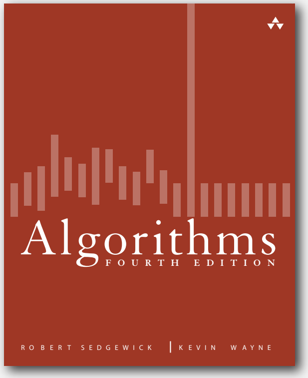
5. Strings



String processing

String. Sequence of characters.

Important fundamental abstraction.
- Information processing.
- Genomic sequences.
- Communication systems (e.g., email).
- Programming systems (e.g., Java programs).
- …

> *“ The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. ”* — *M. V. Olson*

The char data type

C char data type. Typically an 8-bit integer.
- Supports 7-bit ASCII.
- Need more bits to represent certain characters.

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2	SP	!	“	#	$	%	&	‘	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

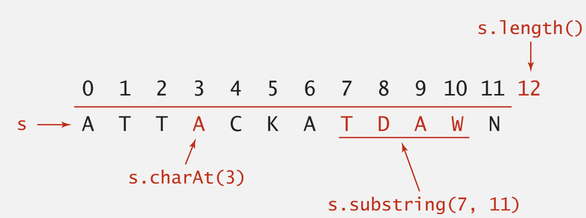
Hexadecimal to ASCII conversion table

Java char data type. A 16-bit unsigned integer.
- Supports original 16-bit Unicode.
- Supports 21-bit Unicode 3.0 (awkwardly).

The String data type

String data type. Sequence of characters (immutable).

Indexing. Get the i^{th} character.
Substring extraction. Get a contiguous sequence of characters from a string.
String concatenation. Append one character to end of another string.



The String data type: Java implementation

```
public final class String implements Comparable<String>
{
   private char[] val;    // characters
   private int offset;    // index of first char in array
   private int length;    // length of string
   private int hash;      // cache of hashCode()

   public int length()
   {  return length; }

   public char charAt(int i)
   {  return value[i + offset];  }

   private String(int offset, int length, char[] val)
   {
      this.offset = offset;
      this.length = length;
      this.val    = val;
   }

   public String substring(int from, int to)
   {  return new String(offset + from, to - from, val);  }
   ...
```

val[]

x	x	s	t	r	i	n	g	x
0	1	2	3	4	5	6	7	8

length (indices 2–7)

offset (index 2)

copy of reference to original char array

The String data type: performance

String data type. Sequence of characters (immutable).

Underlying implementation. Immutable `char[]` array, offset, and length.

	String	
operation	guarantee	extra space
`charAt()`	1	1
`length()`	1	1
`substring()`	1	1
`concat()`	N	N

Memory. $40 + 2N$ bytes for a virgin `String` of length N.

can use `byte[]` or `char[]` instead of `String` to save space (but lose convenience of `String` data type)

The StringBuilder data type

StringBuilder data type. Sequence of characters (mutable).

Underlying implementation. Resizing `char[]` array and length.

	String		StringBuilder	
operation	guarantee	extra space	guarantee	extra space
`charAt()`	1	1	1	1
`length()`	1	1	1	1
`substring()`	1	1	N	N
`concat()`	N	N	1 *	1 *

* amortized

Remark. `StringBuffer` data type is similar, but thread safe (and slower).

String vs. StringBuilder

Q. How to efficiently reverse a string?

A.

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
      rev += s.charAt(i);
   return rev;
}
```

← quadratic time

B.

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

← linear time

String challenge: array of suffixes

Q. How to efficiently form array of suffixes?

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

String vs. StringBuilder

Q. How to efficiently form array of suffixes?

A.

```
public static String[] suffixes(String s)
{
   int N = s.length();
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);
   return suffixes;
}
```

linear time and linear space

B.

```
public static String[] suffixes(String s)
{
   int N = s.length();
   StringBuilder sb = new StringBuilder(s);
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = sb.substring(i, N);
   return suffixes;
}
```

quadratic time and quadratic space

Longest common prefix

Q. How long to compute length of longest common prefix?

p	r	e	f	e	t	c	h
0	1	2	3	4	5	6	7
p	r	e	f	i	x		

