Clustering and the *k*-means Algorithm

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 - A museum catalog according to image similarity

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- Goal: segment the data into k groups

 $\{z_1,...,z_N\}$ where $z_i \in \{1,...,K\}$.



500 2-dimensional data points: $\mathbf{x}_n = \langle x_{n,1}, x_{n,2} \rangle$



• What is a good distance function here?



- What is a good distance function here?
- Squared Euclidean distance is reasonable

$$d(\mathbf{x}_n, \mathbf{x}_m) = \sum_{i=1}^{p} (x_{n,i} - x_{m,i})^2 = ||x_n - x_m||^2$$



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- What should k be?
- Automatically choosing *k* is complicated; for now, 4.

k-means



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- The goal of *k*-means is to assign data to clusters and define these clusters with their means.

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Objective function

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- The *k*-means objective function is the sum of the squared distances of each point to each assigned mean

$$F(z_{1:N},\mathbf{m}_{1:k}) = \frac{1}{2} \sum_{n=1}^{N} ||\mathbf{x}_n - \mathbf{m}_{z_n}||^2$$















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- Thus, *k*-means is a *coordinate descent* algorithm.
- It finds a *local minimum*. (Multiple restarts are often necessary.)

Objective for the example data



Round of k-means

Compressing images



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- · Each pixel is associated with a red, green, and blue value
- A 1024 × 1024 image is a collection of 1048576 values (x₁, x₂, x₃), which requires 3M of storage
- How can we use k-means to compress this image?

Vector quantization





• Replace each pixel \mathbf{x}_n with its assignment \mathbf{m}_{z_n} ("paint by numbers").

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- With k = 100, we need 7 bits per pixel plus 100×3 bits ≈ 897 K.

















Measure of distortion



Charlie Brown and Linus VQ Objective

• The objective gives a measure of how distorted the compressed picture is relative to the original picture
Measure of distortion



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- For more clusters, the picture is less distorted.

k-medoids

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- No need to define the mean.
- Each of the clusters is associated with its most typical example

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Outil assignments z_{1:N} do not change

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- Choosing k is a nagging problem in cluster analysis
- Sometimes, the problem determines k
 - A certain required compression in VQ
 - Clustering customers for k salespeople in a business
- Usually, we seek the "natural" clustering, but what does this mean?
- It is not well-defined.

















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- Tibshirani (2001) presents a method for finding this kink.

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- Cluster the location of archeological sites in Israel
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- Choose *k* very carefully, with a complicated computational technique.



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- Exposed carp to different levels of cold
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- (No mention of how k = 23 was chosen.)



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- Chose the number of clusters to get nice results

	Cluster 1	Cluster 2	Cluster 3	Cluster 4	Cluster 5	
Teacher caring	5	– .5 to .5	– .5 to .5	5	1.0	
Peers' academic support	1.0	5	1.0	5	5 to .5	
Parents' academic support	.5	-1.0	5 to .5	5 to .5	1.0	

TABLE 3. Five-Cluster Solution: Z scores on Each Clustering Variable

TABLE 4. Means and Standard Deviations for Each Cluster on Grade 8 Motivational Variables

	Academic Self-Efficacy		Intrinsic Valuing of Education		Teacher-Rated Effort	
Cluster	М	SD	М	SD	М	SD
1. All positive	3.59	.48ª	2.99	.55ª	3.74	.26ª
2. Peer negative, parents very negative	2.44	.66 ^b	2.16	.51 ^b	3.05	.61 ^b
3. Peer positive	3.01	.73°	2.43	.66 ^b	3.26	.66 ^b
Negative teacher and peer	2.47	.63 ^b	2.24	.51 ^b	3.17	.59 ^b
5. Positive teacher and parents	3.19	.65°	2.89	.62ª	3.54	.47ª

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- Draw the conclusion that patterns exist. What's wrong with this?
- *k*-means will find patterns everywhere!

TABLE 2. Percentage distribution of participants, by cluster, and behavioral patterns defining each cluster Cluster type and behavioral patterns 96 Light substance dabblers-infrequent or no current use of substancest 24.4 None have had sev Abstainers-none have ever used substances† or had sex 22.7 Sex dabblers-all have had sex 14.5 Median no. of partners=1 60% used a condom at last sex Infrequent use of substancest Drinkers-all consumed alcohol in past 12 mos. 74 49% report binge drinking Infrequent or no illicit drug use None have had sex Smokers-all smoke cigarettes daily 7.3 Infrequent use of alcohol/illicit drugs 62% have had sex Alcohol-and-sex dabblers-all drink occasionally; all have had sex 5.4 Infrequent tobacco/illicit drug use Binge drinkers-all binge frequently 4.4 Infrequent cigarette, marijuana and other drug use 60% binge ≥1 time/wk. 45% have had sev Heavy dabbiers-all smoke, drink and binge drink with moderate frequency 3.6 45% use mariluana; few use other illicit drugs 91% have had sev Combination sex and drug use-all have had sex: all used alcohol/illicit drug at last sex 3.4 Marijuana users-all use marijuana frequently; few have used other illicit drugs 1.7 94% use alcohol 79% smoke cigarettes 74% have had sex Multiple partners—all report ≥14 sexual partners 1.3 75% report low or moderate use of substances† Sex for drugs or money-all have had sex for drugs or money 1.2 50% report low or moderate use of substances† Median no. of partners=3 High marijuana use and sex-all use marijuana frequently; all have had sex 1.1 All used alcohol/other drug at last sex 82% have had >1 partner (median=6) Marijuana and other drug users-95% report heavy marijuana use; all use other illicit drugs 0.6 68% have had sey 28% used alcohol/other drug at last sex Injection-drug users-all have injected drugs 0.6 82% have had sex Median no. of partners=4 Males who have sex with males—all are males who have had sex with another male 0.3 78% have had multiple partners (median=5) 40% used marijuana in past 30 days 50% use alcohol≥1 time/mo. 17% have had sex for drugs or money

Summary