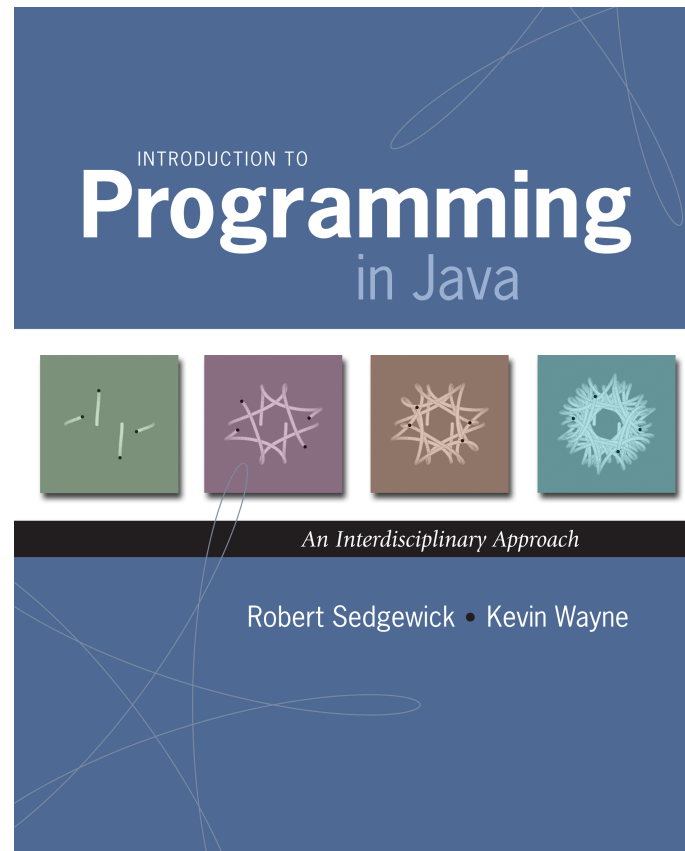


2.3 Recursion



Overview

What is recursion? When one function calls **itself** directly or indirectly.

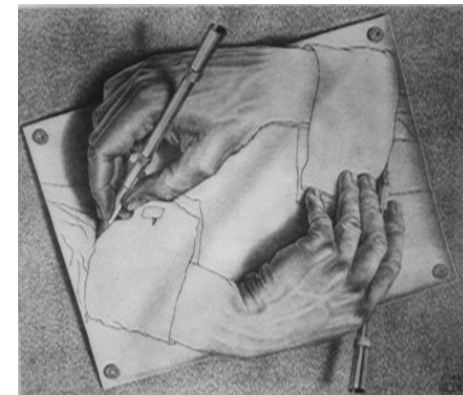
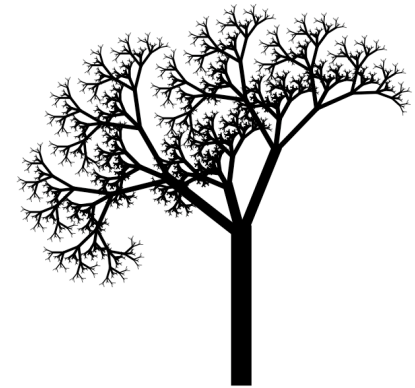
Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.

- Mergesort, FFT, gcd, depth-first search.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.



Reproductive Parts
M. C. Escher, 1948

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. $\text{gcd}(4032, 1272) = 24$.

$$4032 = 2^6 \times 3^2 \times 7^1$$

$$1272 = 2^3 \times 3^1 \times 53^1$$

$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.

- Simplify fractions: $1272/4032 = 53/168$.
- RSA cryptosystem.

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q .

Euclid's algorithm. [Euclid 300 BCE]

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case

← reduction step,
converges to base case

$$\begin{aligned} \text{gcd}(4032, 1272) &= \text{gcd}(1272, 216) \\ &= \text{gcd}(216, 192) \\ &= \text{gcd}(192, 24) \\ &= \text{gcd}(24, 0) \\ &= 24. \end{aligned}$$

$$4032 = 3 \times 1272 + 216$$

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case

← reduction step,
converges to base case

<i>p</i>							
<i>q</i>			<i>q</i>			<i>p % q</i>	
x	x	x	x	x	x	x	x

↑
gcd

$$\begin{aligned} p &= 8x \\ q &= 3x \\ \text{gcd}(p, q) &= x \end{aligned}$$

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q .

$$\text{gcd}(p, q) = \begin{cases} p & \text{if } q = 0 \\ \text{gcd}(q, p \% q) & \text{otherwise} \end{cases}$$

← base case

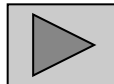
← reduction step,
converges to base case

Java implementation.