```
public static int lcp(String s, String t)
{
   int N = Math.min(s.length(), t.length());
   for (int i = 0; i < N; i++)
      if (s.charAt(i) != t.charAt(i))
         return i;
   return N;
}
```

linear time (worst case)
sublinear time (typical case)

Running time. Proportional to length D of longest common prefix.

Remark. Also can compute `compareTo()` in sublinear time.

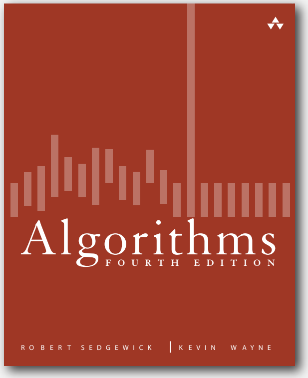
Alphabets

Digital key. Sequence of digits over fixed alphabet.

Radix. Number of digits R in alphabet.

name	R()	lgR()	characters
BINARY	2	1	01
OCTAL	8	3	01234567
DECIMAL	10	4	0123456789
HEXADECIMAL	16	4	0123456789ABCDEF
DNA	4	2	ACTG
LOWERCASE	26	5	abcdefghijklmnopqrstuvwxyz
UPPERCASE	26	5	ABCDEFGHIJKLMNOPQRSTUVWXYZ
PROTEIN	20	5	ACDEFGHIKLMNPQRSTVWY
BASE64	64	6	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef ghijklmnopqrstuvwxyz0123456789+/
ASCII	128	7	*ASCII characters*
EXTENDED_ASCII	256	8	*extended ASCII characters*
UNICODE16	65536	16	*Unicode characters*

5.1 String Sorts



- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

Review: summary of the performance of sorting algorithms

Frequency of operations = key compares.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 / 2$	$N^2 / 4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`

* probabilistic

Lower bound. ~ $N \lg N$ compares are required by any compare-based algorithm.

Q. Can we do better (despite the lower bound)?

A. Yes, if we don't depend on key compares.

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

Key-indexed counting: assumptions about keys

Assumption. Keys are integers between 0 and $R - 1$.

Implication. Can use key as an array index.

Applications.

- Sort string by first letter.
- Sort class roster by section.
- Sort phone numbers by area code.
- Subroutine in a sorting algorithm. [stay tuned]

Remark. Keys may have associated data ⇒ can't just count up number of keys of each value.

input name	section	sorted result (by section)	
Anderson	2	Harris	1
Brown	3	Martin	1
Davis	3	Moore	1
Garcia	4	Anderson	2
Harris	1	Martinez	2
Jackson	3	Miller	2
Johnson	4	Robinson	2
Jones	3	White	2
Martin	1	Brown	3
Martinez	2	Davis	3
Miller	2	Jackson	3
Moore	1	Jones	3
Robinson	2	Taylor	3
Smith	4	Williams	3
Taylor	3	Garcia	4
Thomas	4	Johnson	4
Thompson	4	Smith	4
White	2	Thomas	4
Williams	3	Thompson	4
Wilson	4	Wilson	4

↑ *keys are small integers*

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- **Count frequencies of each letter using key as index.**
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

count frequencies

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

offset by 1 [stay tuned]

r	count[r]
a	0
b	2
c	3
d	1
e	2
f	1
-	3

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- **Compute frequency cumulates which specify destinations.**
- Access cumulates using key as index to move items.
- Copy back into original array.

compute cumulates

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

6 keys < d, 8 keys < e so d's go in a[6] and a[7]

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- **Access cumulates using key as index to move items.**
- Copy back into original array.

move items

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- **Copy back into original array.**

copy back

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting: analysis

Proposition. Key-indexed counting uses $\sim 11\,N + 4\,R$ array accesses to sort N items whose keys are integers between 0 and $R - 1$.

Proposition. Key-indexed counting uses extra space proportional to $N + R$.

Stable? ✔

i	a[i]	key	aux[i]	key
0	Anderson	2	Harris	1
1	Brown	3	Martin	1
2	Davis	3	Moore	1
3	Garcia	4	Anderson	2
4	Harris	1	Martinez	2
5	Jackson	3	Miller	2
6	Johnson	4	Robinson	2
7	Jones	3	White	2
8	Martin	1	Brown	3
9	Martinez	2	Davis	3
10	Miller	2	Jackson	3
11	Moore	1	Jones	3
12	Robinson	2	Taylor	3
13	Smith	4	Williams	3
14	Taylor	3	Garcia	4
15	Thomas	4	Johnson	4
16	Thompson	4	Smith	4
17	White	2	Thomas	4
18	Williams	3	Thompson	4
19	Wilson	4	Wilson	4

- key-indexed counting
- **LSD radix sort**
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

Least-significant-digit-first string sort

LSD string (radix) sort.
- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

i	original	sort key: 3rd char	sort key: 2nd char	sort key: 1st char
0	d a b	d a b	d a b	a c e
1	a d d	c a b	c a b	a d d
2	c a b	e b b	f a d	b a d
3	f a d	a d d	b a d	b e d
4	f e e	f a d	d a d	b e e
5	b a d	b a d	e b b	c a b
6	d a d	d a d	a c e	d a b
7	b e e	f e d	a d d	d a d
8	f e d	b e d	f e d	e b b
9	b e d	f e e	b e d	f a d
10	e b b	b e e	f e e	f e d
11	a c e	a c e	b e e	f e e

sort must be stable (arrows do not cross)

LSD string sort: correctness proof

Proposition. LSD sorts fixed-length strings in ascending order.

Pf. [by induction]

Invariant. After pass i, the strings are sorted by last i characters.
- If two strings differ on sort key, key-indexed sort puts them in proper relative order.
- If two strings agree on sort key, stability keeps them in proper relative order.

i	sorted from previous passes (by induction)	after sort on key (1st char)
0	d a b	a c e
1	c a b	a d d
2	f a d	b a d
3	b a d	b e d
4	d a d	b e e
5	e b b	c a b
6	a c e	d a b
7	a d d	d a d
8	f e d	e b b
9	b e d	f a d
10	f e e	f e d
11	b e e	f e e

LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256;
      int N = a.length;
      String[] aux = new String[N];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < N; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < N; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < N; i++)
            a[i] = aux[i];
      }
   }
}
```

fixed-length W strings

radix R

do key-indexed counting for each digit from right to left

key-indexed counting

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 / 2$	$N^2 / 4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 W N	2 W N	N + R	yes	`charAt()`

