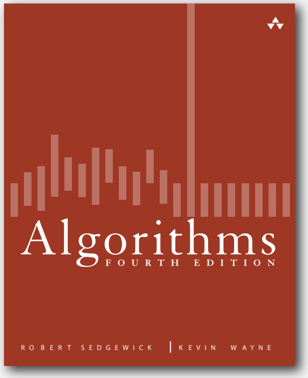
4.1 Undirected Graphs



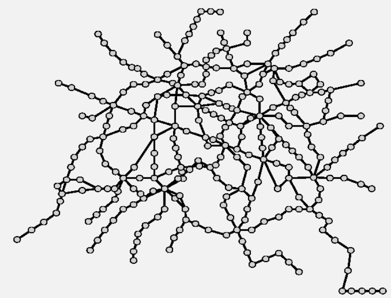
- graph API
- depth-first search
- breadth-first search
- connected components
- challenges

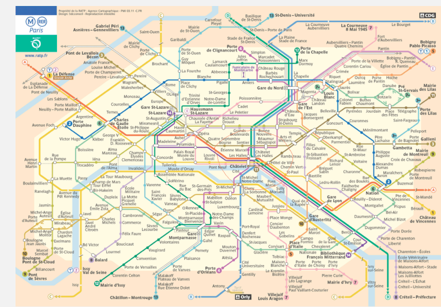
Undirected graphs

Graph. Set of vertices connected pairwise by edges.

Why study graph algorithms?

- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.

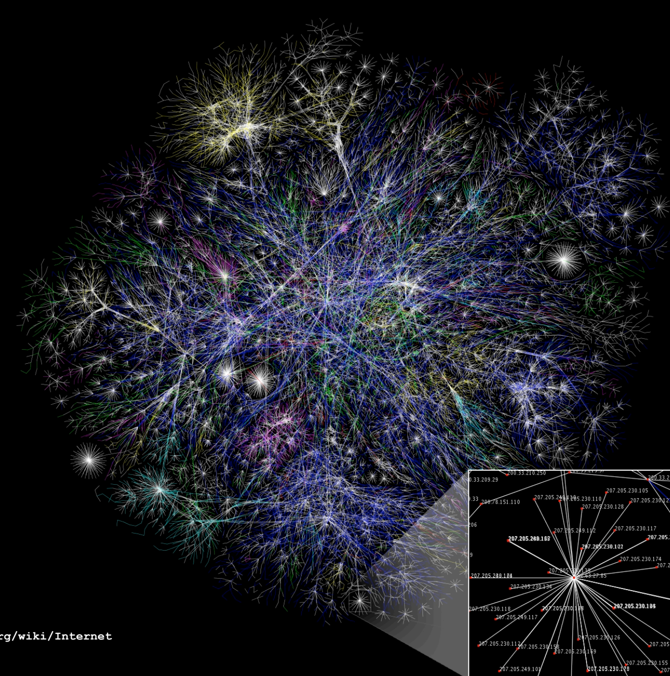




Protein-protein interaction network

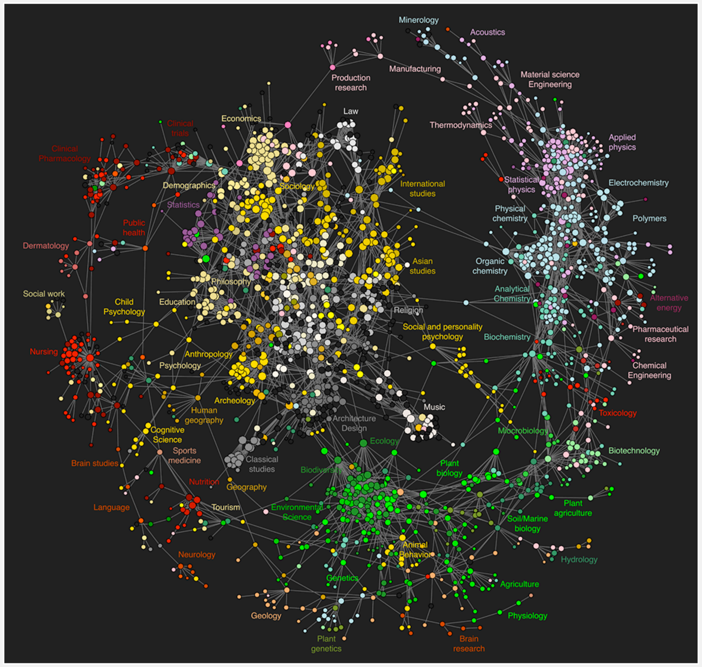
Reference: Jeong et al, Nature Review | Genetics