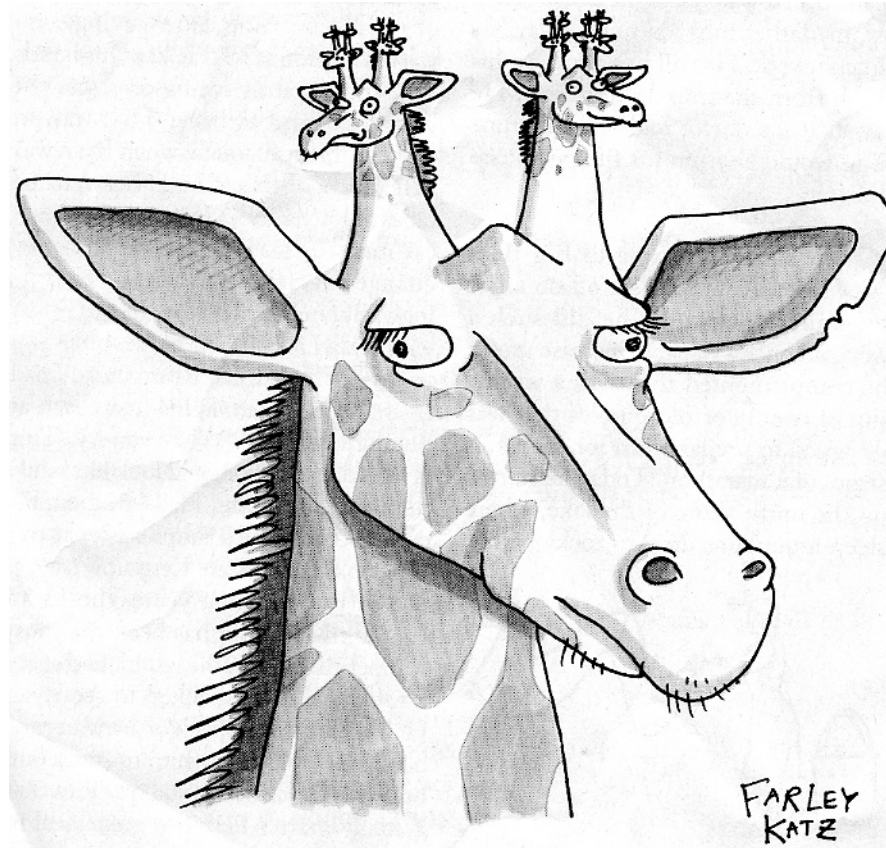
```
public static int gcd(int p, int q) {  
    if (q == 0) return p;  
    else return gcd(q, p % q);  
}
```

