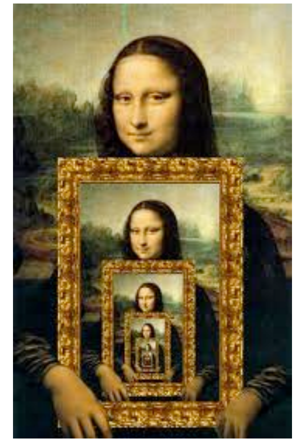
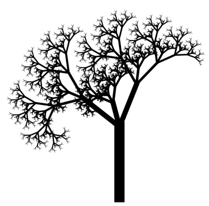
2.3 Recursion



Overview

What is recursion? When one function calls itself directly or indirectly.



Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.

- Binary search, mergesort, FFT, GCD.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

M. C. Escher, 1956

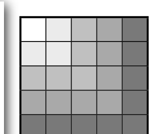
Mathematical Induction

Mathematical induction. Prove a statement involving an integer N by

- base case: Prove it for some specific N (usually 0 or 1).
- induction step: Assume it to be true for all positive integers less than N, use that fact to prove it for N.

Ex. Sum of the first N odd integers is N^2.

1	= 1
1 + 3	= 4
1 + 3 + 5	= 9
1 + 3 + 5 + 7	= 16
1 + 3 + 5 + 7 + 9	= 25
...	...



Base case: True for N = 1.

Induction step:

- Let T(N) be the sum of the first N odd integers: 1 + 3 + 5 + ... + (2N - 1).
- Assume that $T(N-1) = (N-1)^2$.
- $T(N) = T(N-1) + (2N - 1)$
  $= (N-1)^2 + (2N - 1)$
  $= N^2 - 2N + 1 + (2N - 1)$
  $= N^2$

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Do something specific in response to "base" argument values.
- reduction step: Assume the function works for the base case, and use the function to implement itself for general argument values.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

automatic cast to String (either "0" or "1")

Ex 1. Convert positive `int` to binary `String`.

Base case: return "1" for x = 1.

Reduction step:

- convert x/2 to binary
- append "0" if x even
- append "1" if x odd

```
   37          18
"100101" = "10010" + "1"
```

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Implementing it for some specific values of the arguments.
- reduction step: Assume the function works for smaller values of its arguments and use it to implement it for the given values.

```
public class Binary
{
   public static int convert(int x)
   {
      if (x == 1) return "1";
      else return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));
   }
}
```

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```

x = 6

environment

convert(6)

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

x = 6

environment

convert(6)

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}                    "110"
```

```
public class Binary
{
   public static int convert(int x)
   {
      if (x == 0) return "";
      else return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));
                              "110"
   }
}
```

```
% java Binary 6
110
```

Recursion vs. Iteration

Every program with 1 recursive call corresponds to a loop.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

```
public static String convertNR(int x)
{
   String s = "1";
   while (x > 1)
   {
      s = (x % 2) + s;
      x = x/2;
   }
   return s;
}
```

Reasons to use recursion:

- code more compact
- easier to understand
- easier to reason about correctness
- easy to add multiple recursive calls (stay tuned)

Reasons not to use recursion: (stay tuned)

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$4032 = 2^6 \times 3^2 \times 7^1$$
$$1272 = 2^3 \times 3^1 \times 53^1$$
$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.

- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

GCD. Find largest integer that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q, p \,\%\, q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case

```
gcd(4032, 1272)    = gcd(1272, 216)
                   = gcd(216, 192)
                   = gcd(192, 24)
                   = gcd(24, 0)
                   = 24.
```

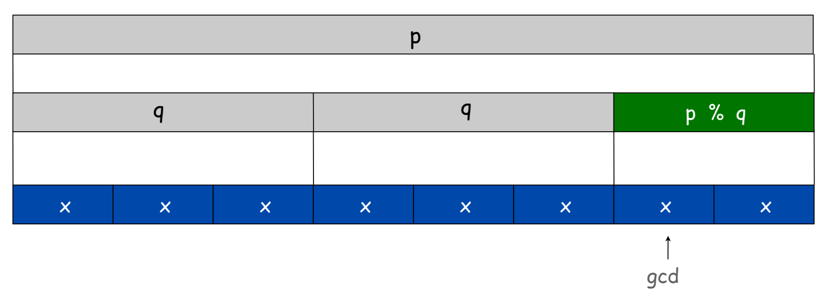
4032 = 3 × 1272 + 216

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q, p \,\%\, q) & \text{otherwise} \end{cases}$$

← base case
← reduction step, converges to base case



p = 8x
q = 3x

gcd(p, q) = gcd(3x, 2x) = x

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

Recursive program