The Internet as mapped by the Opte Project



http://en.wikipedia.org/wiki/Internet

Map of science clickstreams

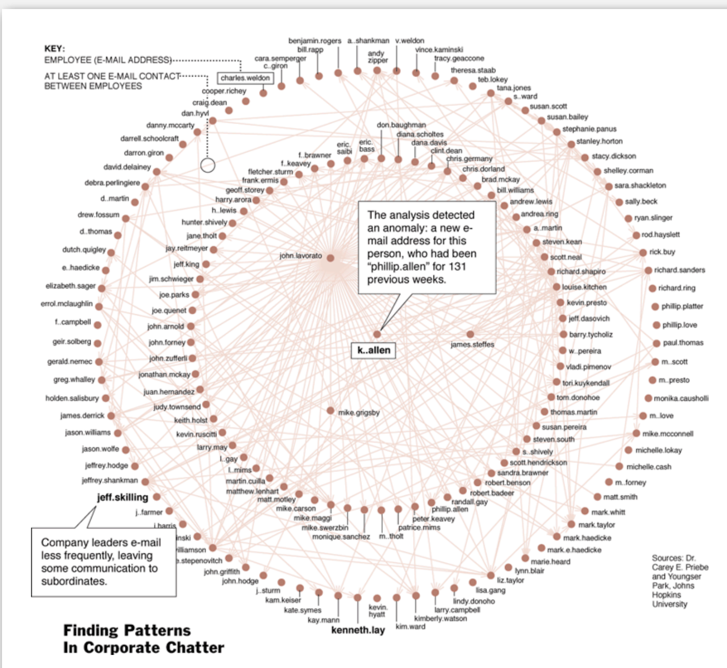


http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

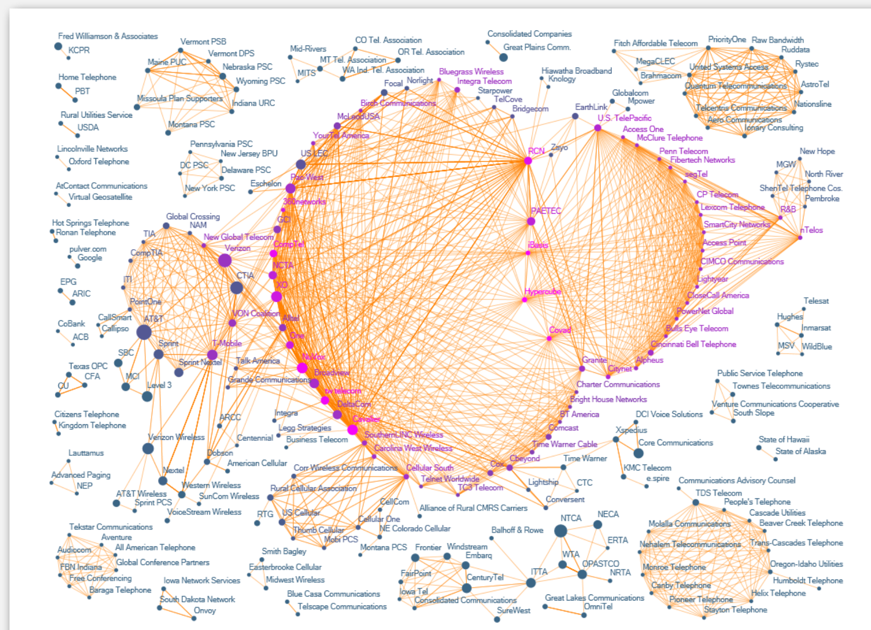
10 million Facebook friends

"Visualizing Friendships" by Paul Butler

One week of Enron emails



The evolution of FCC lobbying coalitions



"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in JoSS Visualization Symposium 2010

Framingham heart study

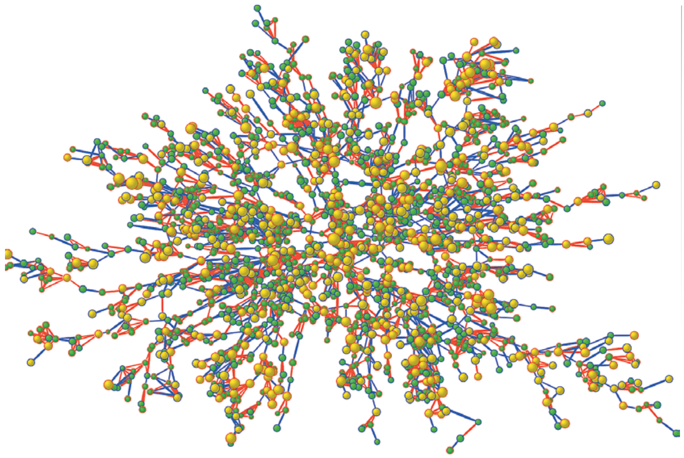


Figure 1. Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.
Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

"The Spread of Obesity in a Large Social Network over 32 Years" by Christakis and Fowler in New England Journal of Medicine, 2007

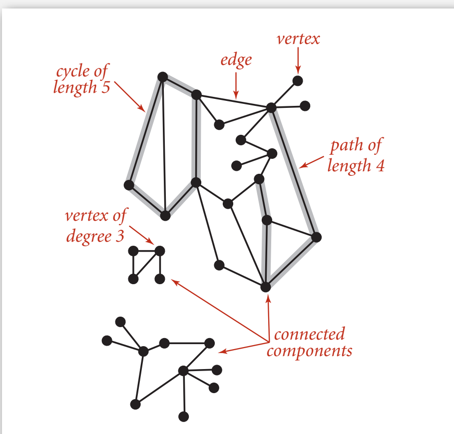
Graph applications

graph	vertex	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
chemical compound	molecule	bond

Graph terminology

Path. Sequence of vertices connected by edges.
Cycle. Path whose first and last vertices are the same.

Two vertices are connected if there is a path between them.



Some graph-processing problems

Path. Is there a path between s and t?
Shortest path. What is the shortest path between s and t?

Cycle. Is there a cycle in the graph?
Euler tour. Is there a cycle that uses each edge exactly once?
Hamilton tour. Is there a cycle that uses each vertex exactly once?

Connectivity. Is there a way to connect all of the vertices?
MST. What is the best way to connect all of the vertices?
Biconnectivity. Is there a vertex whose removal disconnects the graph?

Planarity. Can you draw the graph in the plane with no crossing edges?
Graph isomorphism. Do two adjacency lists represent the same graph?

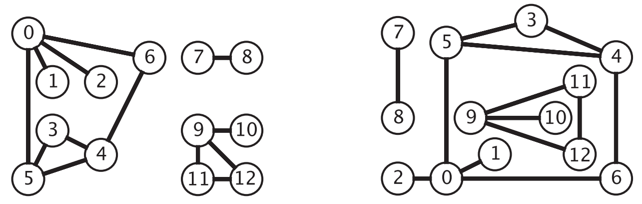
Challenge. Which of these problems are easy? difficult? intractable?

- graph API
- depth-first search
- breadth-first search
- connected components
- challenges

Graph representation

Graph drawing. Provides intuition about the structure of the graph.

Caveat. Intuition can be misleading.

