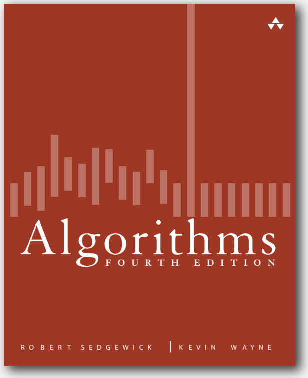
# 4.4 Shortest Paths




- edge-weighted digraph API
- shortest-paths properties
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

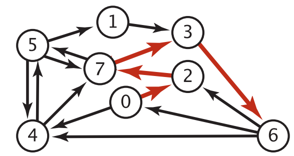


## Shortest paths in a weighted digraph

Given an edge-weighted digraph, find the shortest (directed) path from $s$ to $t$.

edge-weighted digraph

| | |
|---|---|
| 4->5 | 0.35 |
| 5->4 | 0.35 |
| 4->7 | 0.37 |
| 5->7 | 0.28 |
| 7->5 | 0.28 |
| 5->1 | 0.32 |
| 0->4 | 0.38 |
| 0->2 | 0.26 |
| 7->3 | 0.39 |
| 1->3 | 0.29 |
| 2->7 | 0.34 |
| 6->2 | 0.40 |
| 3->6 | 0.52 |
| 6->0 | 0.58 |
| 6->4 | 0.93 |

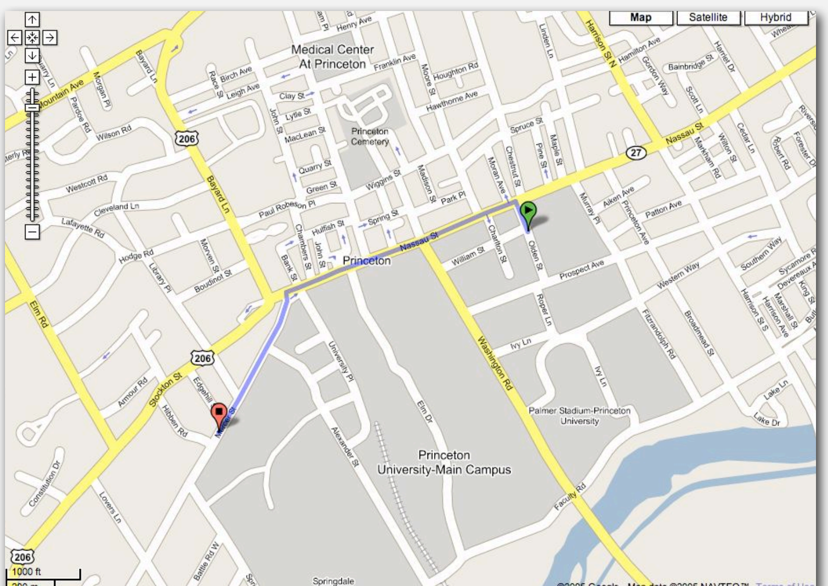



shortest path from 0 to 6

| | |
|---|---|
| 0->2 | 0.26 |
| 2->7 | 0.34 |
| 7->3 | 0.39 |
| 3->6 | 0.52 |

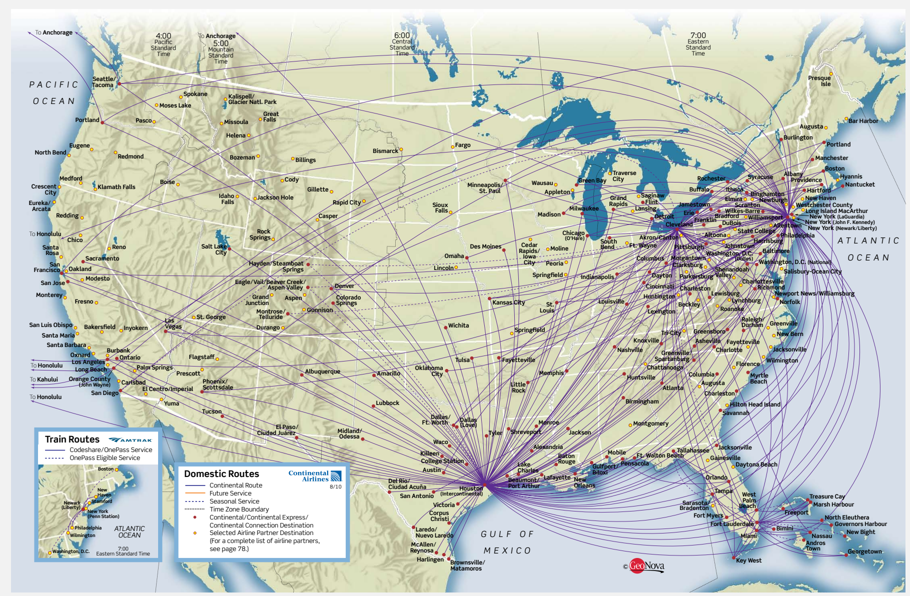


## Google maps






## Continental U.S. routes (August 2010)




http://www.continental.com/web/en-US/content/travel/routes

## Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.



## Shortest path variants

### Which vertices?

- Source-sink: from one vertex to another.
- Single source: from one vertex to every other.
- All pairs: between all pairs of vertices.

### Restrictions on edge weights?

- Nonnegative weights.
- Arbitrary weights.
- Euclidean weights.