* probabilistic
† fixed-length W keys

Q. What if strings do not have same length?

String sorting challenge 1

Problem. Sort a huge commercial database on a fixed-length key.
Ex. Account number, date, Social Security number, ...

Which sorting method to use?
- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

256 (or 65,536) counters;
Fixed-length strings sort in W passes.

B14-99-8765
756-12-AD46
CX6-92-0112
332-WX-9877
375-99-QWAX
CV2-59-0221
387-SS-0321
KJ-0. 12388
715-YT-013C
MJ0-PP-983F
908-KK-33TY
BBN-63-23RE
48G-BM-912D
982-ER-9P1B
WBL-37-PB81
810-F4-J87Q
LE9-N8-XX76
908-KK-33TY
B14-99-8765
CX6-92-0112
CV2-59-0221
332-WX-23SQ
332-6A-9877

String sorting challenge 2a

Problem. Sort one million 32-bit integers.
Ex. Google (or presidential) interview.

Which sorting method to use?
- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

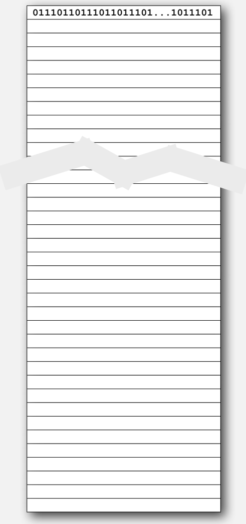
Google CEO Eric Schmidt interviews Barack Obama

String sorting challenge 2b

Problem. Sort huge array of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.



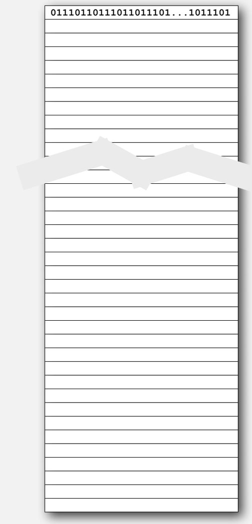
String sorting challenge 2b

Problem. Sort huge array of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

Divide each word into eight 16-bit "chars"
2^{16} = 65,536 counters.
Sort in 8 passes.



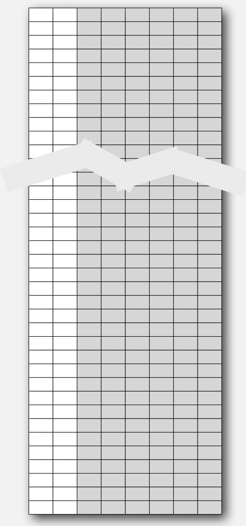
String sorting challenge 2b

Problem. Sort huge array of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- ✓ Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

Divide each word into eight 16-bit "chars"
2^{16} = 65,536 counters
LSD sort on leading 32 bits in 2 passes
Finish with insertion sort
Examines only ~25% of the data



How to take a census in 1900s?

1880 Census. Took 1,500 people 7 years to manually process data.

Herman Hollerith. Developed counting and sorting machine to automate.

- Use punch cards to record data (e.g., gender, age).
- Machine sorts one column at a time (into one of 12 bins).
- Typical question: how many women of age 20 to 30?

Hollerith tabulating machine and sorter

punch card (12 holes per column)

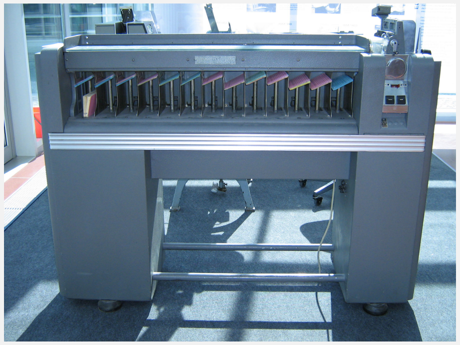
1890 Census. Finished months early and under budget!

How to get rich sorting in 1900s?

Punch cards. [1900s to 1950s]

- Also useful for accounting, inventory, and business processes.
- Primary medium for data entry, storage, and processing.

Hollerith's company later merged with 3 others to form Computing Tabulating Recording Corporation (CTRC); the company was renamed in 1924.