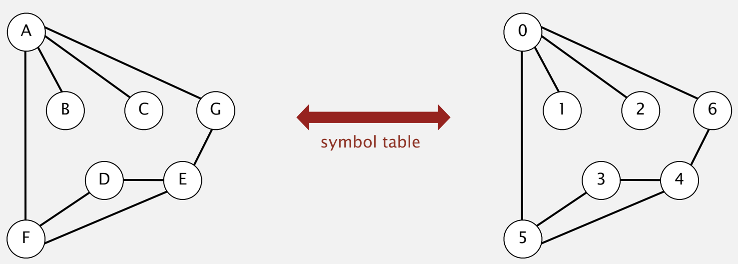


two drawings of the same graph

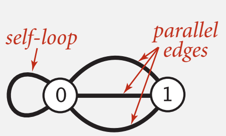
Graph representation

Vertex representation.

- This lecture: use integers between 0 and $V-1$.
- Applications: convert between names and integers with symbol table.



Anomalies.



Graph API

public class	**Graph**	
	Graph(int V)	*create an empty graph with V vertices*
	Graph(In in)	*create a graph from input stream*
void	addEdge(int v, int w)	*add an edge v-w*
Iterable<Integer>	adj(int v)	*vertices adjacent to v*
int	V()	*number of vertices*
int	E()	*number of edges*
String	toString()	*string representation*

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "-" + w);
```

read graph from input stream

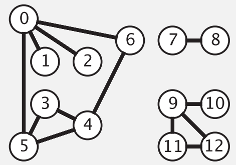
print out each edge (twice)

Graph API: sample client

Graph input format.

tinyG.txt

```
V → 13
    13 ← E
    0 5
    4 3
    0 1
    9 12
    6 4
    5 4
    0 2
    11 12
    9 10
    0 6
    7 8
    9 11
    5 3
```



```
% java Test tinyG.txt
0-6
0-2
0-1
0-5
1-0
2-0
3-5
3-4
…
12-11
12-9
```

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "-" + w);
```

read graph from input stream

print out each edge (twice)

Typical graph-processing code

compute the degree of v

```
public static int degree(Graph G, int v)
{
   int degree = 0;
   for (int w : G.adj(v)) degree++;
   return degree;
}
```

compute maximum degree

```
public static int maxDegree(Graph G)
{
   int max = 0;
   for (int v = 0; v < G.V(); v++)
      if (degree(G, v) > max)
         max = degree(G, v);
   return max;
}
```

compute average degree

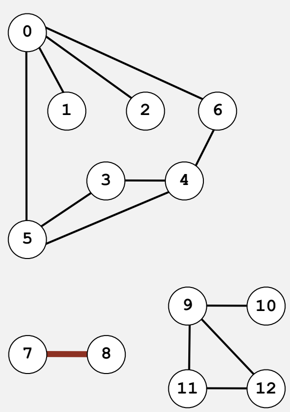
```
public static double averageDegree(Graph G)
{  return 2.0 * G.E() / G.V();  }
```

count self-loops

```
public static int numberOfSelfLoops(Graph G)
{
   int count = 0;
   for (int v = 0; v < G.V(); v++)
      for (int w : G.adj(v))
         if (v == w) count++;
   return count/2;   // each edge counted twice
}
```


Set-of-edges graph representation

Maintain a list of the edges (linked list or array).

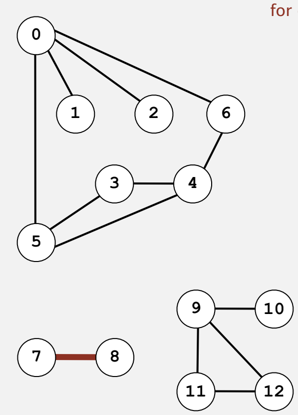


0	1
0	2
0	5
0	6
3	4
3	5
4	5
4	6
7	8
9	10
9	11
9	12
11	12

Adjacency-matrix graph representation

Maintain a two-dimensional V-by-V boolean array;

for each edge v–w in graph: `adj[v][w] = adj[w][v] = true`.

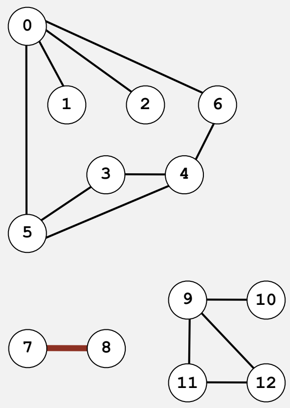


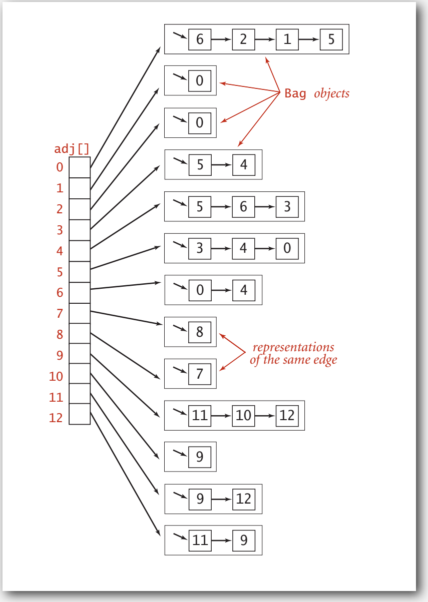
two entries for each edge

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

Adjacency-list graph representation

Maintain vertex-indexed array of lists.





Adjacency-list graph representation: Java implementation

```
public class Graph
{
   private final int V;
   private Bag<Integer>[] adj;

   public Graph(int V)
   {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)
   {
      adj[v].add(w);
      adj[w].add(v);
   }

   public Iterable<Integer> adj(int v)
   {  return adj[v];  }
}
```

adjacency lists (using **Bag** data type)

create empty graph with **V** vertices

add edge **v-w** (parallel edges allowed)

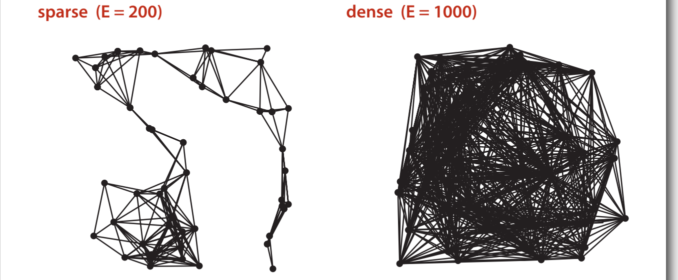
iterator for vertices adjacent to **v**

Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to v.
- Real-world graphs tend to be sparse.

