4.1 Performance

Running Time

“As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—by what course of calculation can these results be arrived at by the machine in the shortest time?” – Charles Babbage

The Challenge

Q. Will my program be able to solve a large practical problem?

Scientific Method

Scientific method.
- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.
- Experiments must be reproducible.
- Hypothesis must be falsifiable.

Key insight. [Knuth 1970s]
Use the scientific method to understand performance.
Reasons to Analyze Algorithms

**Predict performance.**
- Will my program finish?
- When will my program finish?

**Compare algorithms.**
- Will this change make my program faster?
- How can I make my program faster?

**Basis for inventing new ways to solve problems.**
- Enables new technology.
- Enables new research.

Discrete Fourier transform.
- Break down waveform of \( N \) samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ...
- Brute force: \( N^2 \) steps.
- FFT algorithm: \( N \log N \) steps, enables new technology.

Three-Sum Problem

**Three-sum problem.** Given \( N \) integers, how many triples sum to 0?

**Context.** Deeply related to problems in computational geometry.

% more 8ints.txt
30 -30 -20 -10 40 0 10 5
% java ThreeSum < 8ints.txt
4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10

Q. How would you write a program to solve the problem?
Empirical Analysis

**Empirical analysis.** Run the program for various input sizes.

<table>
<thead>
<tr>
<th>$N$</th>
<th>time $^\dagger$</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>0.03</td>
</tr>
<tr>
<td>1024</td>
<td>0.26</td>
</tr>
<tr>
<td>2048</td>
<td>2.16</td>
</tr>
<tr>
<td>4096</td>
<td>17.18</td>
</tr>
<tr>
<td>8192</td>
<td>136.76</td>
</tr>
</tbody>
</table>

$^\dagger$ Running Linux on Sun-Fire-X4100 with 16GB RAM

**Caveat.** If $N$ is too small, you will measure mainly noise.

---

Stopwatch

**Q.** How to time a program?

**A.** A stopwatch.

% java ThreeSum < 1Kints.txt

% java ThreeSum < 2Kints.txt

$2$

391930676 -763182495 371251819
-326747290 802431422 -475684132

$2$

nick nick nick

0

nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
nick nick nick nick nick
Stopwatch

Q. How to time a program?
A. A Stopwatch object.

```java
public class Stopwatch {
    private final long start;
    public Stopwatch() {
        start = System.currentTimeMillis();
    }
    public double elapsedTime() {
        return (System.currentTimeMillis() - start) / 1000.0;
    }
}
```

Q. How to time a program?
A. A Stopwatch object.

```java
public static void main(String[] args) {
    int[] a = StdArrayIO.readInt1D();
    Stopwatch timer = new Stopwatch();
    StdOut.println(count(a));
    StdOut.println(timer.elapsedTime());
}
```

Empirical Analysis

Data analysis. Plot running time vs. input size \(N\).

![Graph showing running time vs. input size](https://example.com/graph.png)

Q. How fast does running time grow as a function of input size \(N\)?

Initial hypothesis. Running time approximately obeys a power law \( T(N) = a N^b \).

Data analysis. Plot running time vs. input size \(N\) on a log-log scale.

Consequence. Power law yields straight line.

Refined hypothesis. Running time grows as cube of input size: \( a N^3 \).
Doubling Hypothesis

**Doubling hypothesis.** Quick way to estimate $b$ in a power law hypothesis.

Run program, doubling the size of the input?

<table>
<thead>
<tr>
<th>$N$</th>
<th>time</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>0.033</td>
<td>-</td>
</tr>
<tr>
<td>1024</td>
<td>0.26</td>
<td>7.88</td>
</tr>
<tr>
<td>2048</td>
<td>2.16</td>
<td>8.43</td>
</tr>
<tr>
<td>4096</td>
<td>17.18</td>
<td>7.96</td>
</tr>
<tr>
<td>8192</td>
<td>136.76</td>
<td>7.96</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ratio</th>
<th>time †</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.96</td>
<td>17.18</td>
<td>4096</td>
</tr>
<tr>
<td>8.43</td>
<td>2.16</td>
<td>2048</td>
</tr>
<tr>
<td>7.88</td>
<td>0.26</td>
<td>1024</td>
</tr>
<tr>
<td>-</td>
<td>0.033</td>
<td>512</td>
</tr>
<tr>
<td>7.96</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

seems to converge to a constant $c = 8$

**Hypothesis.** Running time is about $a N^b$ with $b = \lg c$.

Performance Challenge 1

Let $T(N)$ be running time of `main()` as a function of input size $N$.

```java
public static void main(String[] args) {
    ...
    int N = Integer.parseInt(args[0]);
    ...
}
```