IBM 80 Series Card Sorter (650 cards per minute)

IBM®

LSD string sort: a moment in history (1960s)

card punch

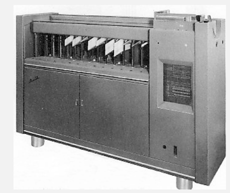
punched cards

card reader

mainframe

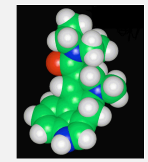
line printer

To sort a card deck
- start on right column
- put cards into hopper
- machine distributes into bins
- pick up cards (stable)
- move left one column
- continue until sorted



card sorter

not related to sorting



Lysergic Acid Diethylamide
(Lucy in the Sky with Diamonds)

- key-indexed counting
- LSD radix sort
- **MSD radix sort**
- 3-way radix quicksort
- suffix arrays

Most-significant-digit-first string sort

MSD string (radix) sort.

- Partition array into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort key

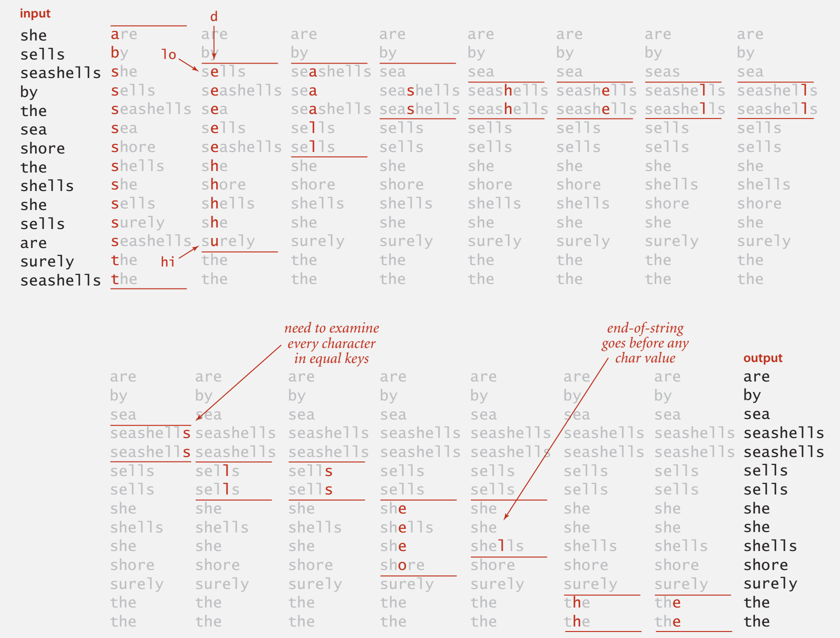
count[]

a	0
b	2
c	5
d	6
e	8
f	9
–	12

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort subarrays recursively

MSD string sort: example



Trace of recursive calls for MSD string sort (no cutoff for small subarrays, subarrays of size 0 and 1 omitted)

Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?

0	s	e	a	-1						
1	s	e	a	s	h	e	l	l	s	-1
2	s	e	l	l	s	-1				
3	s	h	e	-1						
4	s	h	e	-1						
5	s	h	e	l	l	s	-1			
6	s	h	o	r	e	-1				
7	s	u	r	e	l	y	-1			

she before **shells**

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char `'\0'` at end ⇒ no extra work needed.

MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int[] count = new int[R+2];
   for (int i = lo; i <= hi; i++)
      count[charAt(a[i], d) + 2]++;
   for (int r = 0; r < R+1; r++)
      count[r+1] += count[r];
   for (int i = lo; i <= hi; i++)
      aux[count[charAt(a[i], d) + 1]++] = a[i];
   for (int i = lo; i <= hi; i++)
      a[i] = aux[i - lo];

   for (int r = 0; r < R; r++)
      sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);
}
```

can recycle `aux[]` array but not `count[]` array

key-indexed counting

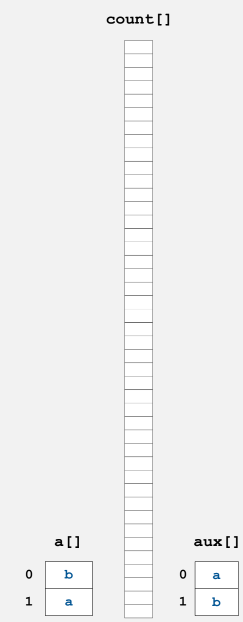
sort R subarrays recursively

MSD string sort: potential for disastrous performance

Observation 1. Much too slow for small subarrays.

- Each function call needs its own `count[]` array.
- ASCII (256 counts): 100x slower than copy pass for $N = 2$.
- Unicode (65,536 counts): 32,000x slower for $N = 2$.

Observation 2. Huge number of small subarrays because of recursion.



Cutoff to insertion sort

Solution. Cutoff to insertion sort for small subarrays.

- Insertion sort, but start at d^{th} character.
- Implement `less()` so that it compares starting at d^{th} character.