huge number of vertices, small average vertex degree



Two graphs (V = 50)

Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to v.
- Real-world graphs tend to be sparse.

huge number of vertices, small average vertex degree

representation	space	add edge	edge between v and w?	iterate over vertices adjacent to v?
list of edges	E	1	E	E
adjacency matrix	V^2	1 *	1	V
adjacency lists	E + V	1	degree(v)	degree(v)

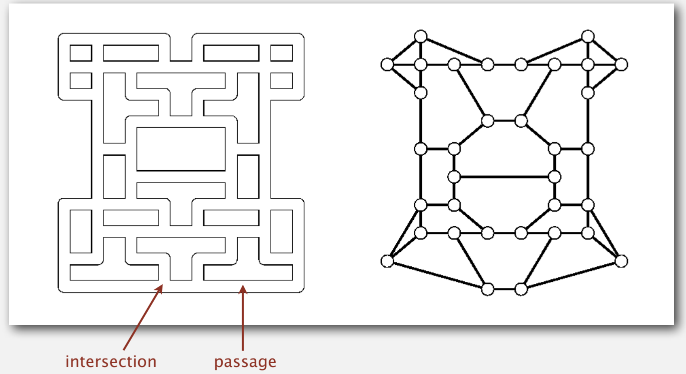
* disallows parallel edges

- graph API
- **depth-first search**
- breadth-first search
- connected components
- challenges

Maze exploration

Maze graphs.

- Vertex = intersection.
- Edge = passage.

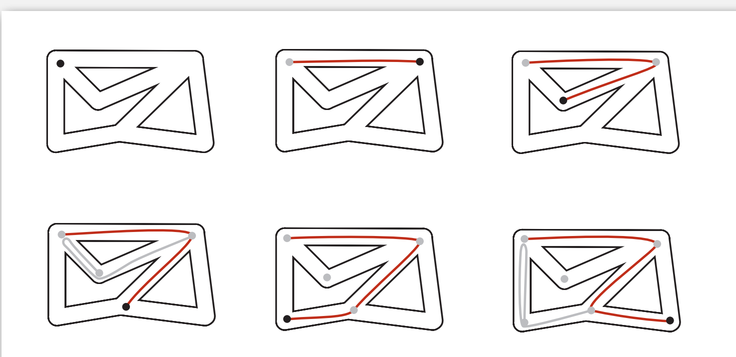


Goal. Explore every intersection in the maze.

Trémaux maze exploration

Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.

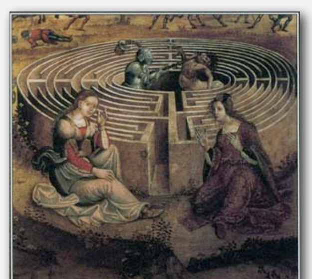


Trémaux maze exploration

Algorithm.

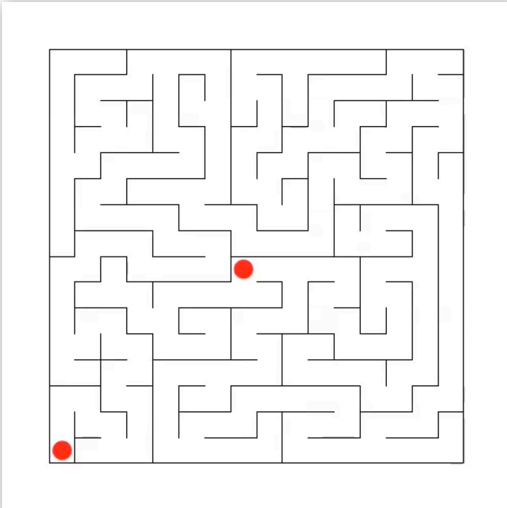
- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.

First use? Theseus entered labyrinth to kill the monstrous Minotaur; Ariadne held ball of string.

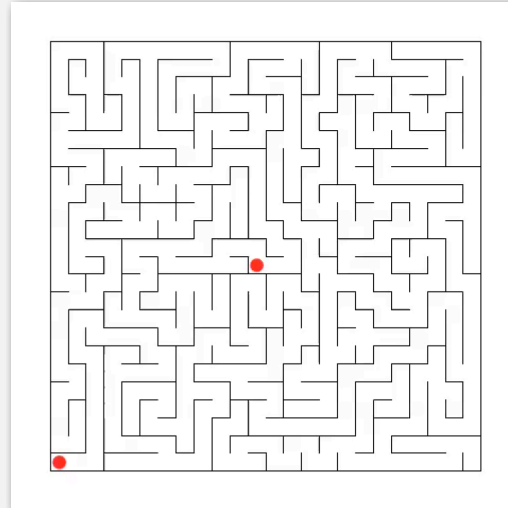


Claude Shannon (with Theseus mouse)

Maze exploration



Maze exploration



Depth-first search

Goal. Systematically search through a graph.
Idea. Mimic maze exploration.

DFS (to visit a vertex v)

Mark v as visited.
Recursively visit all unmarked vertices w adjacent to v.

Typical applications.

- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

Design pattern for graph processing

Design pattern. Decouple graph data type from graph processing.

- Create a `Graph` object.
- Pass the `Graph` to a graph-processing routine, e.g., `Paths`.
- Query the graph-processing routine for information.

	`public class Paths`	
	`Paths(Graph G, int s)`	*find paths in G from source s*
`boolean`	`hasPathTo(int v)`	*is there a path from s to v?*
`Iterable<Integer>`	`pathTo(int v)`	*path from s to v; null if no such path*

```
Paths paths = new Paths(G, s);
for (int v = 0; v < G.V(); v++)
   if (paths.hasPathTo(v))
      StdOut.println(v);
```

print all vertices connected to s

Depth-first search demo

Depth-first search

Goal. Find all vertices connected to s (and a path).

Idea. Mimic maze exploration.

Algorithm.

- Use recursion (ball of string).
- Mark each visited vertex (and keep track of edge taken to visit it).
- Return (retrace steps) when no unvisited options.