← base case

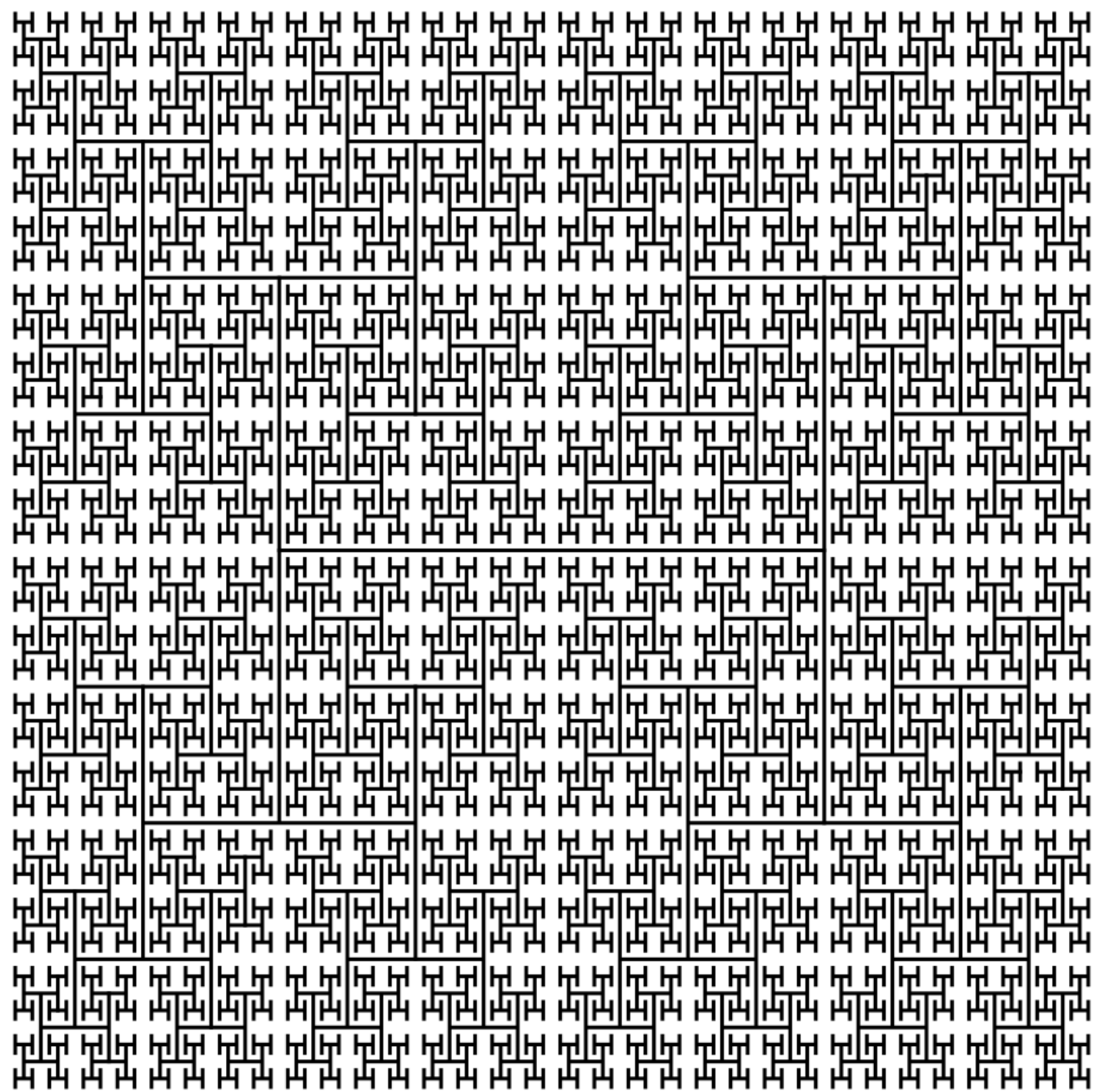
← reduction step



Recursive Graphics



New Yorker Magazine, August 11, 2008

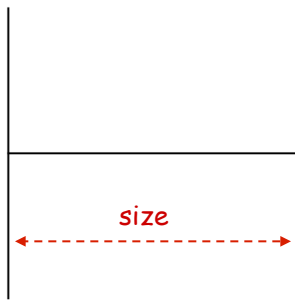


Htree

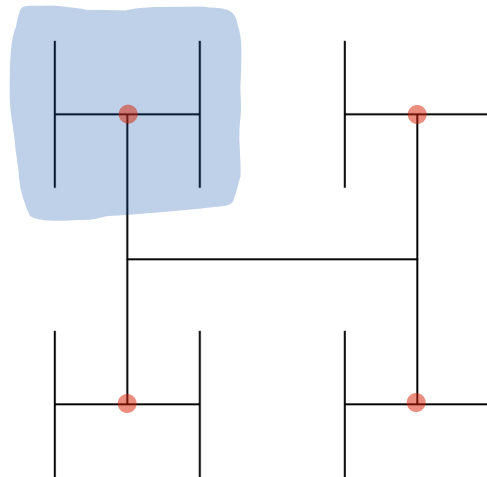
H-tree of order n .

- Draw an H.
- Recursively draw 4 H-trees of order $n-1$, one connected to each tip.

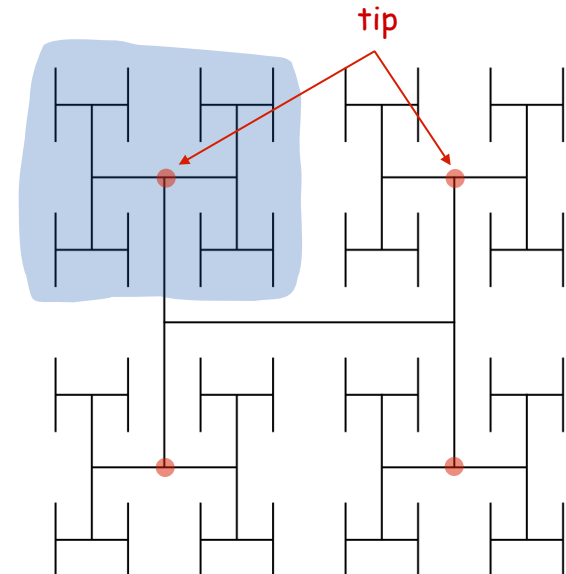
and half the size



order 1



order 2



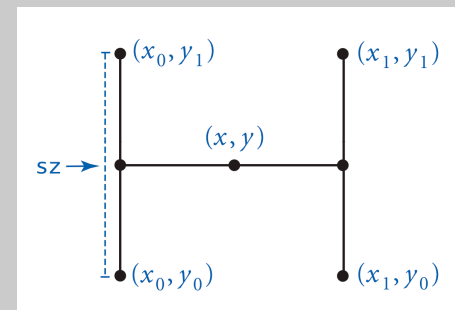
order 3

Htree in Java

```
public class Htree {  
    public static void draw(int n, double sz, double x, double y) {  
        if (n == 0) return;  
        double x0 = x - sz/2, x1 = x + sz/2;  
        double y0 = y - sz/2, y1 = y + sz/2;  
  
        StdDraw.line(x0, y, x1, y);  
        StdDraw.line(x0, y0, x0, y1);  
        StdDraw.line(x1, y0, x1, y1);  
  
        draw(n-1, sz/2, x0, y0);  
        draw(n-1, sz/2, x0, y1);  
        draw(n-1, sz/2, x1, y0);  
        draw(n-1, sz/2, x1, y1);  
    }  
  
    public static void main(String[] args) {  
        int n = Integer.parseInt(args[0]);  
        draw(n, .5, .5, .5);  
    }  
}
```