```
public static void sort(String[] a, int lo, int hi, int d)
{
   for (int i = lo; i <= hi; i++)
      for (int j = i; j > lo && less(a[j], a[j-1], d); j--)
         exch(a, j, j-1);
}

private static boolean less(String v, String w, int d)
{  return v.substring(d).compareTo(w.substring(d)) < 0;  }
```

in Java, forming and comparing substrings is faster than directly comparing chars with charAt()

MSD string sort: performance

Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear in input size!

compareTo() based sorts can also be sublinear!

Random (sublinear)	Non-random with duplicates (nearly linear)	Worst case (linear)
1EIO402	are	1DNB377
1HYL490	by	1DNB377
1ROZ572	sea	1DNB377
2HXE734	seashells	1DNB377
2IYE230	seashells	1DNB377
2XOR846	sells	1DNB377
3CDB573	sells	1DNB377
3CVP720	she	1DNB377
3IGJ319	she	1DNB377
3KNA382	shells	1DNB377
3TAV879	shore	1DNB377
4CQP781	surely	1DNB377
4QGI284	the	1DNB377
4YHV229	the	1DNB377

Characters examined by MSD string sort

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 / 2$	$N^2 / 4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`

D = function-call stack depth (length of longest prefix match)

* probabilistic
† fixed-length W keys
‡ average-length W keys

MSD string sort vs. quicksort for strings

Disadvantages of MSD string sort.

- Accesses memory "randomly" (cache inefficient).
- Inner loop has a lot of instructions.
- Extra space for `count[]`.
- Extra space for `aux[]`.

Disadvantage of quicksort.

- Linearithmic number of string compares (not linear).
- Has to rescan many characters in keys with long prefix matches.

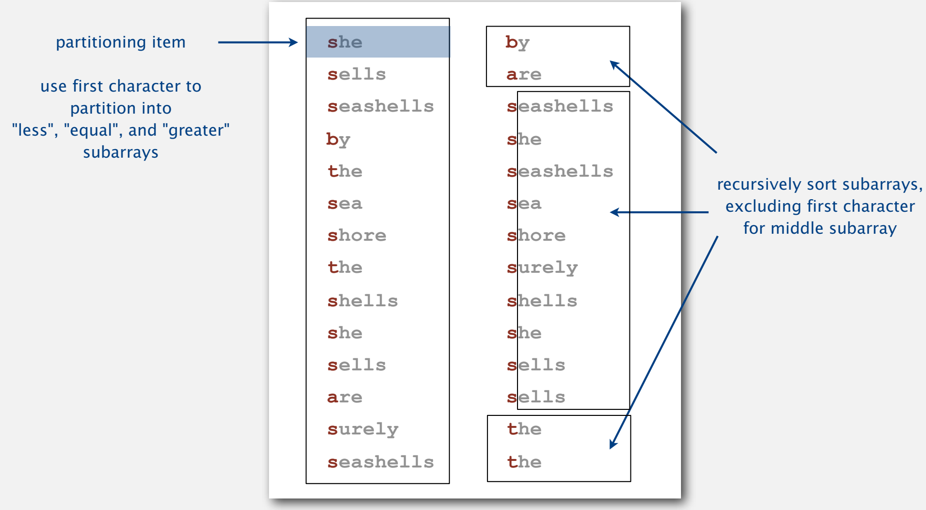
Goal. Combine advantages of MSD and quicksort.

- key-indexed counting
- LSD radix sort
- MSD radix sort
- **3-way radix quicksort**
- suffix arrays

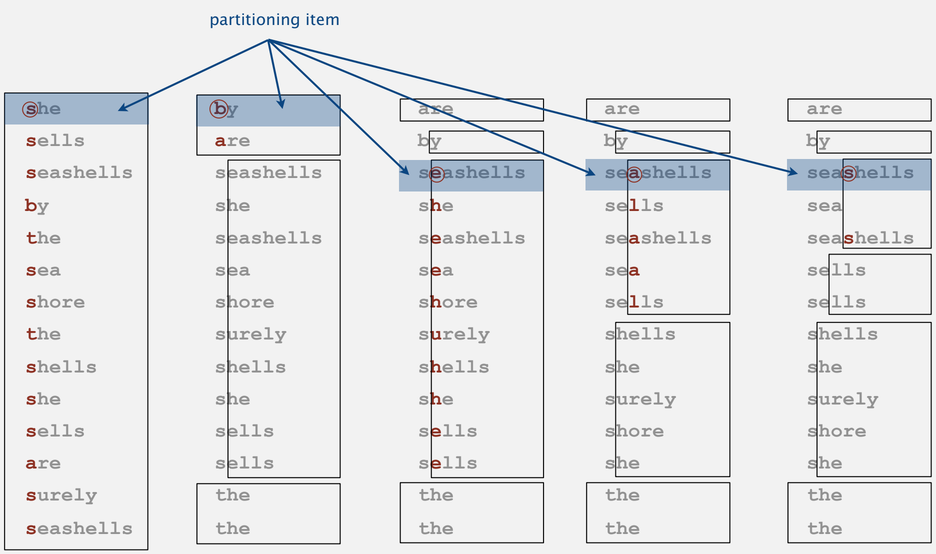
3-way string quicksort (Bentley and Sedgewick, 1997)

Overview. Do 3-way partitioning on the d^{th} character.

- Less overhead than R-way partitioning in MSD string sort.
- Does not re-examine characters equal to the partitioning char (but does re-examine characters not equal to the partitioning char).



3-way string quicksort: trace of recursive calls



Trace of first few recursive calls for 3-way string quicksort (subarrays of size 1 not shown)

3-way string quicksort: Java implementation