Data structures.

- `boolean[] marked` to mark visited vertices.
- `int[] edgeTo` to keep tree of paths.
  `(edgeTo[w] == v)` means that edge `v-w` taken to visit `w` for first time

Depth-first search

```
public class DepthFirstPaths
{
   private boolean[] marked;
   private int[] edgeTo;
   private int s;

   public DepthFirstSearch(Graph G, int s)
   {
      ...
      dfs(G, s);
   }

   private void dfs(Graph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w])
         {
            dfs(G, w);
            edgeTo[w] = v;
         }
   }
}
```

marked[v] = true if v connected to s

edgeTo[v] = previous vertex on path from s to v

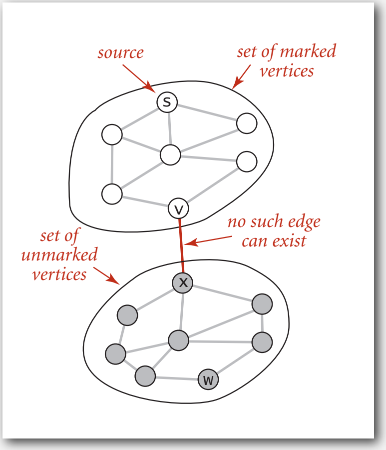
initialize data structures

find vertices connected to s

recursive DFS does the work

Depth-first search properties

Proposition. DFS marks all vertices connected to s in time proportional to the sum of their degrees.



Pf.

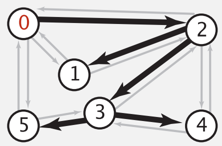
- Correctness:
  - if w marked, then w connected to s (why?)
  - if w connected to s, then w marked
    (if w unmarked, then consider last edge on a path from s to w that goes from a marked vertex to an unmarked one)
- Running time: each vertex connected to s is visited once.

Depth-first search properties

Proposition. After DFS, can find vertices connected to s in constant time and can find a path to s (if one exists) in time proportional to its length.

Pf. `edgeTo[]` is a parent-link representation of a tree rooted at `s`.

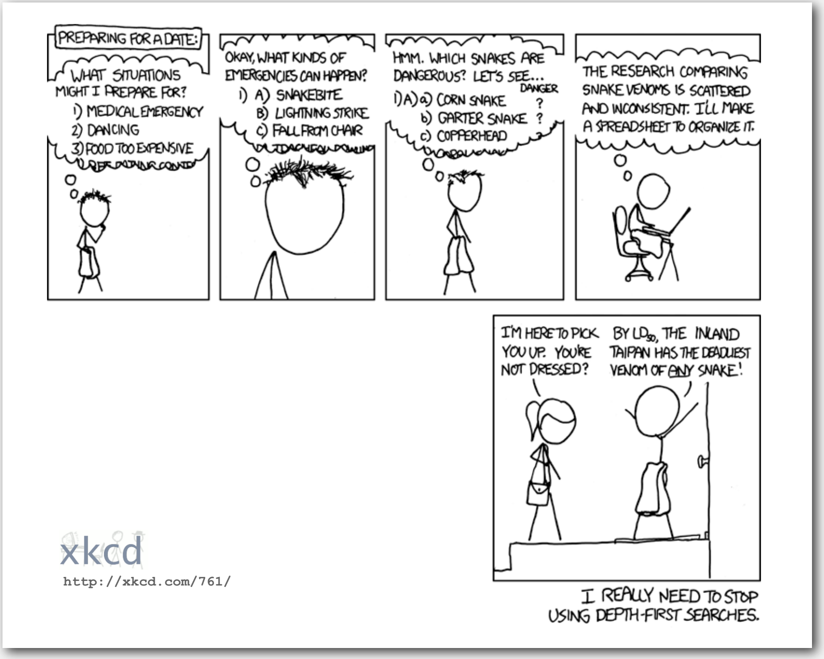
```
public boolean hasPathTo(int v)
{  return marked[v];  }

public Iterable<Integer> pathTo(int v)
{
   if (!hasPathTo(v)) return null;
   Stack<Integer> path = new Stack<Integer>();
   for (int x = v; x != s; x = edgeTo[x])
      path.push(x);
   path.push(s);
   return path;
}
```



edgeTo[]

0	
1	2
2	0
3	2
4	3
5	3

Depth-first search application: preparing for a date

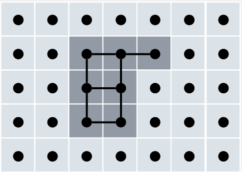


Depth-first search application: flood fill

Challenge. Flood fill (Photoshop magic wand).
Assumptions. Picture has millions to billions of pixels.

Solution. Build a grid graph.

- Vertex: pixel.
- Edge: between two adjacent gray pixels.
- Blob: all pixels connected to given pixel.



Breadth-first search demo

Breadth-first search

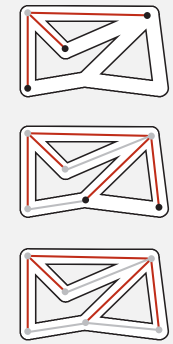
Depth-first search. Put unvisited vertices on a stack.

Breadth-first search. Put unvisited vertices on a queue.

Shortest path. Find path from s to t that uses fewest number of edges.

BFS (from source vertex s)

Put s onto a FIFO queue, and mark s as visited.
Repeat until the queue is empty:
- **remove the least recently added vertex v**
- **add each of v's unvisited neighbors to the queue, and mark them as visited.**



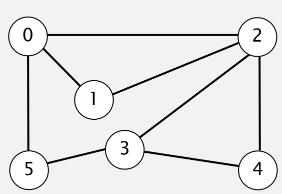
Intuition. BFS examines vertices in increasing distance from s.

Breadth-first search properties

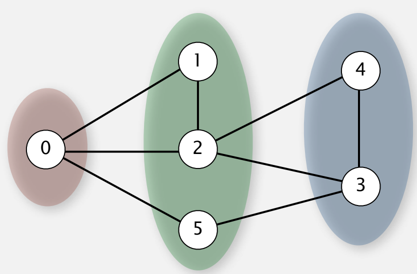
Proposition. BFS computes shortest path (number of edges) from s in a connected graph in time proportional to $E + V$.