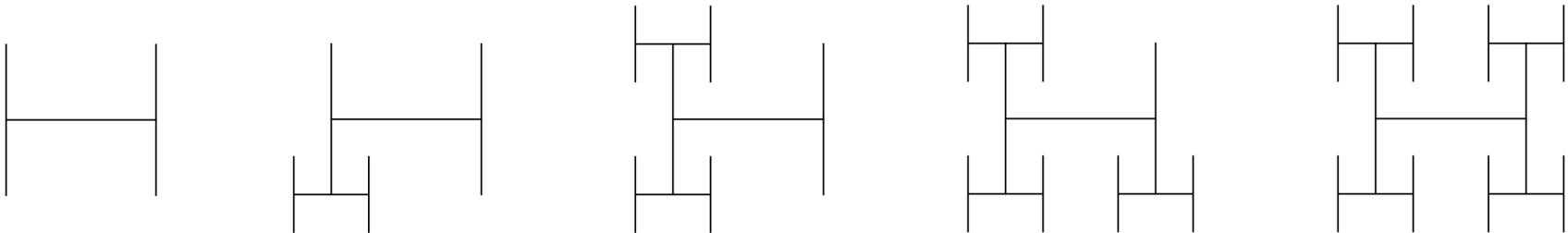
← draw the H, centered on (x, y)

← recursively draw 4 half-size Hs



Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.