```
public static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case

← reduction step

p = 1272, q = 216

environment

gcd(1272, 216)

```
static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

p = 1272, q = 216

environment

gcd(1272, 216)

```
static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

24

```
public class Euclid
{
   public static int gcd(int p, int q)
   {
      if (q == 0) return p;
      else return gcd(q, p % q);
   }
   public static void main(String[] args)
   {
      int p = Integer.parseInt(args[0]);
      int q = Integer.parseInt(args[1]);
      System.out.println(gcd(p, q));
   }
}
```

24

```
% java Euclid 1272 216
24
```

Possible debugging challenges with recursion

Missing base case.

```
public static double BAD(int N)
{
   return BAD(N-1) + 1.0/N;
}
```

No convergence guarantee.

```
public static double BAD(int N)
{
   if (N == 1) return 1.0;
   return BAD(1 + N/2) + 1.0/N;
}
```

Both lead to INFINITE RECURSIVE LOOP (bad news).

Try it! ← so that you can recognize and deal with it if it later happens to you

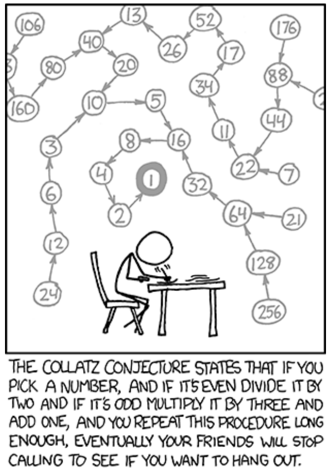
Collatz Sequence

Collatz sequence.

- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

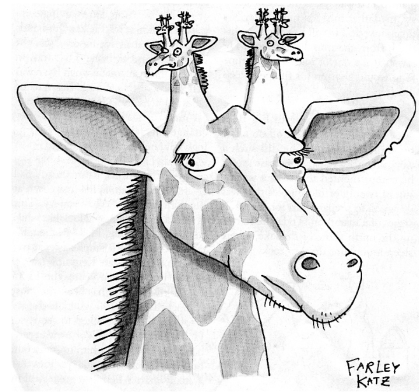
```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    collatz(3*N + 1);
 }
```



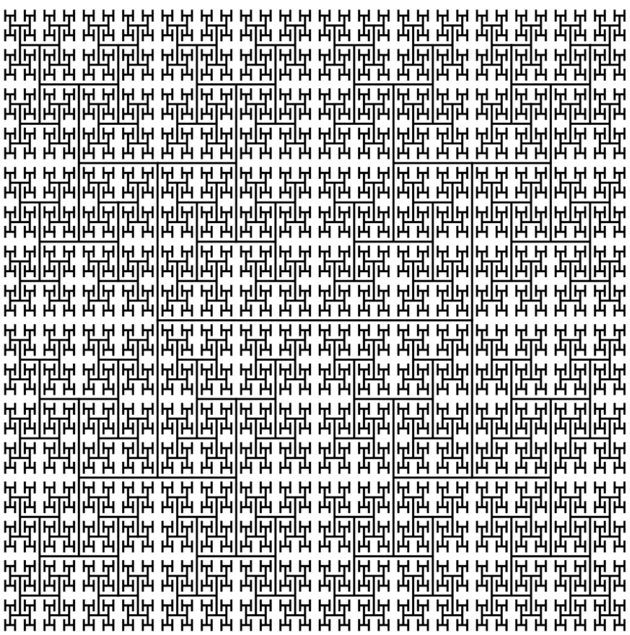
No one knows whether or not this function terminates for all N (!)