Pf.

- Correctness: queue always consists of zero or more vertices of distance k from s, followed by zero or more vertices of distance $k + 1$.
- Running time: each vertex connected to s is visited once.



standard drawing



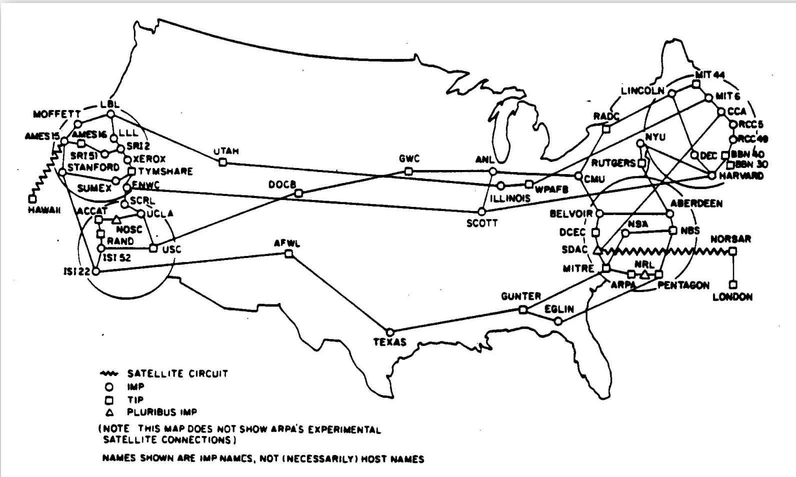
dist = 0 dist = 1 dist = 2

Breadth-first search

```
public class BreadthFirstPaths
{
   private boolean[] marked;
   private boolean[] edgeTo[];
   private final int s;
   …

   private void bfs(Graph G, int s)
   {
     Queue<Integer> q = new Queue<Integer>();
      q.enqueue(s);
      marked[s] = true;
      while (!q.isEmpty())
      {
         int v = q.dequeue();
         for (int w : G.adj(v))
         {
            if (!marked[w])
            {
               q.enqueue(w);
               marked[w] = true;
               edgeTo[w] = v;
            }
         }
      }
   }
}
```

Breadth-first search application: routing

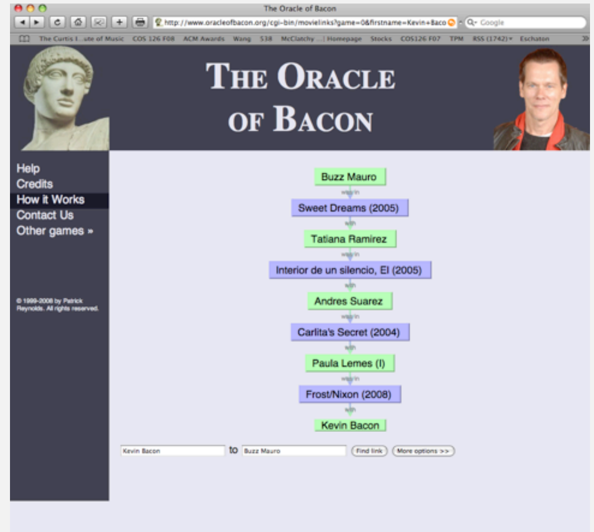
Fewest number of hops in a communication network.



ARPANET, July 1977

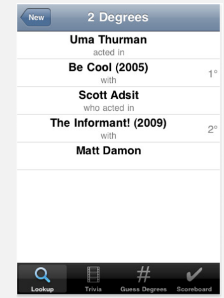
Breadth-first search application: Kevin Bacon numbers

Kevin Bacon numbers.



http://oracleofbacon.org

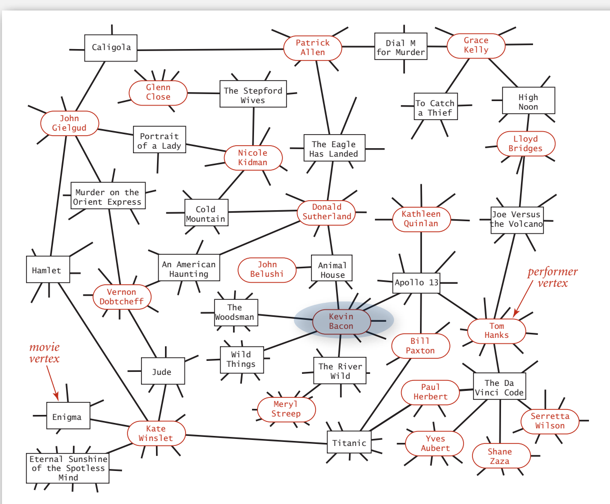
Endless Games board game



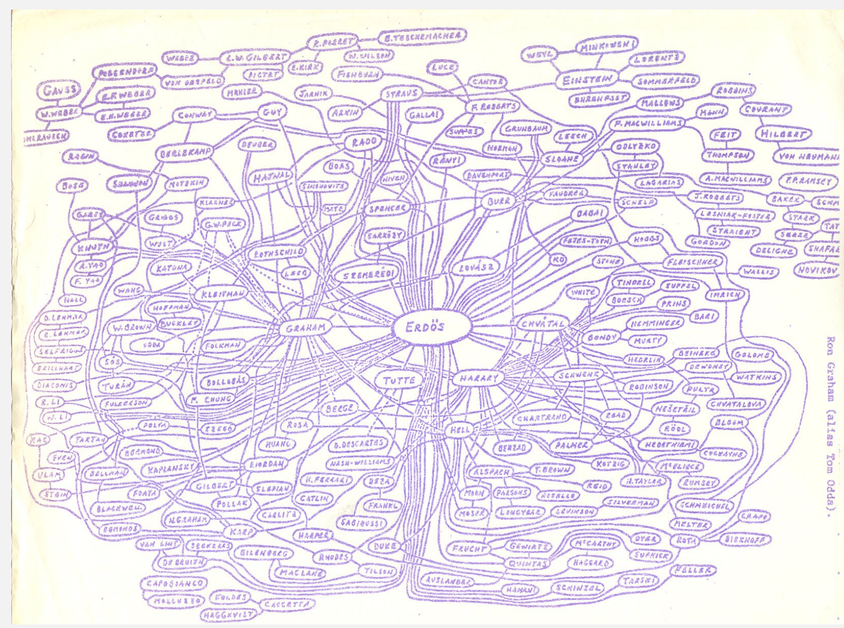
SixDegrees iPhone App

Kevin Bacon graph

- Include a vertex for each performer and for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest path from s = Kevin Bacon.