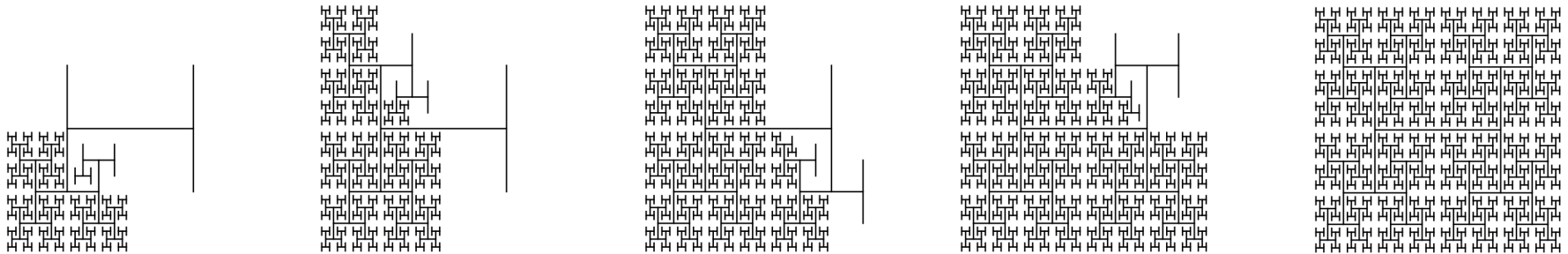
20%

40%

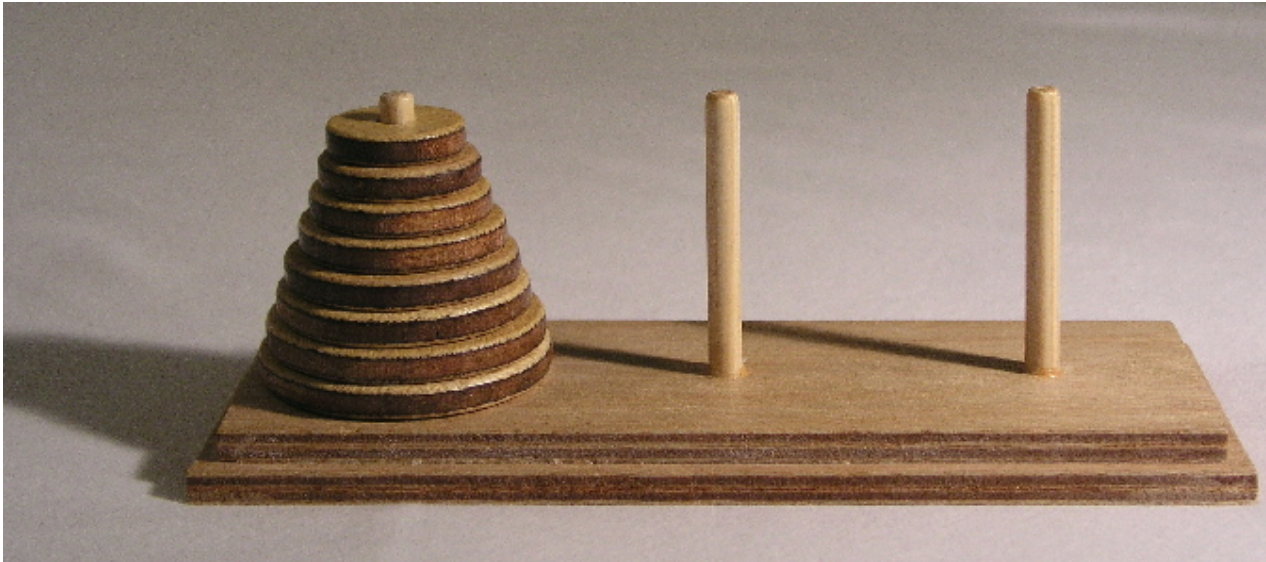
60%

80%

100%



Towers of Hanoi

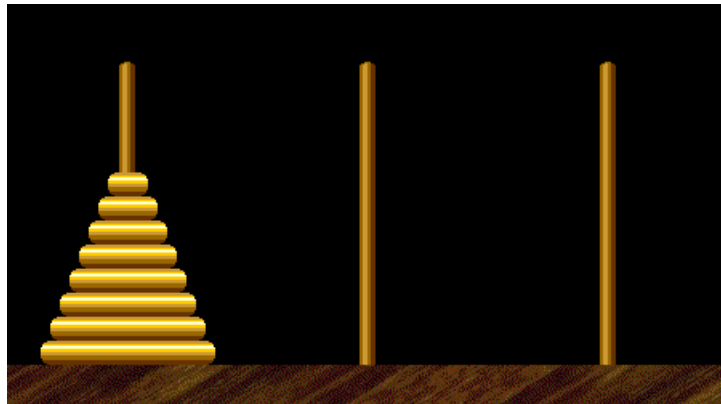


<http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg>

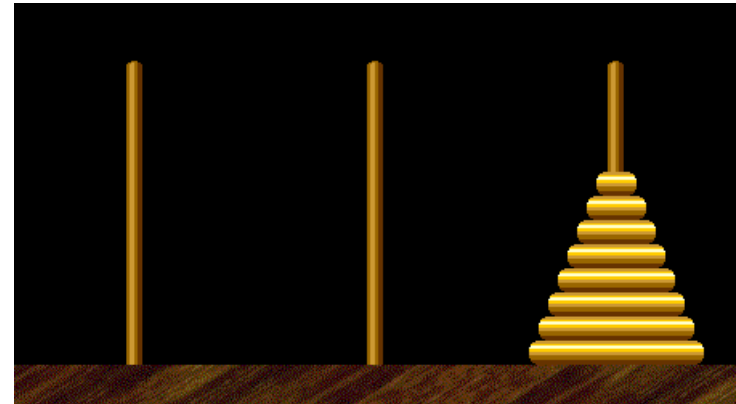
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

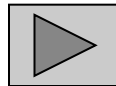
- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



start



finish

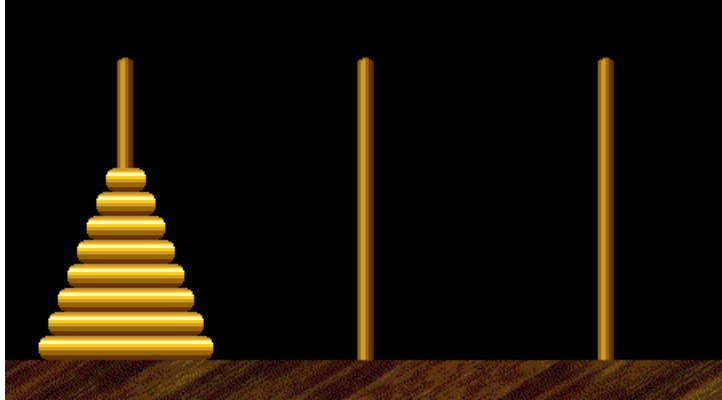


Towers of Hanoi demo

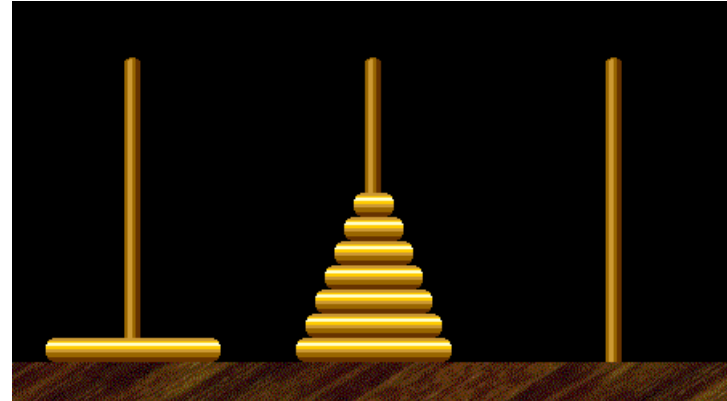


Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

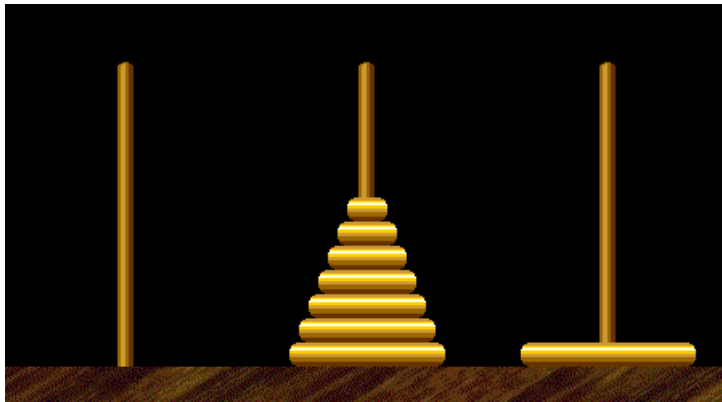


Move $n-1$ smallest discs right.