[usually we decrease N for all recursive calls]

Recursive Graphics



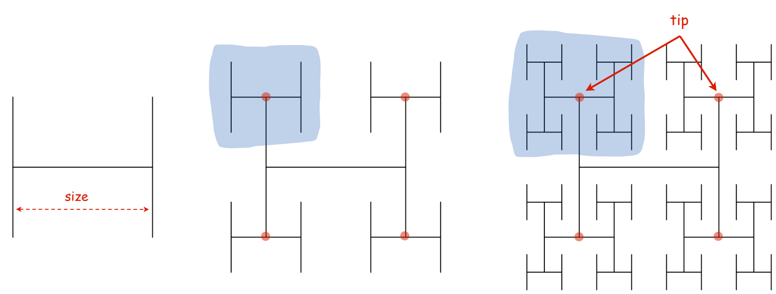
New Yorker Magazine, August 11, 2008



Htree

H-tree of order n.

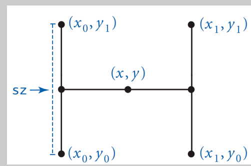
- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip. (and half the size)



order 1 order 2 order 3

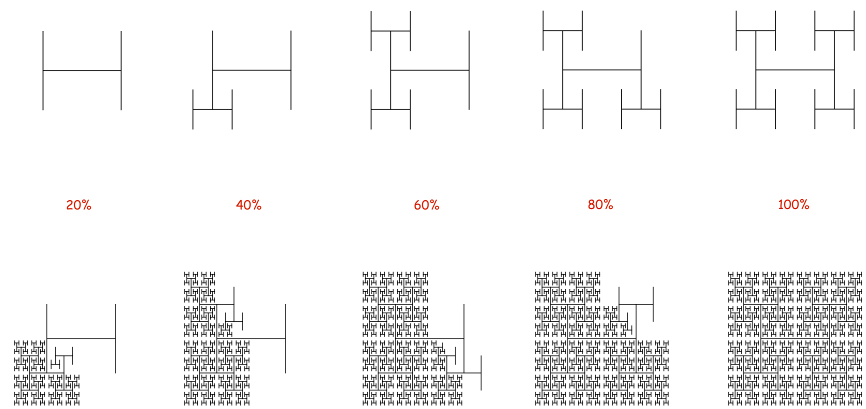
Htree in Java

```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);     ← draw the H, centered on (x, y)
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);          ← recursively draw 4 half-size Hs
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

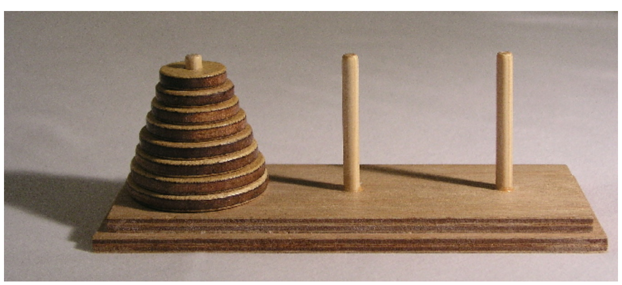


Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



Towers of Hanoi

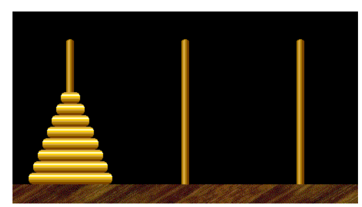


http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

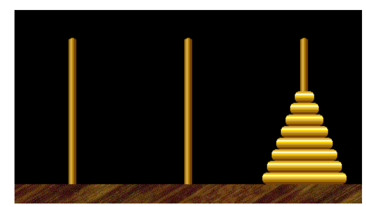
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



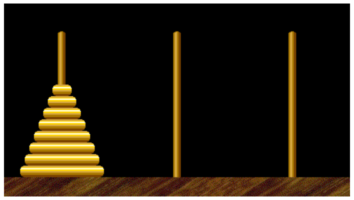
start



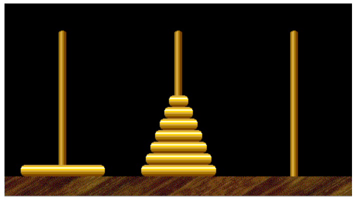
finish

Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

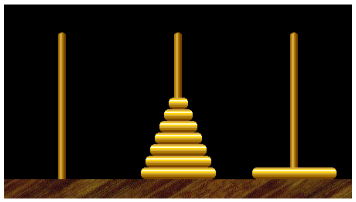


Move n-1 smallest discs right.

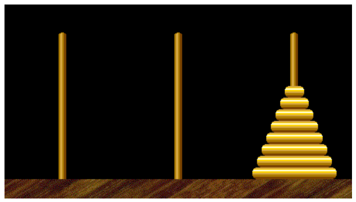


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi Legend

Q. Is world going to end (according to legend)?
- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi
{
   public static void moves(int n, boolean left)
   {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

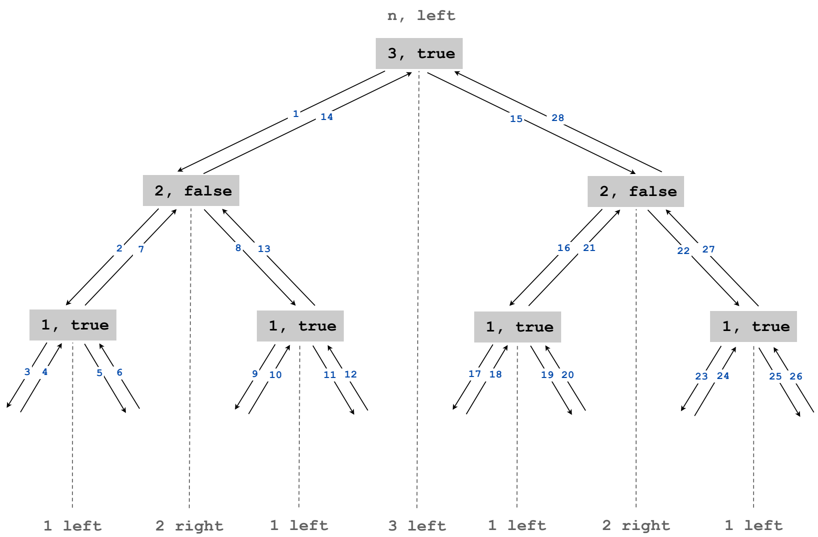