```
private static void sort(String[] a)
{  sort(a, 0, a.length - 1, 0);  }

private static void sort(String[] a, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   int v = charAt(a[lo], d);
   int i = lo + 1;
   while (i <= gt)
   {
      int t = charAt(a[i], d);
      if      (t < v) exch(a, lt++, i++);
      else if (t > v) exch(a, i, gt--);
      else            i++;
   }

   sort(a, lo, lt-1, d);
   if (v >= 0) sort(a, lt, gt, d+1);
   sort(a, gt+1, hi, d);
}
```

3-way partitioning (using d^{th} character)

to handle variable-length strings

sort 3 subarrays recursively

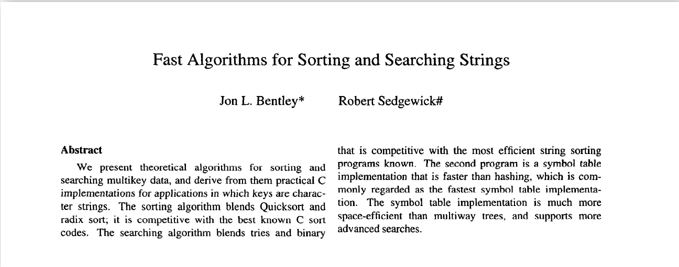
3-way string quicksort vs. standard quicksort

Standard quicksort.

- Uses $\sim 2N \ln N$ string compares on average.
- Costly for keys with long common prefixes (and this is a common case!)

3-way string (radix) quicksort.

- Uses $\sim 2N \ln N$ character compares on average for random strings.
- Avoids re-comparing long common prefixes.



Fast Algorithms for Sorting and Searching Strings

Jon L. Bentley* Robert Sedgewick#

Abstract

We present theoretical algorithms for sorting and searching multikey data, and derive from them practical C implementations for applications in which keys are character strings. The sorting algorithm blends Quicksort and radix sort; it is competitive with the best known C sort codes. The searching algorithm blends tries and binary that is competitive with the most efficient string sorting programs known. The second program is a symbol table implementation that is faster than hashing, which is commonly regarded as the fastest symbol table implementation. The symbol table implementation is much more space-efficient than multiway trees, and supports more advanced searches.

3-way string quicksort vs. MSD string sort

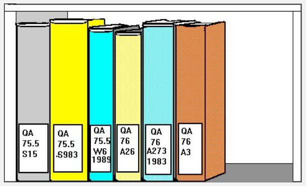
MSD string sort.

- Is cache-inefficient.
- Too much memory storing `count[]`.
- Too much overhead reinitializing `count[]` and `aux[]`.

3-way string quicksort.

- Has a short inner loop.
- Is cache-friendly.
- Is in-place.

library of Congress call numbers



Bottom line. 3-way string quicksort is the method of choice for sorting strings.

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	$N^2 / 2$	$N^2 / 4$	1	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N *	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	1	no	`compareTo()`
LSD †	2 N W	2 N W	N + R	yes	`charAt()`
MSD ‡	2 N W	$N \log_R N$	N + D R	yes	`charAt()`
3-way string quicksort	1.39 W N lg N *	1.39 N lg N	log N + W	no	`charAt()`

* probabilistic
† fixed-length W keys
‡ average-length W keys

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- **suffix arrays**

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

Applications. Bioinformatics, cryptanalysis, data compression, ...

Longest repeated substring: a musical application

Visualize repetitions in music. http://www.bewitched.com

Mary Had a Little Lamb

Bach's Goldberg Variations

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Brute-force algorithm.

- Try all indices i and j for start of possible match.
- Compute longest common prefix (LCP) for each pair.

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
	i								j					

Analysis. Running time ≤ $D N^2$, where D is length of longest match.

Longest repeated substring: a sorting solution

input string

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

form suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

sort suffixes to bring repeated substrings together

0	a a c a a g t t t a c a a g c
11	a a g c
3	a a g t t t a c a a g c
9	a c a a g c
1	a c a a g t t t a c a a g c
12	a g c
4	a g t t t a c a a g c
14	c
10	c a a g c
2	c a a g t t t a c a a g c
13	g c
5	g t t t a c a a g c
8	t a c a a g c
7	t t a c a a g c
6	t t t a c a a g c

compute longest prefix between adjacent suffixes

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Longest repeated substring: Java implementation

```
public String lrs(String s)
{
  int N = s.length();