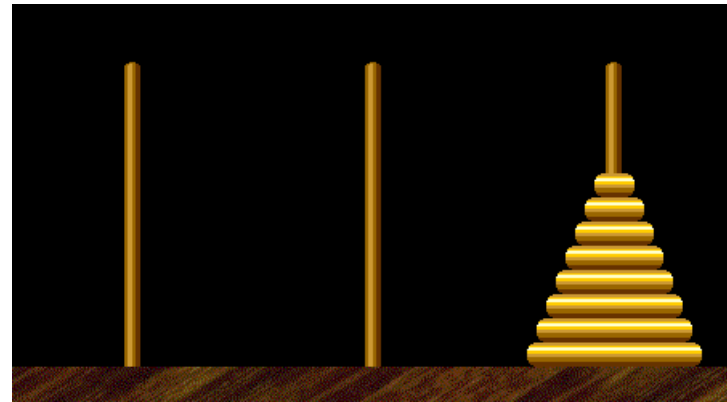


Move largest disc left.

cyclic wrap-around



Move $n-1$ smallest discs right.



Towers of Hanoi Legend

- Q. Is world going to end (according to legend)?
 - 64 golden discs on 3 diamond pegs.
 - World ends when certain group of monks accomplish task.

- Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi {  
  
    public static void moves(int n, boolean left) {  
        if (n == 0) return;  
        moves(n-1, !left);  
        if (left) System.out.println(n + " left");  
        else      System.out.println(n + " right");  
        moves(n-1, !left);  
    }  
  
    public static void main(String[] args) {  
        int N = Integer.parseInt(args[0]);  
        moves(N, true);  
    }  
}
```

moves(n, true) : move discs 1 to n one pole to the left
moves(n, false): move discs 1 to n one pole to the right

 smallest disc

Towers of Hanoi: Recursive Solution