**Scenario 1.** $T(2N) / T(N)$ converges to about 4.

**Q.** What is order of growth of the running time?

$I$ | $N$ | $N^2$ | $N^3$ | $N^4$ | $2^N$
---|-----|------|------|------|------
1 | 1   | 1    | 1    | 1    | 1    
2 | 2   | 4    | 8    | 16   | 16   
4 | 4   | 16   | 64   | 256  | 256  
8 | 8   | 64   | 512  | 1024 | 1024 

**Prediction and Validation**

**Hypothesis.** Running time is about $a N^3$ for input of size $N$.

**Q.** How to estimate $a$?

**A.** Run the program!

<table>
<thead>
<tr>
<th>$N$</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>4096</td>
<td>17.18</td>
</tr>
<tr>
<td>4096</td>
<td>17.15</td>
</tr>
<tr>
<td>4096</td>
<td>17.17</td>
</tr>
</tbody>
</table>

$17.17 \div 4096^3 \Rightarrow a = 2.5 \times 10^{-10}$

**Refined hypothesis.** Running time is about $2.5 \times 10^{-10} \times N^3$ seconds.

**Prediction.** 1,100 seconds for $N = 16,384$.

**Observation.**

<table>
<thead>
<tr>
<th>$N$</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16384</td>
<td>1118.86</td>
</tr>
</tbody>
</table>
**Mathematical Analysis**

**Running time.** Count up frequency of execution of each instruction and weight by its execution time.

```c
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0) count++;
```

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>$N + 2$</td>
</tr>
<tr>
<td>variable assignment</td>
<td>$N + 2$</td>
</tr>
<tr>
<td>less than comparison</td>
<td>$\frac{1}{2} (N + 1) (N + 2)$</td>
</tr>
<tr>
<td>equal to comparison</td>
<td>$\frac{1}{2} N (N - 1)$</td>
</tr>
<tr>
<td>array access</td>
<td>$N (N - 1)$</td>
</tr>
<tr>
<td>increment</td>
<td>$\leq 2N$</td>
</tr>
</tbody>
</table>

0 + 1 + 2 + ... + $(N-1) = \frac{1}{2} \cdot N (N - 1)$

becoming very tedious to count

---

**Tilde Notation**

- Estimate running time as a function of input size $N$.
- Ignore lower order terms.
  - when $N$ is large, terms are negligible
  - when $N$ is small, we don’t care

**Ex 1.** $6N^3 + 17N^2 + 56 \sim 6N^3$
**Ex 2.** $6N^3 + 100N\frac{4}{3} + 56 \sim 6N^3$
**Ex 3.** $6N^3 + 17N^2 \log N \sim 6N^3$

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$
**Mathematical Analysis**

**Running time.** Count up frequency of execution of each instruction and weight by its execution time.

**Inner loop.** Focus on instructions in "inner loop."

---

**Constants in Power Law**

**Power law.** Running time of a typical program is $\sim a N^b$.

**Exponent $b$ depends on:** algorithm.

**Leading constant $a$ depends on:**
- Algorithm.
- Input data.
- Caching.
- Machine.
- Compiler.
- Garbage collection.
- Just-in-time compilation.
- CPU use by other applications.

**Our approach.** Use doubling hypothesis (or mathematical analysis) to estimate exponent $b$, run experiments to estimate $a$.

---

**Analysis: Empirical vs. Mathematical**

**Empirical analysis.**
- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

**Mathematical analysis.**
- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

**Critical difference.** Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

---

**Order of Growth Classifications**

**Observation.** A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```java
public static void f(int N) {
    if (N == 0) return;
    f(N-1);
    f(N-1);
    ...
}
N \cdot \lg N

\lg N = \log_2 N

public static void g(int N) {
    if (N == 0) return;
    g(N/2);
    g(N/2);
    for (int i = 0; i < N; i++)
        ...
}
N \cdot \lg N