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes 2^n - 1 moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
  - move smallest disc to right if n is even (to left if n is odd)
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

> Divide et impera. Veni, vidi, vici. - *Julius Caesar*

Many important problems succumb to divide-and-conquer.

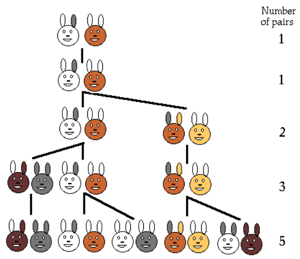
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

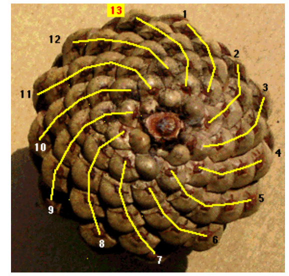
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



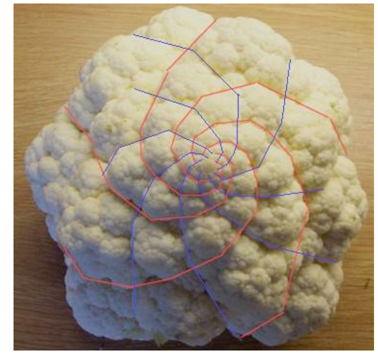
Fibonacci rabbits

L. P. Fibonacci
(1170 - 1250)

Fibonacci Numbers



pinecone



cauliflower

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

FYI (classical math):

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio $\approx$ 1.618

Ex: $F(50) \approx 1.2 \times 10^{10}$

A natural for recursion?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

Recursion Challenge 1 (difficult but important)

Is this an efficient way to compute F(50)?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```

Recursion Challenge 2 (easy and also important)

Is this an efficient way to compute F(50)?

```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (n == 1) return 1;
for (int i = 2; i <= 50; i++)
   F[i] = F[i-1] + F[i-2];
```

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?

Towers of Hanoi by W. A. Schloss.

- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.

Exponential time.

- Easy to specify recursive program that takes exponential time.
- Don't do it unless you plan to (and are working on a small problem).