```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
3 right
1 right
2 left
1 right
```

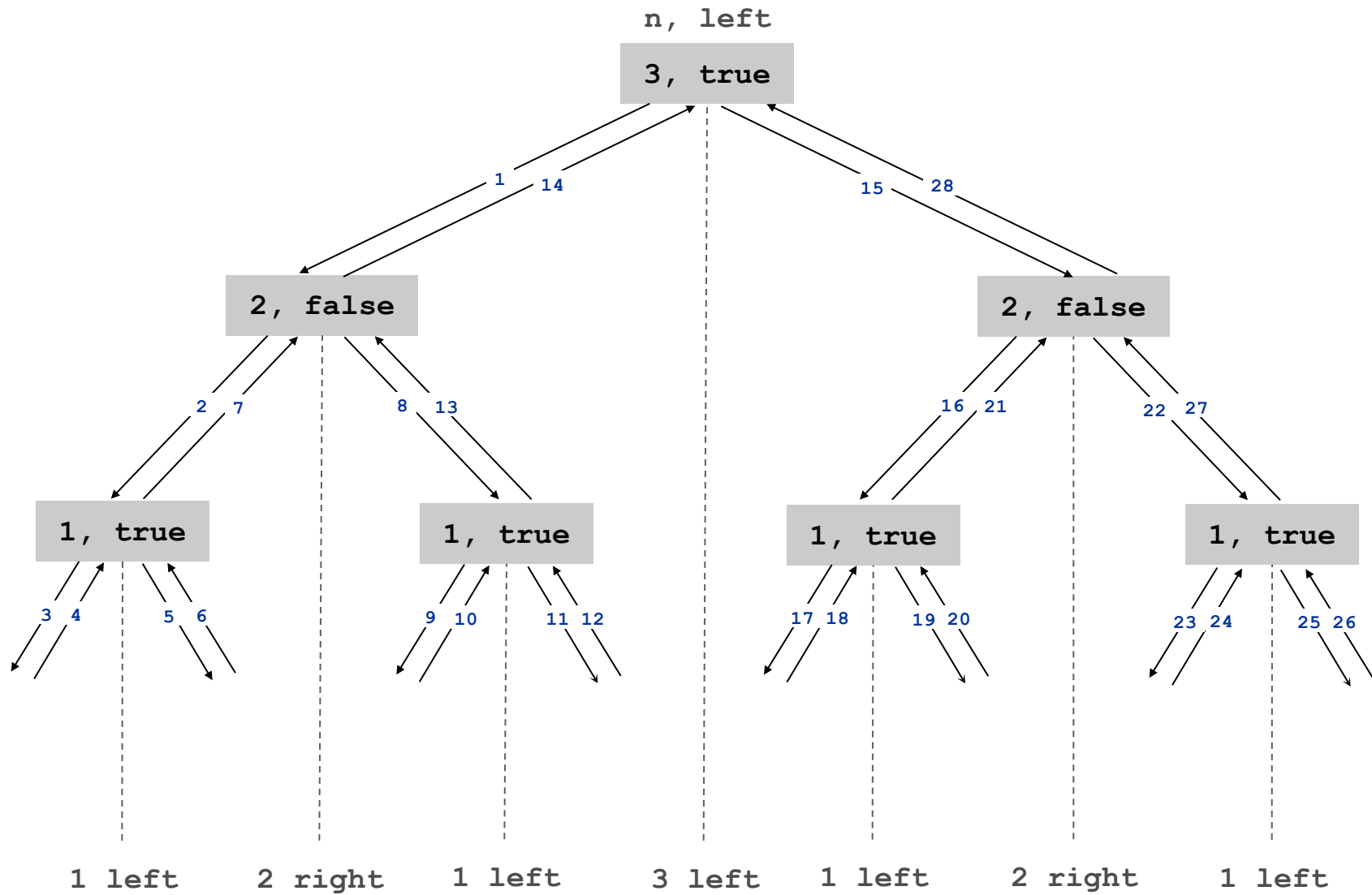
every other move is smallest disc



subdivisions of ruler



Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes $2^n - 1$ moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
 - move smallest disc to right if n is even
 - make only legal move not involving smallest disc
- to left if n is odd

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for $n = 64$ (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Divide et impera. Veni, vidi, vici. - Julius Caesar

Many important problems succumb to divide-and-conquer.

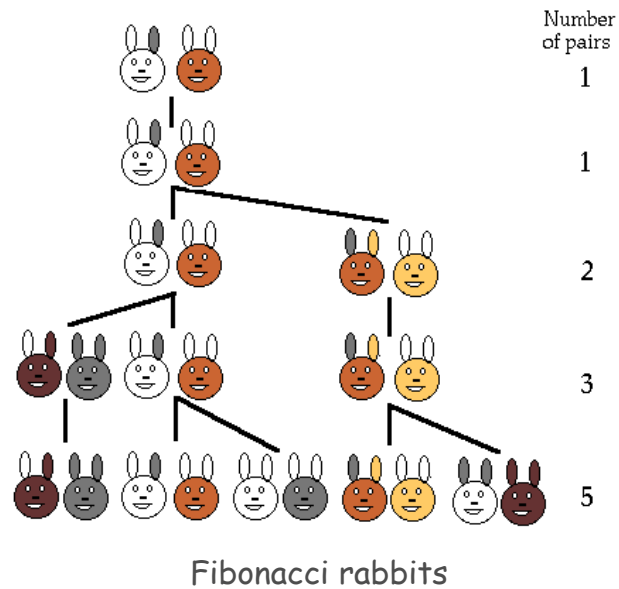
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

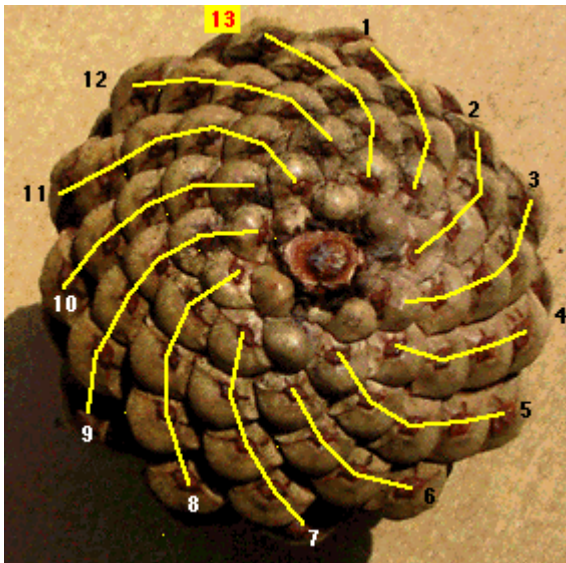


L. P. Fibonacci
(1170 - 1250)

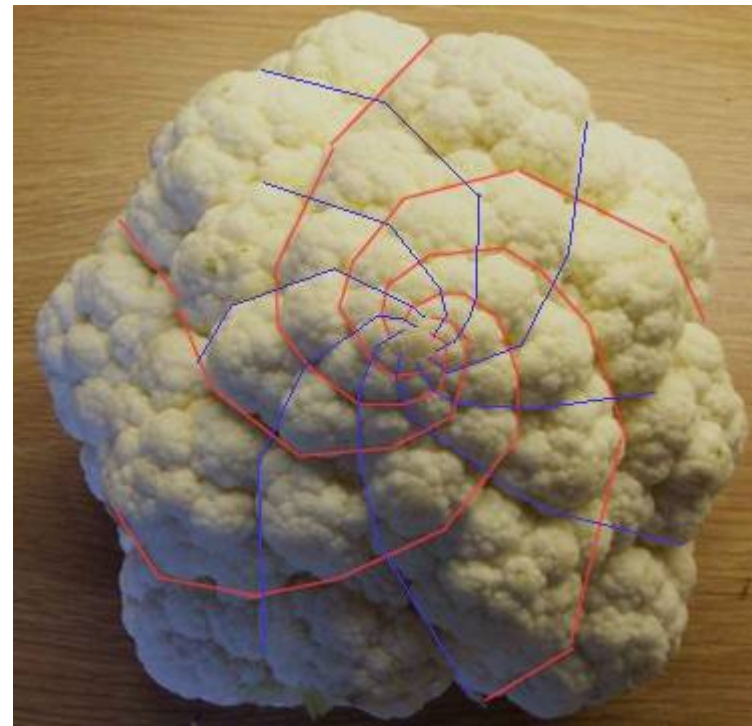
Fibonacci Numbers and Nature

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$



pinecone



cauliflower

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

A natural for recursion?

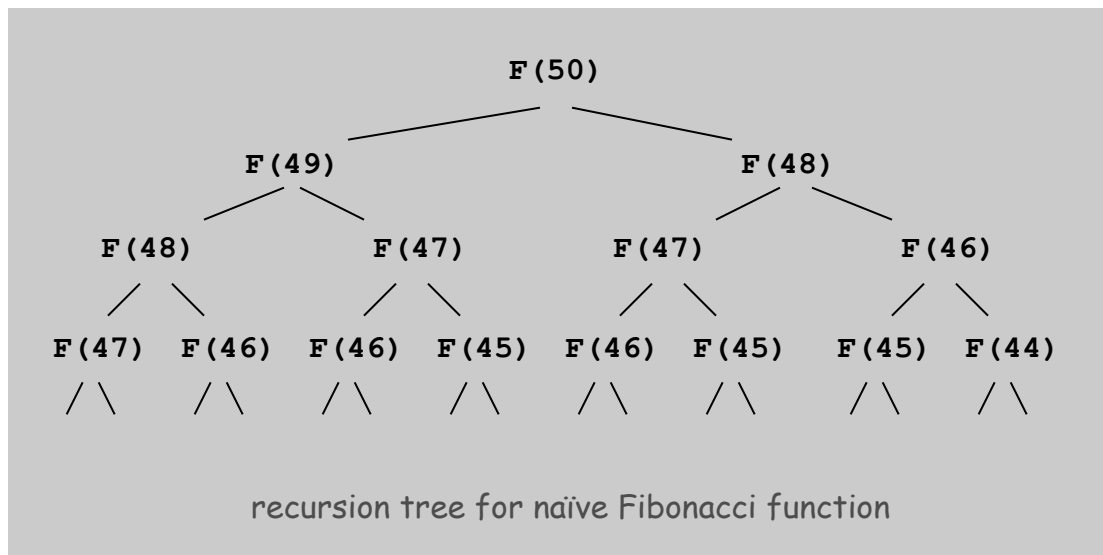
```
public static long F(int n) {  
    if (n == 0) return 0;  
    if (n == 1) return 1;  
    return F(n-1) + F(n-2);  
}
```


Recursion Challenge 1 (difficult but important)

Q. Is this an efficient way to compute $F(50)$?

