PRINCETON UNIVERSITY SPR'12 COS 521: ADVANCED ALGORITHMS

Homework 1

Out: *Feb 16* Due: *Feb 24*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince the grader. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Suppose we throw n balls randomnly into n bits. Show that the expected number of bins with no balls is approximately $1/e$. Give an approximate estimate (a clean number like $1/e$) for the expected number of bins with i balls, where i is small. Also identify a function $c(n)$ such that the number of bins with $c(n)$ balls is zero with high probability.

Repeat the above calculations when we randomly throw $n \log n$ balls into n bins.

§2 The simplest model for a *random graph* consists of n vertices, and tossing a random fair coin for each pair $\{i, j\}$ to decide whether this edge should be present in the graph. For each k compute the expected number of k-cliques in the graph. (A k-clique is a subset of k vertices where every pair $\{i, j\}$ is connected with an edge.) What is the smallest value of k for which this expectation falls below 1?

§3 A family of hash functions $H = \{h : M \to N\}$ is said to be a *perfect hash family* if for each set $S \subseteq M$ of size $s \leq n$, there exists a hash function $h \in H$ that is perfect for S. Assuming that $n = s$, show that any perfect hash family must have size $2^{\Omega(s)}$.

§4 In class we constructed *pairwise independent* hash functions using random linear functions modulo a prime p. It was mentioned that if we use random univariate polynomials of degree $k-1$ then the hash function is *k-wise independent*. Prove this. (Hint: You will need at some point the fact that a certain matrix called the Vandermonde matrix is invertible.)

§5 Hashing can be viewed as throwing n balls into n bins, using the hash value $h(i)$ to determine which bin to thrown the ith ball in to. Prove that if we use a k-wise independent hash function where k is a constant, then with high probability every bin has at most $O(n^{1/k})$ balls.

§6 Describe a randomized update rule for the list update problem that has competitive ratio somewhat better than 2. Give a short analysis.