  String[] suffixes = new String[N];
  for (int i = 0; i < N; i++)
     suffixes[i] = s.substring(i, N);

  Arrays.sort(suffixes);

  String lrs = "";
  for (int i = 0; i < N-1; i++)
  {
     int len = lcp(suffixes[i], suffixes[i+1]);
     if (len > lrs.length())
        lrs = suffixes[i].substring(0, len);
  }
  return lrs;
}
```

create suffixes (linear time and space)

sort suffixes

find LCP between adjacent suffixes in sorted order

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```


Longest repeated substring: empirical analysis

input file	characters	brute	suffix sort	length of LRS
LRS.java	2,162	0.6 sec	0.14 sec	73
amendments.txt	18,369	37 sec	0.25 sec	216
aesop.txt	191,945	1.2 hours	1.0 sec	58
mobydick.txt	1.2 million	43 hours †	7.6 sec	79
chromosome11.txt	7.1 million	2 months †	61 sec	12,567
pi.txt	10 million	4 months †	84 sec	14
pipi.txt	20 million	forever †	???	10 million

† estimated

Suffix sorting: worst-case input

Bad input: longest repeated substring very long.

- Ex: same letter repeated N times.
- Ex: two copies of the same Java codebase.

form suffixes

```
0  a a a a a a a a a a
1  a a a a a a a a a
2  a a a a a a a a
3  a a a a a a a
4  a a a a a a
5  a a a a a
6  a a a a
7  a a a
8  a a
9  a
```

sorted suffixes

```
9  a
8  a a
7  a a a
6  a a a a
5  a a a a a
4  a a a a a a
3  a a a a a a a
2  a a a a a a a a
1  a a a a a a a a a
0  a a a a a a a a a a
```

Running time. Quadratic (or worse) in the length of the longest match.

Suffix sorting challenge

Problem. Suffix sort an arbitrary string of length N.

Q. What is worst-case running time of best algorithm for problem?

- Quadratic.
- ✓ Linearithmic. ← Manber's algorithm
- ✓ Linear. ← suffix trees (see COS 423)
- Nobody knows.

Suffix sorting in linearithmic time

Manber's MSD algorithm overview.

- Phase 0: sort on first character using key-indexed counting sort.
- Phase i: given array of suffixes sorted on first 2^{i-1} characters, create array of suffixes sorted on first 2^i characters.

Worst-case running time. $N \lg N$.

- Finishes after $\lg N$ phases.
- Can perform a phase in linear time. (!) [ahead]

Linearithmic suffix sort example: phase 0

original suffixes

```
 0  b a b a a a a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
 9  b a b a a a a a 0
10  a b a a a a a 0
11  b a a a a a 0
12  a a a a a 0
13  a a a a 0
14  a a a 0
15  a a 0
16  a 0
17  0
```

key-indexed counting sort (first character)

```
17  0
 1  a b a a a a b c b a b a a a a a 0
16  a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
15  a a 0
14  a a a 0
13  a a a a 0
12  a a a a a 0
10  a b a a a a a 0
 0  b a b a a a a b c b a b a a a a a 0
 9  b a b a a a a a 0
11  b a a a a a 0
 7  b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 8  c b a b a a a a a 0
```

↑ *sorted*

Linearithmic suffix sort example: phase 1

original suffixes

```
 0  b a b a a a a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
 9  b a b a a a a a 0
10  a b a a a a a 0
11  b a a a a a 0
12  a a a a a 0
13  a a a a 0
14  a a a 0
15  a a 0
16  a 0
17  0
```

index sort (first two characters)

```
17  0
16  a 0
12  a a a a a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
13  a a a a 0
15  a a 0
14  a a a 0
 6  a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
10  a b a a a a a 0
 0  b a b a a a a b c b a b a a a a a 0
 9  b a b a a a a a 0
11  b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
```

↑ *sorted*

Linearithmic suffix sort example: phase 2

original suffixes

```
 0  b a b a a a a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0
 3  a a a a b c b a b a a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 6  a b c b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
 9  b a b a a a a a 0
10  a b a a a a a 0
11  b a a a a a 0
12  a a a a a 0
13  a a a a 0
14  a a a 0
15  a a 0
16  a 0
17  0
```

index sort (first four characters)