### Cycles?

- No directed cycles.
- No "negative cycles."

Simplifying assumption. There exists a shortest path from $s$ to each vertex $v$.



- edge-weighted digraph API
- shortest-paths properties
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## Weighted directed edge API

`public class DirectedEdge`

| | | |
|---|---|---|
| | `DirectedEdge(int v, int w, double weight)` | *weighted edge v→w* |
| `int` | `from()` | *vertex v* |
| `int` | `to()` | *vertex w* |
| `double` | `weight()` | *weight of this edge* |
| `String` | `toString()` | *string representation* |


Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

## Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

`from()` and `to()` replace `either()` and `other()`



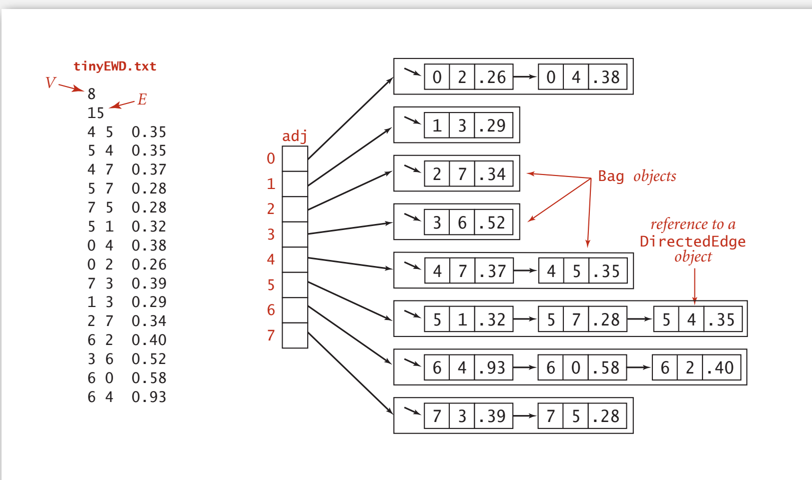
## Edge-weighted digraph API

| | `public class EdgeWeightedDigraph` | |
|---|---|---|
| | `EdgeWeightedDigraph(int V)` | *edge-weighted digraph with V vertices* |
| | `EdgeWeightedDigraph(In in)` | *edge-weighted digraph from input stream* |
| `void` | `addEdge(DirectedEdge e)` | *add weighted directed edge e* |
| `Iterable<DirectedEdge>` | `adj(int v)` | *edges pointing from v* |
| `int` | `V()` | *number of vertices* |
| `int` | `E()` | *number of edges* |
| `Iterable<DirectedEdge>` | `edges()` | *all edges* |
| `String` | `toString()` | *string representation* |

Conventions. Allow self-loops and parallel edges.



## Edge-weighted digraph: adjacency-lists representation






## Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w only to v's adjacency list

## Single-source shortest paths API

**Goal.** Find the shortest path from $s$ to every other vertex.

| public class SP | | |
|---|---|---|
| | `SP(EdgeWeightedDigraph G, int s)` | *shortest paths from s in graph G* |
| `double` | `distTo(int v)` | *length of shortest path from s to v* |
| `Iterable <DirectedEdge>` | `pathTo(int v)` | *shortest path from s to v* |
| `boolean` | `hasPathTo(int v)` | *is there a path from s to v?* |

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
   StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
   for (DirectedEdge e : sp.pathTo(v))
      StdOut.print(e + "  ");
   StdOut.println();
}
```



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| `boolean` | `hasPathTo(int v)` | *is there a path from s to v?* |

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```



- edge-weighted digraph API
- **shortest-paths properties**
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights



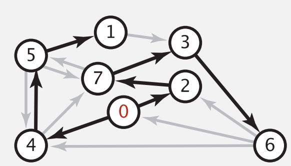
## Data structures for single-source shortest paths

**Goal.** Find the shortest path from $s$ to every other vertex.

**Observation.** A shortest-paths tree (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from $s$ to $v$.
- `edgeTo[v]` is last edge on shortest path from $s$ to $v$.




| | edgeTo[] | | distTo[] |
|---|---|---|---|
| 0 | null | | 0 |
| 1 | 5->1 | 0.32 | 1.05 |
| 2 | 0->2 | 0.26 | 0.26 |
| 3 | 7->3 | 0.37 | 0.97 |
| 4 | 0->4 | 0.38 | 0.38 |
| 5 | 4->5 | 0.35 | 0.73 |
| 6 | 3->6 | 0.52 | 1.49 |
| 7 | 2->7 | 0.34 | 0.60 |

**shortest-paths tree from 0**

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- `distTo[v]` is length of shortest path from $s$ to $v$.
- `edgeTo[v]` is last edge on shortest path from $s$ to $v$.

```
public double distTo(int v)
{  return distTo[v];  }
```

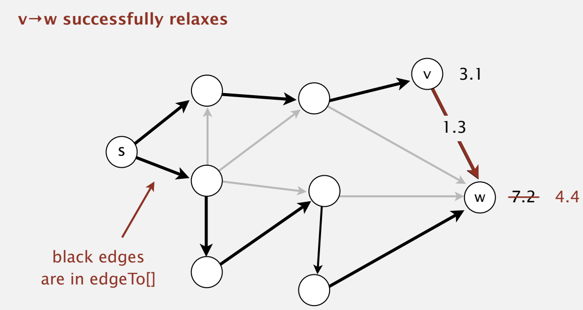
```
public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```



## Edge relaxation

**Relax edge** $e = v \to w$.

- `distTo[v]` is length of shortest known path from $s$ to $v$.
- `distTo[w]` is length of shortest known path from $s$ to $w$.
- `edgeTo[w]` is last edge on shortest known path from $s$ to $w$.
- If $e = v \to w$ gives shorter path to $w$ through $v$, update `distTo[w]` and `edgeTo[w]`.