```
public static long F(int n) {  
    if (n == 0) return 0;  
    if (n == 1) return 1;  
    return F(n-1) + F(n-2);  
}
```

A. No, no, no! This code is **spectacularly inefficient**.



$F(50)$ is called once.

$F(49)$ is called once.

$F(48)$ is called 2 times.

$F(47)$ is called 3 times.

$F(46)$ is called 5 times.

$F(45)$ is called 8 times.

...

$F(1)$ is called 12,586,269,025 times.

↖
 $F(50)$

Recursion Challenge 2 (easy and also important)

Q. Is this an efficient way to compute $F(50)$?

```
public static long(long n) {  
    long[] F = new long[n+1];  
    F[0] = 0; F[1] = 1;  
    for (int i = 2; i <= n; i++)  
        F[i] = F[i-1] + F[i-2];  
    return F[n];  
}
```

FYI: classic math

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$
$$= \left\lfloor \phi^n / \sqrt{5} \right\rfloor$$

ϕ = golden ratio ≈ 1.618

A. Yes. This code does it with 50 additions.

Lesson. Don't use recursion to engage in exponential waste.

Context. This is a special case of an important programming technique known as **dynamic programming** (stay tuned).

Summary

How to write simple recursive programs?

- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.



Why learn recursion?

- New mode of thinking.
- Powerful programming tool.

Towers of Hanoi by W. A. Schloss.

Divide-and-conquer. Elegant solution to many important problems.