Breadth-first search application: Erdös numbers



hand-drawing of part of the Erdös graph by Ron Graham

- graph API
- depth-first search
- breadth-first search
- **connected components**
- challenges

Connectivity queries

Def. Vertices v and w are connected if there is a path between them.

Goal. Preprocess graph to answer queries: is v connected to w ?
in constant time.

public class CC		
	CC(Graph G)	*find connected components in G*
boolean	connected(int v, int w)	*are v and w connected?*
int	count()	*number of connected components*
int	id(int v)	*component identifier for v*

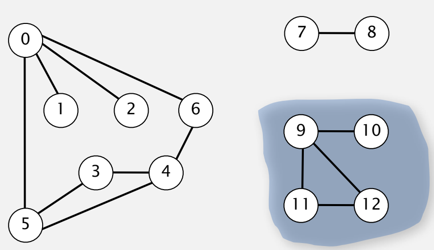
Union-Find? Not quite.
Depth-first search. Yes. [next few slides]

Connected components

The relation "is connected to" is an equivalence relation:

- Reflexive: v is connected to v.
- Symmetric: if v is connected to w, then w is connected to v.
- Transitive: if v connected to w and w connected to x, then v connected to x.

Def. A connected component is a maximal set of connected vertices.



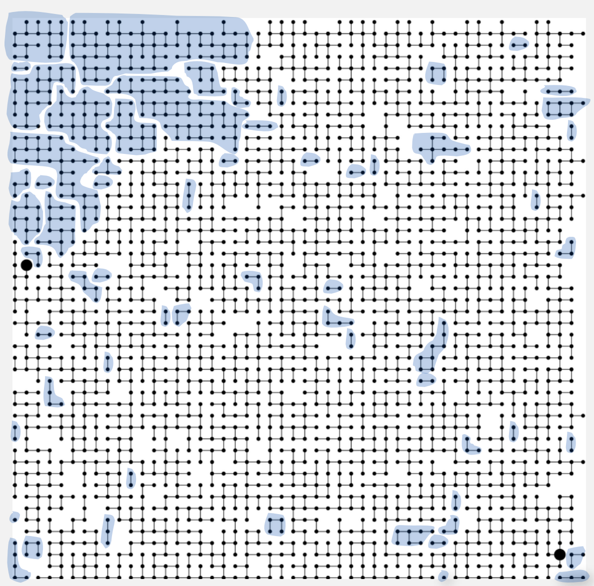
3 connected components

v	id[v]
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	1
8	1
9	2
10	2
11	2
12	2

Remark. Given connected components, can answer queries in constant time.

Connected components

Def. A connected component is a maximal set of connected vertices.


63 connected components

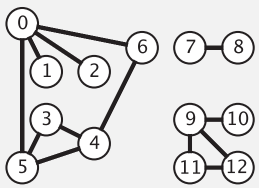
Connected components

Goal. Partition vertices into connected components.

Connected components

Initialize all vertices v as unmarked.

For each unmarked vertex v, run DFS to identify all vertices discovered as part of the same component.



tinyG.txt

```
13
13
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```

V → 13 (first line); E → 13 (second line)

Connected components demo

Finding connected components with DFS

```
public class CC
{
   private boolean marked[];
   private int[] id;
   private int count;

   public CC(Graph G)
   {
      marked = new boolean[G.V()];
      id = new int[G.V()];
      for (int v = 0; v < G.V(); v++)
      {
         if (!marked[v])
         {
            dfs(G, v);
            count++;
         }
      }
   }

   public int count()
   public int id(int v)
   private void dfs(Graph G, int v)

}
```

id[v] = id of component containing v

number of components

run DFS from one vertex in each component

see next slide

Finding connected components with DFS (continued)

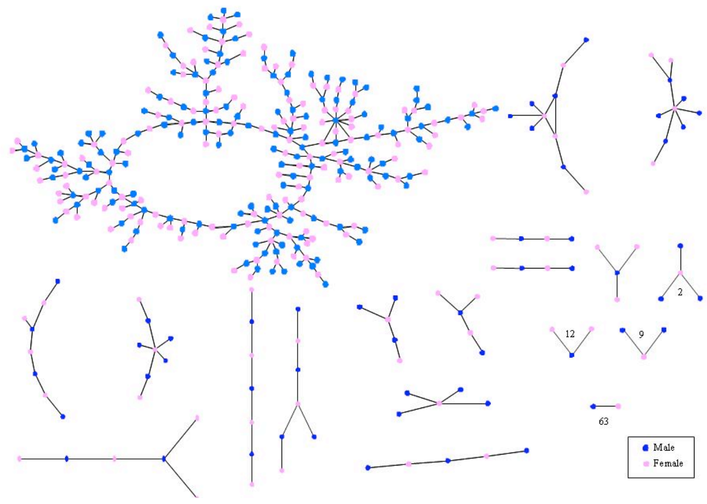
```
public int count()
{  return count;  }

