

- If `f(f)` halts, then `strange(f)` goes into an infinite loop.
- If `f(f)` does not halt, then `strange(f)` halts.

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
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In other words:

- If `f(f)` halts, then `strange(f)` goes into an infinite loop.
- If `f(f)` does not halt, then `strange(f)` halts.

Call `strange()` with ITSELF as input.

- If `strange(strange)` halts then `strange(strange)` does not halt.
- If `strange(strange)` does not halt then `strange(strange)` halts.

Halting Problem Proof

Assume the existence of `halt(f,x)`:

- Input: a function `f` and its input `x`.
- Output: `true` if `f(x)` halts, and `false` otherwise.

Construct function `strange(f)` as follows:

- If `halt(f,f)` returns `true`, then `strange(f)` goes into an infinite loop.
- If `halt(f,f)` returns `false`, then `strange(f)` halts.

In other words:

- If `f(f)` halts, then `strange(f)` goes into an infinite loop.
- If `f(f)` does not halt, then `strange(f)` halts.

Call `strange()` with ITSELF as input.

- If `strange(strange)` halts then `strange(strange)` does not halt.
- If `strange(strange)` does not halt then `strange(strange)` halts.

Either way, a contradiction. Hence `halt(f,x)` cannot exist.

Consequences

Q. Why is debugging hard?

A. All problems below are undecidable.

Halting problem. Give a function f, does it halt on a given input x?

Totality problem. Give a function f, does it halt on every input x?

No-input halting problem. Give a function f with no input, does it halt?

Program equivalence. Do two functions f and g always return same value?

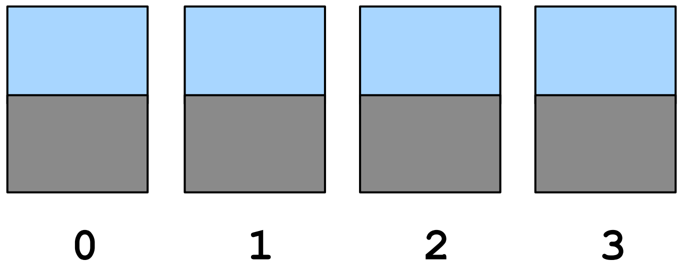
Uninitialized variables. Is the variable x initialized before it's used?

Dead-code elimination. Does this statement ever get executed?

Post's Correspondence Problem

Given a set of cards:

- N card types (can use as many copies of each type as needed).
- Each card has a top string and bottom string.



Puzzle:

- Is it possible to arrange cards so that top and bottom strings match?

Challenge:

- Write a program to take cards as input and solve the puzzle.

is UNDECIDABLE

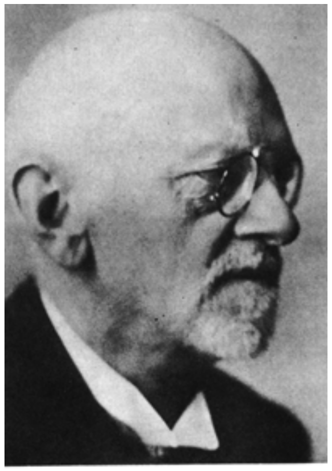
More Undecidable Problems

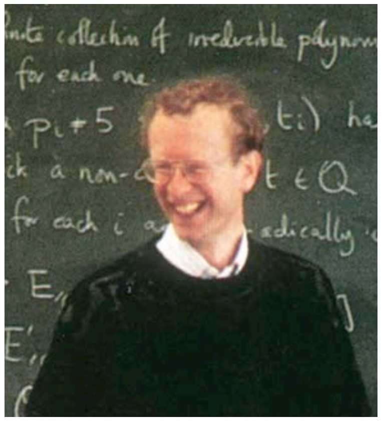
Hilbert's 10th problem.

- "Devise a process according to which it can be determined by a finite number of operations whether a given multivariate polynomial has an integral root."

Examples.