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- `edgeTo[w]` is last edge on shortest known path from $s$ to $w$.
- If $e = v \to w$ gives shorter path to $w$ through $v$, update `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```



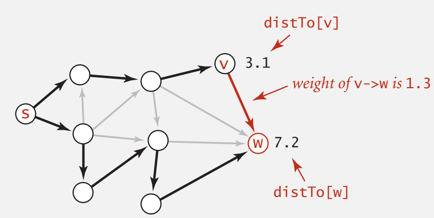
## Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from $s$ iff:

- For each vertex $v$, `distTo[v]` is the length of some path from $s$ to $v$.
- For each edge $e = v \to w$, `distTo[w] ≤ distTo[v] + e.weight()`.

**Pf.** $\Leftarrow$ [ necessary ]

- Suppose that `distTo[w] > distTo[v] + e.weight()` for some edge $e = v \to w$.
- Then, $e$ gives a path from $s$ to $w$ (through $v$) of length less than `distTo[w]`.



## Shortest-paths optimality conditions

**Proposition.** Let $G$ be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from $s$ iff:

- For each vertex $v$, `distTo[v]` is the length of some path from $s$ to $v$.
- For each edge $e = v \rightarrow w$, `distTo[w] ≤ distTo[v] + e.weight()`.

**Pf.** $\Rightarrow$ [ sufficient ]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from $s$ to $w$.
- Then,

$$\left.\begin{array}{lll} \texttt{distTo}[v_k] & \le & \texttt{distTo}[v_{k-1}] + e_k\texttt{.weight()} \\ \texttt{distTo}[v_{k-1}] & \le & \texttt{distTo}[v_{k-2}] + e_{k-1}\texttt{.weight()} \\ \ldots & & \\ \texttt{distTo}[v_1] & \le & \texttt{distTo}[v_0] + e_1\texttt{.weight()} \end{array}\right\} \quad e_i = i^{th} \text{ edge on shortest path from s to w}$$

- Add inequalities; simplify; and substitute `distTo[`$v_0$`] = distTo[s] = 0`:

$$\underbrace{\texttt{distTo[w]}}_{\text{weight of some path from s to w}} = \texttt{distTo}[v_k] \le \underbrace{e_k\texttt{.weight()} + e_{k-1}\texttt{.weight()} + \ldots + e_1\texttt{.weight()}}_{\text{weight of shortest path from s to w}}$$

- Thus, `distTo[w]` is the weight of shortest path to $w$. ▪



## Generic shortest-paths algorithm

**Generic algorithm (to compute SPT from s)**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat until optimality conditions are satisfied:**
- **Relax any edge.**

**Proposition.** Generic algorithm computes SPT (if it exists) from $s$.

**Pf sketch.**

- Throughout algorithm, `distTo[v]` is the length of a simple path from $s$ to $v$ (and `edgeTo[v]` is last edge on path).
- Each successful relaxation decreases `distTo[v]` for some $v$.
- The entry `distTo[v]` can decrease at most a finite number of times. ▪



## Generic shortest-paths algorithm

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**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat until optimality conditions are satisfied:**
- **Relax any edge.**

**Efficient implementations.** How to choose which edge to relax?

**Ex 1.** Dijkstra's algorithm (nonnegative weights).

**Ex 2.** Topological sort algorithm (no directed cycles).

**Ex 3.** Bellman-Ford algorithm (no negative cycles).

## Edsger W. Dijkstra: select quotes

*“ Do only what only you can do. ”*

*“ In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. ”*

*“ The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. ”*

*“ It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. ”*

*“ APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. ”*

Edsger W. Dijkstra
Turing award 1972

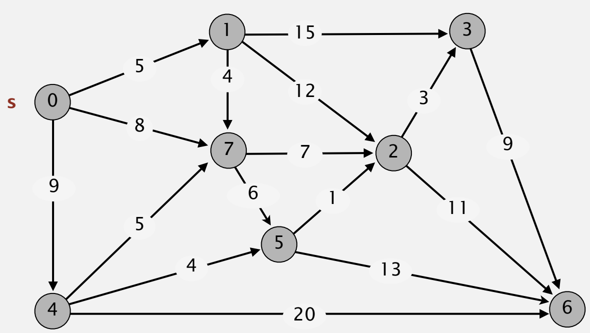


## Edsger W. Dijkstra: select quotes




## Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.




an edge-weighted digraph

| | |
|---|---|
| 0→1 | 5.0 |
| 0→4 | 9.0 |
| 0→7 | 8.0 |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
| 1→7 | 4.0 |
| 2→3 | 3.0 |
| 2→6 | 11.0 |
| 3→6 | 9.0 |
| 4→5 | 4.0 |
| 4→6 | 20.0 |
| 4→7 | 5.0 |
| 5→2 | 1.0 |
| 5→6 | 13.0 |
| 7→5 | 6.0 |
| 7→2 | 7.0 |



## Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from $s$ (non-tree vertex with the lowest `distTo[]` value).
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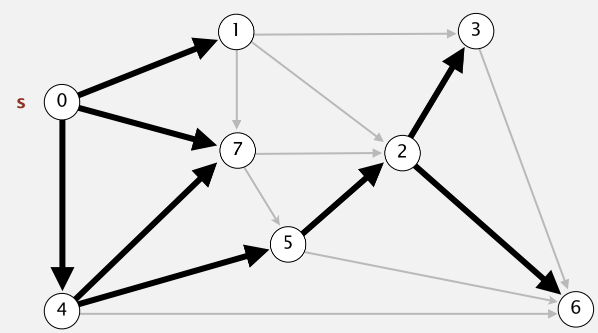
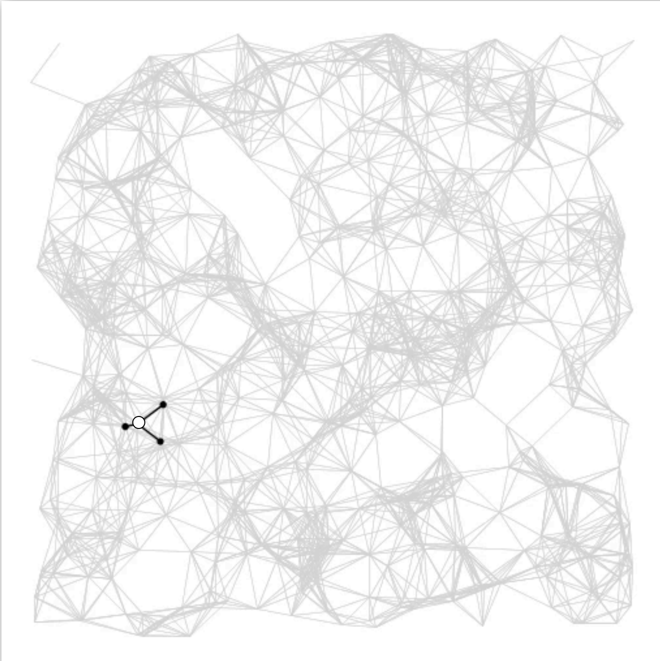



shortest-paths tree from vertex s

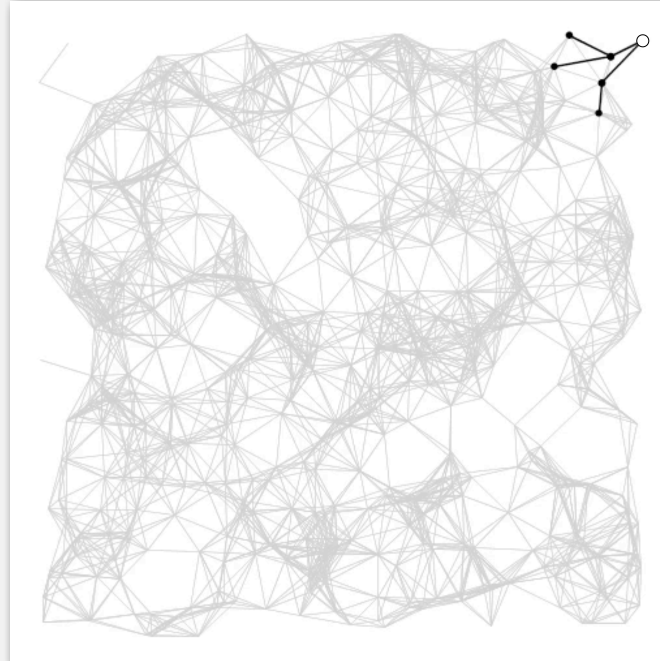
| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

## Dijkstra's algorithm visualization





## Dijkstra's algorithm visualization





## Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge $e = v \to w$ is relaxed exactly once (when $v$ is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← edge weights are nonnegative and we choose lowest `distTo[]` value at each step
- Thus, upon termination, shortest-paths optimality conditions hold. ▪



## Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
   }
 }
```

relax vertices in order of distance from s

## Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ



## Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: $V$ insert, $V$ delete-min, $E$ decrease-key.

| PQ implementation | insert | delete-min | decrease-key | total |
|---|---|---|---|---|
| array | 1 | V | 1 | $V^2$ |
| binary heap | log V | log V | log V | E log V |
| d-way heap (Johnson 1975) | d $\log_d$ V | d $\log_d$ V | $\log_d$ V | E $\log_{E/V}$ V |
| Fibonacci heap (Fredman-Tarjan 1984) | 1 † | log V † | 1 † | E + V log V |

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- d-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

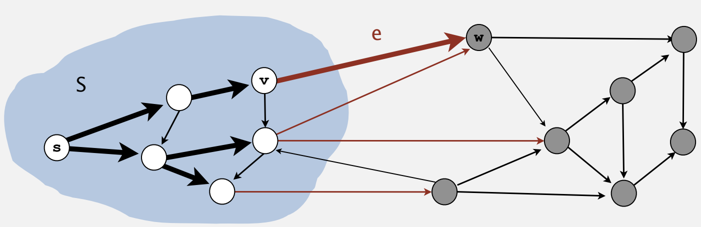


## Priority-first search

Insight. Four of our graph-search methods are the same algorithm!

- Maintain a set of explored vertices $S$.
- Grow $S$ by exploring edges with exactly one endpoint leaving $S$.

DFS. Take edge from vertex which was discovered most recently.
BFS. Take edge from vertex which was discovered least recently.
Prim. Take edge of minimum weight.
Dijkstra. Take edge to vertex that is closest to $S$.




Challenge. Express this insight in reusable Java code.



- edge-weighted digraph API
- shortest-paths properties
- Dijkstra's algorithm
- **edge-weighted DAGs**
- negative weights

## Acyclic edge-weighted digraphs

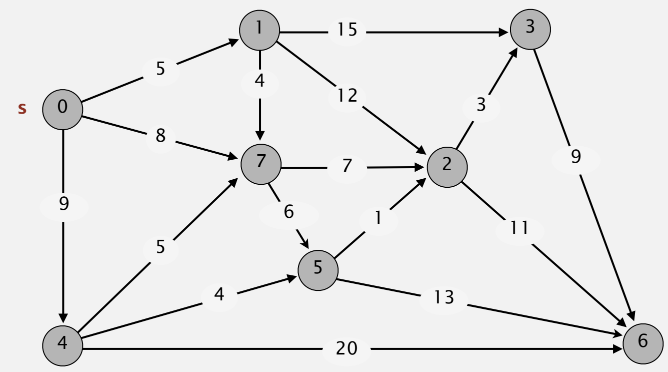
Q. Suppose that an edge-weighted digraph has no directed cycles. Is it easier to find shortest paths than in a general digraph?

A. Yes!



## Topological sort algorithm demo

- Consider vertices in topologically order.
- Relax all edges pointing from vertex.




an edge-weighted DAG

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|---|---|
| 0→1 | 5.0 |
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| 1→7 | 4.0 |
| 2→3 | 3.0 |
| 2→6 | 11.0 |
| 3→6 | 9.0 |
| 4→5 | 4.0 |
| 4→6 | 20.0 |
| 4→7 | 5.0 |
| 5→2 | 1.0 |
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| 7→5 | 6.0 |
| 7→2 | 7.0 |



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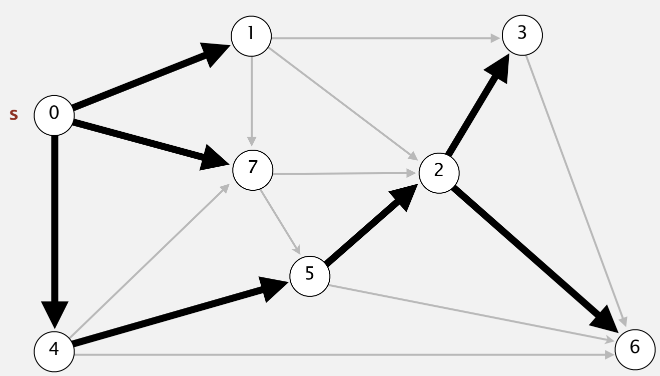



shortest-paths tree from vertex s

| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |



## Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights can be negative!

Pf.

- Each edge $e = v \to w$ is relaxed exactly once (when $v$ is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

## Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

topological order



## Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.




## Content-aware resizing

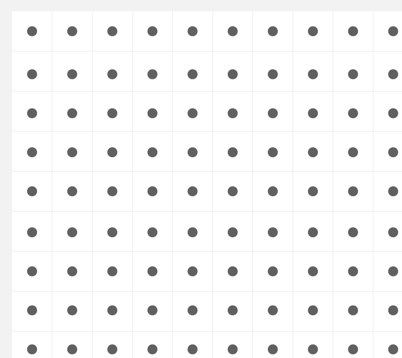
Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.


In the wild. Photoshop CS 5, Imagemagick, GIMP, ...



## Content-aware resizing

To find vertical seam:
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path from top to bottom.



## Content-aware resizing

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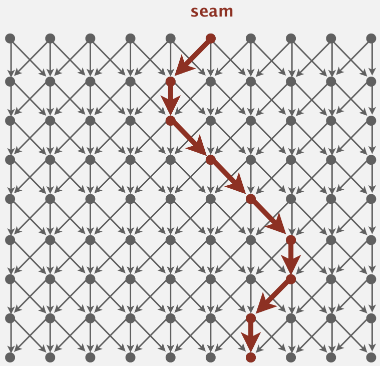