public int id(int v)
{  return id[v];  }

private void dfs(Graph G, int v)
{
   marked[v] = true;
   id[v] = count;
   for (int w : G.adj(v))
      if (!marked[w])
         dfs(G, w);
}
```

number of components

id of component containing v

all vertices discovered in same call of dfs have same id

Connected components application: study spread of STDs



Relationship graph at "Jefferson High"

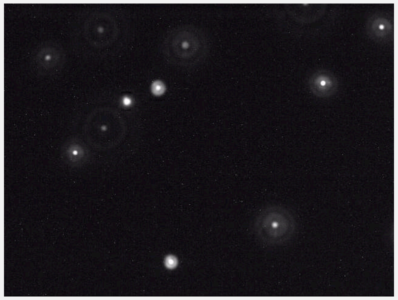
Peter Bearman, James Moody, and Katherine Stovel. Chains of affection: The structure of adolescent romantic and sexual networks. American Journal of Sociology, 110(1): 44–99, 2004.

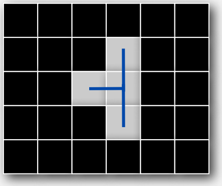
Connected components application: particle detection

Particle detection. Given grayscale image of particles, identify "blobs."

- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value ≥ 70.
- Blob: connected component of 20-30 pixels.

black = 0
white = 255





Particle tracking. Track moving particles over time.

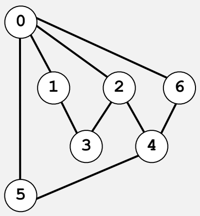
- graph API
- depth-first search
- breadth-first search
- connected components
- challenges

Graph-processing challenge 1

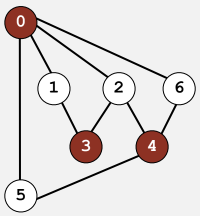
Problem. Is a graph bipartite?

How difficult?

- Any COS 126 student could do it.
- Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.



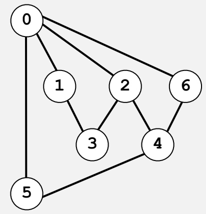
0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

Graph-processing challenge 1

Problem. Is a graph bipartite?

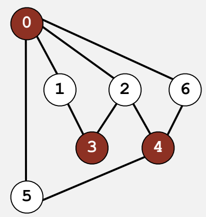


0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

How difficult?

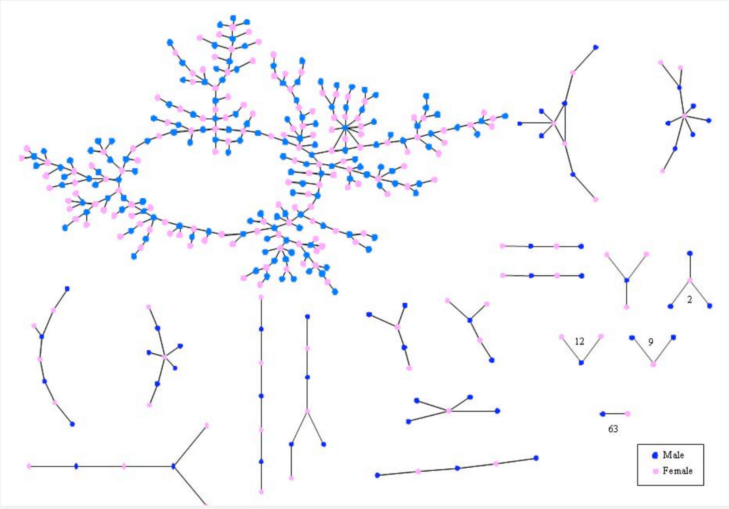
- Any COS 126 student could do it.
- ✓ Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

simple DFS-based solution (see textbook)



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

Bipartiteness application

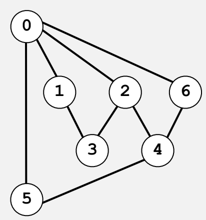


Relationship graph at "Jefferson High"

Peter Bearman, James Moody, and Katherine Stovel. Chains of affection: The structure of adolescent romantic and sexual networks. American Journal of Sociology, 110(1): 44–99, 2004.

Graph-processing challenge 2

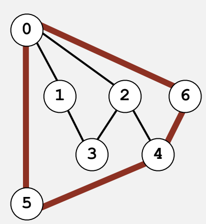
Problem. Find a cycle.



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

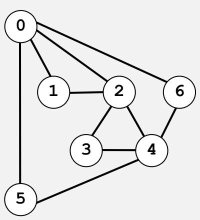
How difficult?

- Any COS 126 student could do it.
- Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.



Graph-processing challenge 3

Problem. Find a cycle that uses every edge.
Assumption. Need to use each edge exactly once.



0-1
0-2
0-5
0-6
1-2
2-3
2-4
3-4
4-5
4-6

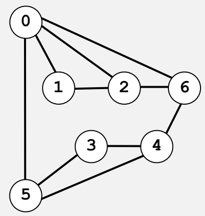
0-1-2-3-4-2-0-6-4-5-0

How difficult?

- Any COS 126 student could do it.
- Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Graph-processing challenge 4

Problem. Find a cycle that visits every vertex.
Assumption. Need to visit each vertex exactly once.



0-5-3-4-6-2-1-0

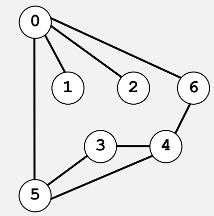
0-1
0-2
0-5
0-6
1-2
2-6
3-4
3-5
4-5
4-6

How difficult?

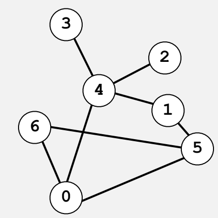
- Any COS 126 student could do it.
- Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Graph-processing challenge 5

Problem. Are two graphs identical except for vertex names?



0-1
0-2
0-5
0-6
3-4
3-5
4-5
4-6



0-4
0-5
0-6
1-4
1-5
2-4
3-4
5-6

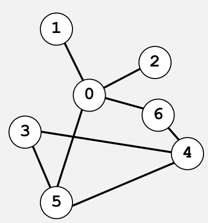
0↔4, 1↔3, 2↔2, 3↔6, 4↔5, 5↔0, 6↔1

How difficult?

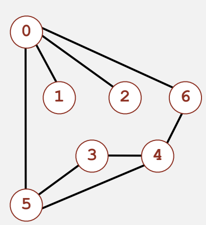
- Any COS 126 student could do it.
- Need to be a typical diligent COS 226 student.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Graph-processing challenge 6

Problem. Lay out a graph in the plane without crossing edges?



0-1
0-2
0-5
0-6
3-4
3-5
4-5
4-6



How difficult?

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- Impossible.