public static void h(int N) {
    if (N == 0) return;
    h(N-1);
    h(N-1);
    for (int j = 0; j < N; j++)
        ...
} 2^N
```
Order of Growth Classifications

<table>
<thead>
<tr>
<th>order of growth</th>
<th>function</th>
<th>factor for doubling hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>logarithmic</td>
<td>\log N</td>
<td>1</td>
</tr>
<tr>
<td>linear</td>
<td>N</td>
<td>2</td>
</tr>
<tr>
<td>linearithmic</td>
<td>N \log N</td>
<td>2</td>
</tr>
<tr>
<td>quadratic</td>
<td>N^2</td>
<td>4</td>
</tr>
<tr>
<td>cubic</td>
<td>N^3</td>
<td>8</td>
</tr>
<tr>
<td>exponential</td>
<td>2^N</td>
<td>2^N</td>
</tr>
</tbody>
</table>

Commonly encountered growth functions

Order of Growth: Consequences

**Effect of increasing problem size for a program that runs for a few seconds**

- **linear**
  - a few minutes
- **linearithmic**
  - a few minutes
- **quadratic**
  - several hours
- **cubic**
  - a few weeks
- **exponential**
  - forever

**Effect of increasing computer speed on problem size that can be solved in a fixed amount of time**

- **linear**
  - 10
- **linearithmic**
  - 10
- **quadratic**
  - 3-4
- **cubic**
  - 2-3
- **exponential**
  - 1

Binary Search

**Sequential search in an unordered array.**

- Examine each entry until finding a match (or reaching the end).
- Takes time proportional to length of array in worst case.

**Binary search in an ordered array.**

- Examine the middle entry.
  - If equal, return index.
  - If too large, search in left half (recursively).
  - If too small, search in right half (recursively).
**Invariant.** If key appears in the array, then $a[lo] \leq key \leq a[hi]$.

```java
// precondition: array a[] is sorted
public static int search(int key, int[] a) {
    int lo = 0;
    int hi = a.length - 1;
    while (lo <= hi) {
        int mid = lo + (hi - lo) / 2;
        if (key < a[mid]) hi = mid - 1;
        else if (key > a[mid]) lo = mid + 1;
        else return mid;
    }
    return -1;  // not found
}
```

Java library implementation. `Arrays.binarySearch()`.

**Proposition.** Binary search in an ordered array of size $N$ takes at most $1 + \log_2 N$ 3-way compares.

**Pf.** After each 3-way compare, problem size decreases by a factor of 2.

$N \rightarrow N/2 \rightarrow N/4 \rightarrow N/8 \rightarrow \ldots \rightarrow 1$

**Q.** How many times can you divide $N$ by 2 until you reach 1?

**A.** About $\log_2 N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2$</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^3$</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^4$</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^5$</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^6$</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^7$</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^8$</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^9$</td>
<td>256</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^{10}$</td>
<td>512</td>
<td>256</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Typical Memory Requirements for Primitive Types

<table>
<thead>
<tr>
<th>Primitive Type</th>
<th>Byte Size</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>boolean</code></td>
<td>1</td>
</tr>
<tr>
<td><code>byte</code></td>
<td>1</td>
</tr>
<tr>
<td><code>char</code></td>
<td>2</td>
</tr>
<tr>
<td><code>int</code></td>
<td>4</td>
</tr>
<tr>
<td><code>float</code></td>
<td>4</td>
</tr>
<tr>
<td><code>long</code></td>
<td>8</td>
</tr>
<tr>
<td><code>double</code></td>
<td>8</td>
</tr>
</tbody>
</table>

**Q.** How much memory (in bytes) does your computer have?
Typical Memory Requirements for Reference Types

**Memory of an object.**
- Memory for each instance variable, plus
- Object overhead = 8 bytes on a 32-bit machine.
  - 16 bytes on a 64-bit machine

**Memory of a reference.** 4 byte pointer on a 32-bit machine.
- 8 bytes on a 64-bit machine

Typical Memory Requirements for Array Types

**Memory of an array.**
- Memory for each array entry.
- Array overhead = 16 bytes on a 32-bit machine.
  - 24 bytes on a 64-bit machine

<table>
<thead>
<tr>
<th>type</th>
<th>formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>int[]</td>
<td>4N + 16</td>
</tr>
<tr>
<td>double[]</td>
<td>8N + 16</td>
</tr>
<tr>
<td>Charge[]</td>
<td>36N + 16</td>
</tr>
<tr>
<td>int[][]</td>
<td>4N^2 + 20N + 16</td>
</tr>
<tr>
<td>double[][]</td>
<td>8N^2 + 20N + 16</td>
</tr>
<tr>
<td>String</td>
<td>2N + 40</td>
</tr>
</tbody>
</table>

Q. What’s the biggest double[] array you can store on your computer?

Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis, **scientific method**.

Q. What if it’s not fast enough? Not enough memory?
- Understand why.
- Buy a faster computer or more memory.
- Learn a better algorithm. — see COS 226
- Discover a new algorithm. — see COS 423

<table>
<thead>
<tr>
<th>attribute</th>
<th>better machine</th>
<th>better algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>cost</td>
<td>$$$ or more</td>
<td>$ or less</td>
</tr>
<tr>
<td>applicability</td>
<td>makes “everything” run faster</td>
<td>does not apply to some problems</td>
</tr>
<tr>
<td>improvement</td>
<td>quantitative improvements</td>
<td>dramatic qualitative improvements possible</td>
</tr>
</tbody>
</table>