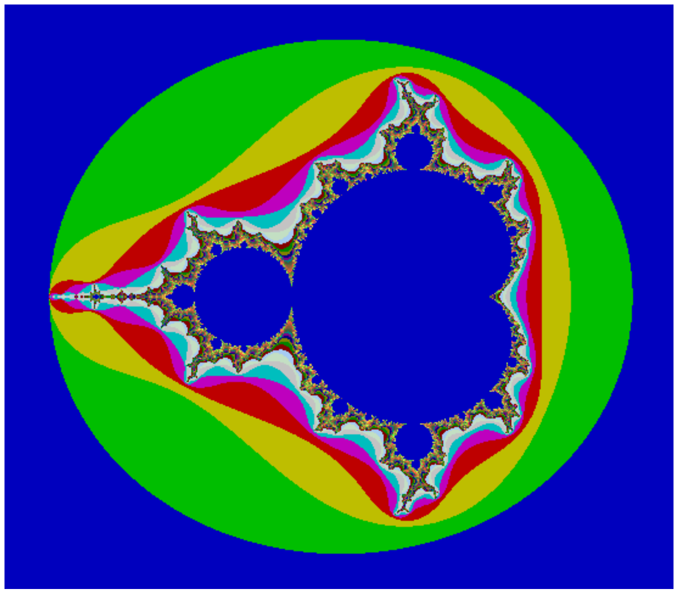
- $f(x, y, z) = 6x^3 y z^2 + 3xy^2 - x^3 - 10.$ ⬅ yes: $f(5, 3, 0) = 0$
- $f(x, y) = x^2 + y^2 - 3.$ ⬅ no
- $f(x, y, z) = x^n + y^n - z^n$ ⬅ yes if $n = 2, x = 3, y = 4, z = 5$
  ⬅ no if $n \geq 3$ and $x, y, z > 0$. (Fermat's Last Theorem)


Hilbert


Andrew Wiles, 1995

More Undecidable Problems

Optimal data compression. Find the shortest program to produce a given string or picture.



Mandelbrot set (40 lines of code)

More Undecidable Problems

Virus identification. Is this program a virus?

```
Private Sub AutoOpen()
On Error Resume Next
If System.PrivateProfileString("", CURRENT_USER\Software\Microsoft\Office\9.0\Word\Security",
                               "Level") <> "" Then
CommandBars("Macro").Controls("Security...").Enabled = False
. . .
For oo = 1 To AddyBook.AddressEntries.Count
   Peep = AddyBook.AddressEntries(x)
   BreakUmOffASlice.Recipients.Add Peep
   x = x + 1
   If x > 50 Then oo = AddyBook.AddressEntries.Count
Next oo
. . .
BreakUmOffASlice.Subject = "Important Message From " & Application.UserName
BreakUmOffASlice.Body = "Here is that document you asked for ... don't show anyone else ;-)"
. . .
```

Can write programs in MS Word.
This statement disables security.

Melissa virus
March 28, 1999

Turing's Key Ideas

Turing machine.

formal model of computation

Program and data.

encode program and data as sequence of symbols

Universality.

concept of general-purpose, programmable computers

Church-Turing thesis.

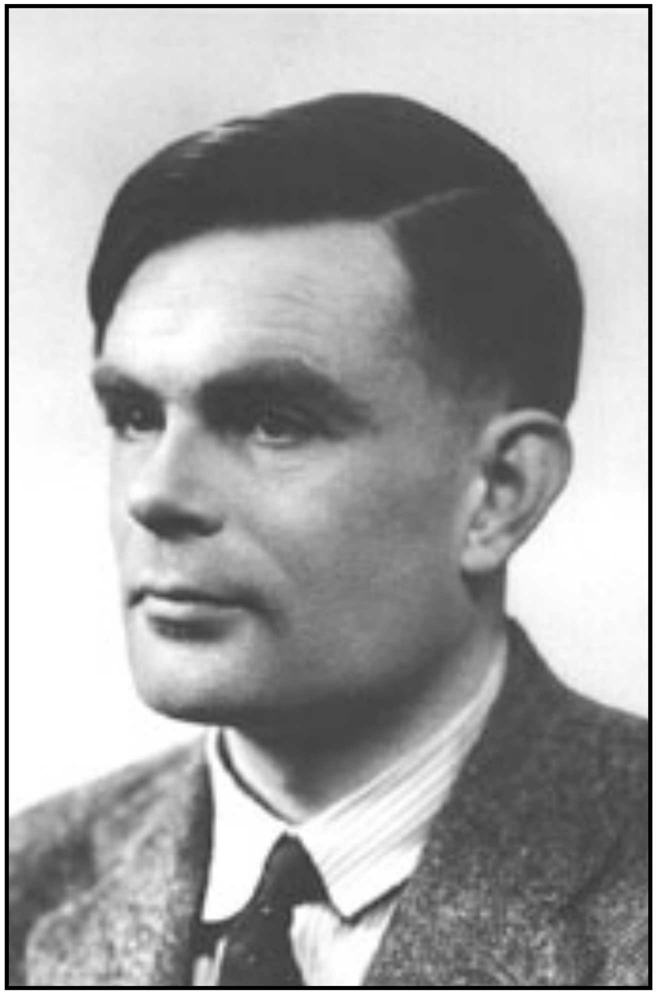
computable at all == computable with a Turing machine

Computability.

inherent limits to computation

Hailed as one of top 10 science papers of 20th century.

Reference: On Computable Numbers, With an Application to the Entscheidungsproblem by A. M. Turing. In Proceedings of the London Mathematical Society, ser. 2. vol. 42 (1936-7), pp.230-265.



Alan Turing
1912-1954