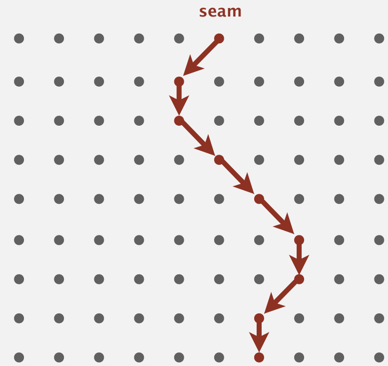





## Content-aware resizing

To remove vertical seam:

- Delete pixels on seam (one in each row).



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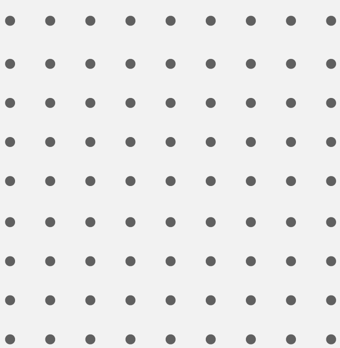




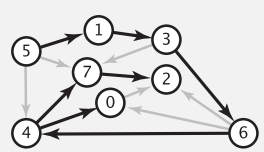
## Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

equivalent: reverse sense of equality in `relax()`

| longest paths input | | shortest paths input | |
|---|---|---|---|
| 5->4 | 0.35 | 5->4 | -0.35 |
| 4->7 | 0.37 | 4->7 | -0.37 |
| 5->7 | 0.28 | 5->7 | -0.28 |
| 5->1 | 0.32 | 5->1 | -0.32 |
| 4->0 | 0.38 | 4->0 | -0.38 |
| 0->2 | 0.26 | 0->2 | -0.26 |
| 3->7 | 0.39 | 3->7 | -0.39 |
| 1->3 | 0.29 | 1->3 | -0.29 |
| 7->2 | 0.34 | 7->2 | -0.34 |
| 6->2 | 0.40 | 6->2 | -0.40 |
| 3->6 | 0.52 | 3->6 | -0.52 |
| 6->0 | 0.58 | 6->0 | -0.58 |
| 6->4 | 0.93 | 6->4 | -0.93 |

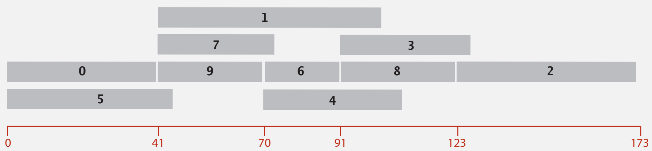



Key point. Topological sort algorithm works even with negative edge weights.

## Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

| job | duration | must complete before | | |
|---|---|---|---|---|
| 0 | 41.0 | 1 | 7 | 9 |
| 1 | 51.0 | 2 | | |
| 2 | 50.0 | | | |
| 3 | 36.0 | | | |
| 4 | 38.0 | | | |
| 5 | 45.0 | | | |
| 6 | 21.0 | 3 | 8 | |
| 7 | 32.0 | 3 | 8 | |
| 8 | 32.0 | 2 | | |
| 9 | 29.0 | 4 | 6 | |



Parallel job scheduling solution

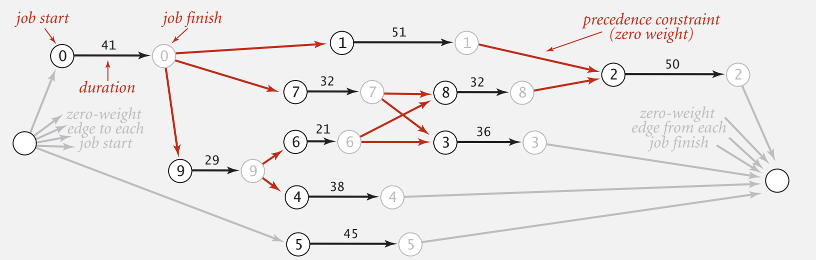


## Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

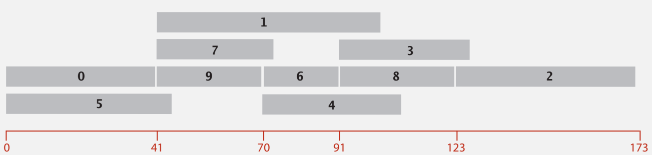
| job | duration | must complete before | | |
|---|---|---|---|---|
| 0 | 41.0 | 1 | 7 | 9 |
| 1 | 51.0 | 2 | | |
| 2 | 50.0 | | | |
| 3 | 36.0 | | | |
| 4 | 38.0 | | | |
| 5 | 45.0 | | | |
| 6 | 21.0 | 3 | 8 | |
| 7 | 32.0 | 3 | 8 | |
| 8 | 32.0 | 2 | | |
| 9 | 29.0 | 4 | 6 | |



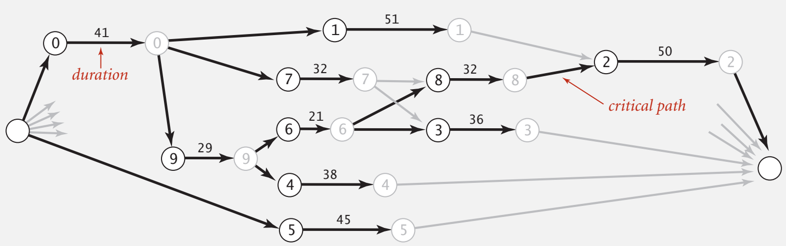



## Critical path method

CPM. Use longest path from the source to schedule each job.

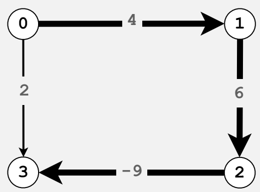


Parallel job scheduling solution



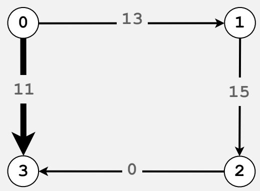
## Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.




Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.




Adding 9 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

Bad news. Need a different algorithm.

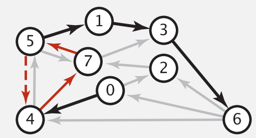


## Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.

digraph

| | |
|---|---|
| 4->5 | 0.35 |
| 5->4 | -0.66 |
| 4->7 | 0.37 |
| 5->7 | 0.28 |
| 7->5 | 0.28 |
| 5->1 | 0.32 |
| 0->4 | 0.38 |
| 0->2 | 0.26 |
| 7->3 | 0.39 |
| 1->3 | 0.29 |
| 2->7 | 0.34 |
| 6->2 | 0.40 |
| 3->6 | 0.52 |
| 6->0 | 0.58 |
| 6->4 | 0.93 |




negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

shortest path from 0 to 6
0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s



## Bellman-Ford algorithm

**Bellman-Ford algorithm**

**Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.**

**Repeat V times:**
- **Relax each edge.**

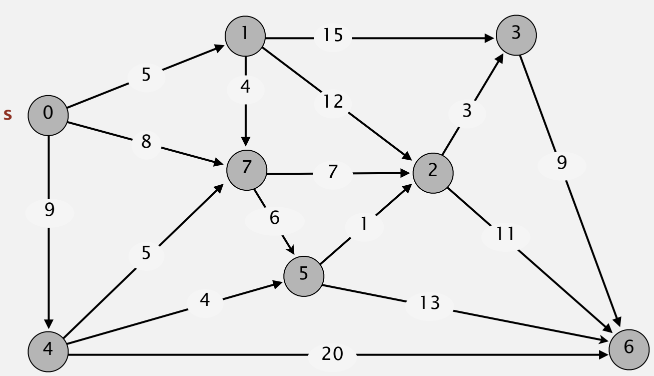
```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)



## Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

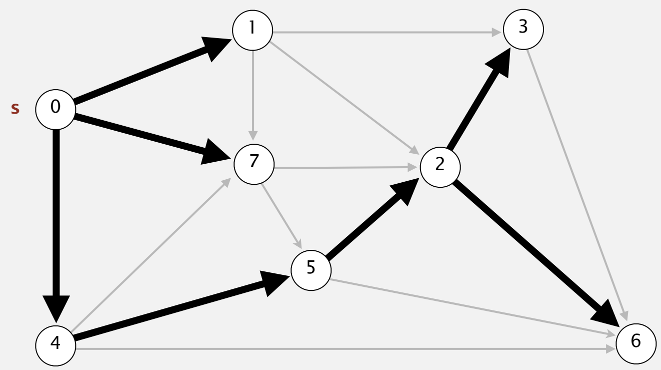



an edge-weighted digraph

| | |
|---|---|
| 0→1 | 5.0 |
| 0→4 | 9.0 |
| 0→7 | 8.0 |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
| 1→7 | 4.0 |
| 2→3 | 3.0 |
| 2→6 | 11.0 |
| 3→6 | 9.0 |
| 4→5 | 4.0 |
| 4→6 | 20.0 |
| 4→7 | 5.0 |
| 5→2 | 1.0 |
| 5→6 | 13.0 |
| 7→5 | 6.0 |
| 7→2 | 7.0 |

## Bellman-Ford algorithm demo

Repeat $V$ times: relax all $E$ edges.

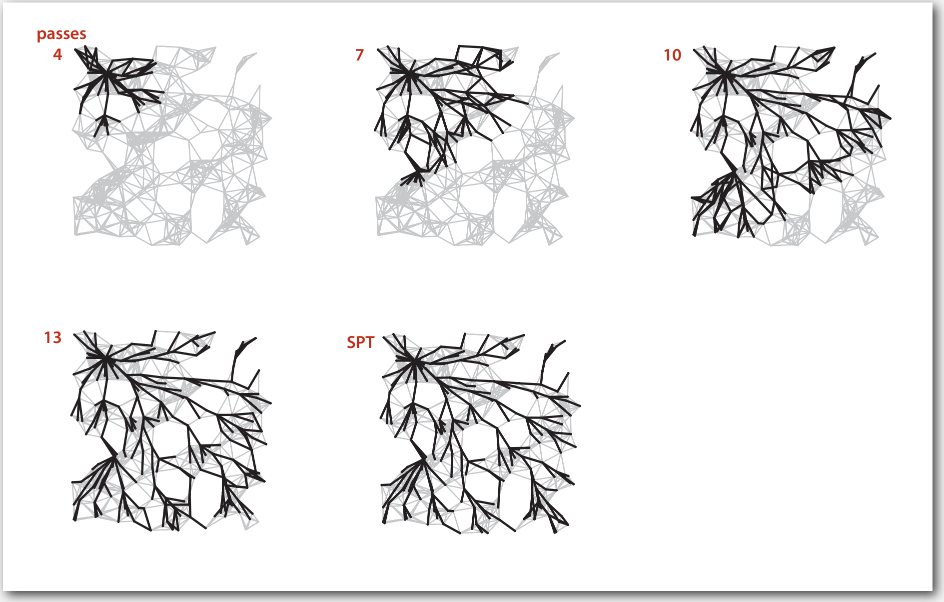



| v | distTo[] | edgeTo[] |
|---|---|---|
| 0 | 0.0 | - |
| 1 | 5.0 | 0→1 |
| 2 | 14.0 | 5→2 |
| 3 | 17.0 | 2→3 |
| 4 | 9.0 | 0→4 |
| 5 | 13.0 | 4→5 |
| 6 | 25.0 | 2→6 |
| 7 | 8.0 | 0→7 |

shortest-paths tree from vertex s



## Bellman-Ford algorithm visualization






## Bellman-Ford algorithm: analysis

**Bellman-Ford algorithm**

**Initialize distTo[s] = 0 and distTo[v] = $\infty$ for all other vertices.**

**Repeat V times:**
- **Relax each edge.**

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass $i$, found shortest path containing at most $i$ edges.



## Bellman-Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass $i$, no need to relax any edge pointing from $v$ in pass $i+1$.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.

- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

## Bellman-Ford algorithm: Java implementation

```
public class BellmanFordSP
{
   private double[] distTo;
   private DirectedEdge[] edgeTo;
   private boolean[] onQ;
   private Queue<Integer> queue;

   public BellmanFordSPT(EdgeWeightedDigraph G, int s)
   {
      distTo = new double[G.V()];
      edgeTo = new DirectedEdge[G.V()];
      onq    = new boolean[G.V()];
      queue  = new Queue<Integer>();

      for (int v = 0; v < V; v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      queue.enqueue(s);
      while (!queue.isEmpty())
      {
         int v = queue.dequeue();
         onQ[v] = false;
         for (DirectedEdge e : G.adj(v))
            relax(e);
      }
   }
}
```

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
      if (!onQ[w])
      {
         queue.enqueue(w);
         onQ[w] = true;
      }
   }
}
```

queue of vertices whose `distTo[]` value changes



## Single source shortest-paths implementation: cost summary

| algorithm | restriction | typical case | worst case | extra space |
|---|---|---|---|---|
| topological sort | no directed cycles | E + V | E + V | V |
| Dijkstra (binary heap) | no negative weights | E log V | E log V | V |
| Bellman-Ford | no negative cycles | E V | E V | V |
| Bellman-Ford (queue-based) | | E + V | E V | V |

Remark 1. Directed cycles make the problem harder.

Remark 2. Negative weights make the problem harder.

Remark 3. Negative cycles makes the problem intractable.



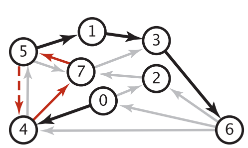
## Finding a negative cycle

Negative cycle. Add two method to the API for `SP`.

| | | |
|---|---|---|
| `boolean` | `hasNegativeCycle()` | *is there a negative cycle?* |
| `Iterable <DirectedEdge>` | `negativeCycle()` | *negative cycle reachable from s* |

digraph

```
4->5   0.35
5->4  -0.66
4->7   0.37
5->7   0.28
7->5   0.28
5->1   0.32
0->4   0.38
0->2   0.26
7->3   0.39
1->3   0.29
2->7   0.34
6->2   0.40
3->6   0.52
6->0   0.58
6->4   0.93
```

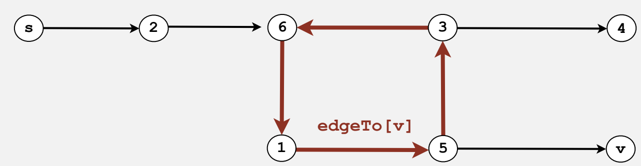



negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5



## Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.




Proposition. If any vertex $v$ is updated in phase $V$, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.

**Problem.** Given table of exchange rates, is there an arbitrage opportunity?

| | USD | EUR | GBP | CHF | CAD |
|---|---|---|---|---|---|
| USD | 1 | 0.741 | 0.657 | 1.061 | 1.011 |
| EUR | 1.350 | 1 | 0.888 | 1.433 | 1.366 |
| GBP | 1.521 | 1.126 | 1 | 1.614 | 1.538 |
| CHF | 0.943 | 0.698 | 0.620 | 1 | 0.953 |
| CAD | 0.995 | 0.732 | 0.650 | 1.049 | 1 |

**Ex.** \$1,000 $\Rightarrow$ 741 Euros $\Rightarrow$ 1,012.206 Canadian dollars $\Rightarrow$ \$1,007.14497.

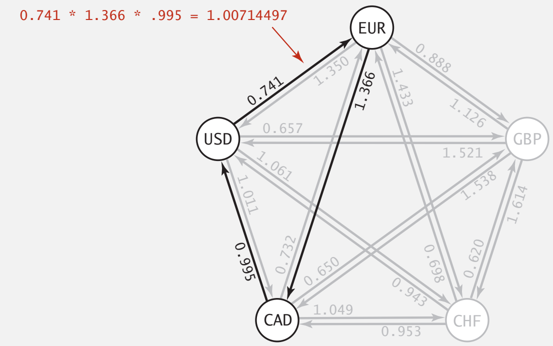
$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$





**Currency exchange graph.**

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is $> 1$.



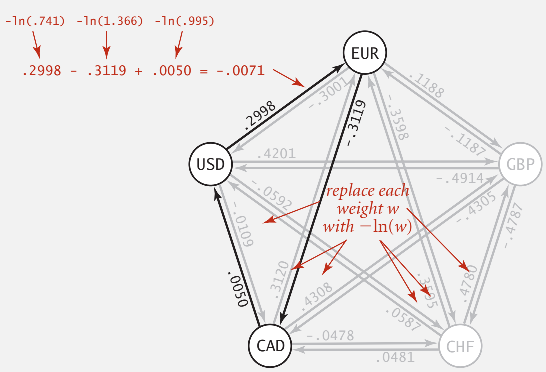

**Challenge.** Express as a negative cycle detection problem.





**Model as a negative cycle detection problem by taking logs.**

- Let weight of edge $v \rightarrow w$ be $-\ln$ (exchange rate from currency $v$ to $w$).
- Multiplication turns to addition; $> 1$ turns to $< 0$.
- Find a directed cycle whose sum of edge weights is $< 0$ (negative cycle).




**Remark.** Fastest algorithm is extraordinarily valuable!





**Dijkstra's algorithm.**

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

**Acyclic edge-weighted digraphs.**

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

**Negative weights and negative cycles.**

- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

**Shortest-paths is a broadly useful problem-solving model.**