```
17  0
16  a 0
15  a a 0
14  a a a 0
 3  a a a a b c b a b a a a a a 0
12  a a a a a 0
13  a a a a 0
 4  a a a b c b a b a a a a a 0
 5  a a b c b a b a a a a a 0
 1  a b a a a a b c b a b a a a a a 0
10  a b a a a a a 0
 6  a b c b a b a a a a a 0
 2  b a a a a b c b a b a a a a a 0 a 0
11  b a a a a a 0
 0  b a b a a a a b c b a b a a a a a 0
 9  b a b a a a a a 0
 7  b c b a b a a a a a 0
 8  c b a b a a a a a 0
```

↑ *sorted*

Linearithmic suffix sort example: phase 3

original suffixes

0	b a b a a a a b c b a b a a a a a 0
1	a b a a a a b c b a b a a a a a 0
2	b a a a a b c b a b a a a a a 0
3	a a a a b c b a b a a a a a 0
4	a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0
6	a b c b a b a a a a a 0
7	b c b a b a a a a a 0
8	c b a b a a a a a 0
9	b a b a a a a a 0
10	a b a a a a a 0
11	b a a a a a 0
12	a a a a a 0
13	a a a a 0
14	a a a 0
15	a a 0
16	a 0
17	0

index sort (first eight characters)

17	0
16	a 0
15	a a 0
14	a a a 0
13	a a a a 0
12	a a a a a 0
3	a a a a b c b a b a a a a a 0
4	a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0
10	a b a a a a a 0
1	a b a a a a b c b a b a a a a a 0
6	a b c b a b a a a a a 0
11	b a a a a a 0
2	b a a a a b c b a b a a a a a 0 a 0
9	b a b a a a a a 0
0	b a b a a a a b c b a b a a a a a 0
7	b c b a b a a a a a 0
8	c b a b a a a a a 0

finished (no equal keys)

Constant-time string compare by indexing into inverse

original suffixes

0	b a b a a a a b c b a b a a a a a 0
1	a b a a a a b c b a b a a a a a 0
2	b a a a a b c b a b a a a a a 0
3	a a a a b c b a b a a a a a 0
4	a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0
6	a b c b a b a a a a a 0
7	b c b a b a a a a a 0
8	c b a b a a a a a 0
9	b a b a a a a a 0
10	a b a a a a a 0
11	b a a a a a 0
12	a a a a a 0
13	a a a a 0
14	a a a 0
15	a a 0
16	a 0
17	0

index sort (first four characters)

17	0
16	a 0
15	a a 0
14	a a a 0
3	a a a a b c b a b a a a a a 0
12	a a a a a 0
13	a a a a 0
4	a a a b c b a b a a a a a 0
5	a a b c b a b a a a a a 0
1	a b a a a a b c b a b a a a a a 0
10	a b a a a a a 0
6	a b c b a b a a a a a 0
2	b a a a a b c b a b a a a a a 0 a 0
11	b a a a a a 0
0	b a b a a a a b c b a b a a a a a 0
9	b a b a a a a a 0
7	b c b a b a a a a a 0
8	c b a b a a a a a 0

0 + 4 = 4

9 + 4 = 13

inverse

0	14
1	9
2	12
3	4
4	7
5	8
6	11
7	16
8	17
9	15
10	10
11	13
12	5
13	6
14	3
15	2
16	1
17	0

`suffixes₄[13]` ≤ `suffixes₄[4]` (*because* `inverse[13]` < `inverse[4]`)
so `suffixes₈[9]` ≤ `suffixes₈[0]`

Keyword-in-context search

Given a text of N characters, preprocess it to enable fast substring search (find all occurrences of query string context).

```
% java KWIC tale.txt 15
search
o st giless to search for contraband
her unavailing search for your fathe
le and gone in search of her husband
t provinces in search of impoverishe
 dispersing in search of other carri
n that bed and search the straw hold

better thing
t is a far far better thing that i do than
 some sense of better things else forgotte
was capable of better things mr carton ent
```

characters of surrounding context

Applications. Linguistics, databases, web search, word processing, ….

Keyword-in-context search: suffix-sorting solution

- Preprocess: suffix sort the text.
- Query: binary search for query; scan until mismatch.

KWIC search for "search" in Tale of Two Cities

	⋮
632698	s e a l e d _ m y _ l e t t e r _ a n d _ …
713727	s e a m s t r e s s _ i s _ l i f t e d _ …
660598	s e a m s t r e s s _ o f _ t w e n t y _ …
67610	s e a m s t r e s s _ w h o _ w a s _ w i …
4430	s e a r c h _ f o r _ c o n t r a b a n d …
42705	s e a r c h _ f o r _ y o u r _ f a t h e …
499797	s e a r c h _ o f _ h e r _ h u s b a n d …
182045	s e a r c h _ o f _ i m p o v e r i s h e …
143399	s e a r c h _ o f _ o t h e r _ c a r r i …
411801	s e a r c h _ t h e _ s t r a w _ h o l d …
158410	s e a r e d _ m a r k i n g _ a b o u t _ …
691536	s e a s _ a n d _ m a d a m e _ d e f a r …
536569	s e a s e _ a _ t e r r i b l e _ p a s s …
484763	s e a s e _ t h a t _ h a d _ b r o u g h …
	⋮

String sorting summary

We can develop linear-time sorts.

- Key compares not necessary for string keys.
- Use characters as index in an array.

We can develop sublinear-time sorts.

- Should measure amount of data in keys, not number of keys.
- Not all of the data has to be examined.

3-way string quicksort is asymptotically optimal.

- $1.39\, N \lg N$ chars for random data.

Long strings are rarely random in practice.

- Goal is often to learn the structure!
- May need specialized algorithms.