2.3 Recursion

Overview

What is recursion? When one function calls itself directly or indirectly.

Why learn recursion?
- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.
- Mergesort, FFT, gcd, depth-first search.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

4032 = $2^6 \times 3^2 \times 7^1$
1272 = $2^3 \times 3^1 \times 53^1$
gcd = $2^3 \times 3^1 = 24$

Applications.
- Simplify fractions: $1272/4032 = 53/168$.
- RSA cryptosystem.

Greatest Common Divisor

Gcd. Find largest integer d that evenly divides into p and q.

Euclid’s algorithm. [Euclid 300 BCE]

$$
\text{gcd}(p, q) = \begin{cases} 
  p & \text{if } q = 0 \\
  \text{gcd}(q, p \% q) & \text{otherwise}
\end{cases}
$$

\[
gcd(4032, 1272) = \begin{cases} 
  4032 & \text{if } 1272 = 0 \\
  \text{gcd}(1272, 216) & \text{otherwise}
\end{cases}
= \begin{cases} 
  1272 & \text{if } 216 = 0 \\
  \text{gcd}(216, 192) & \text{otherwise}
\end{cases}
= \begin{cases} 
  216 & \text{if } 192 = 0 \\
  \text{gcd}(192, 24) & \text{otherwise}
\end{cases}
= \begin{cases} 
  192 & \text{if } 24 = 0 \\
  \text{gcd}(24, 0) & \text{otherwise}
\end{cases}
= \begin{cases} 
  24 & \text{if } 0 = 0 \\
  \text{gcd}(24, 0) & \text{otherwise}
\end{cases}
= 24.
\]
**Greatest Common Divisor**

**Gcd.** Find largest integer \( d \) that evenly divides into \( p \) and \( q \).

\[
\text{gcd}(p, q) = \begin{cases} 
  p & \text{if } q = 0 \\
  \text{gcd}(q, p \mod q) & \text{otherwise}
\end{cases}
\]

- base case
- reduction step, converges to base case

| \( p \) |
| \( q \) |
| \( p \mod q \) |
| \( \times \) | \( \times \) | \( \times \) |

\( p = 8x \)
\( q = 3x \)
\( \text{gcd}(p, q) = x \)

**Java implementation.**

```java
public static int gcd(int p, int q) {
    if (q == 0) return p;
    else return gcd(q, p % q);
}
```

**Recursive Graphics**

New Yorker Magazine, August 11, 2008
H-tree of order $n$.
- Draw an H.
- Recursively draw 4 H-trees of order $n-1$, one connected to each tip.

```java
class Htree {
   public static void draw(int n, double sz, double x, double y) {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0, y, x1, y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

Animated H-tree

Animated H-tree. Pause for 1 second after drawing each H.

Towers of Hanoi

Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.

Towers of Hanoi demo

Edouard Lucas (1883)

Towers of Hanoi Legend

Q. Is world going to end (according to legend)?
- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

public class TowersOfHanoi {
    public static void moves(int n, boolean left) {
        if (n == 0) return;
        moves(n-1, !left);
        if (left) System.out.println(n + " left");
        else System.out.println(n + " right");
        moves(n-1, !left);
    }

    public static void main(String[] args) {
        int N = Integer.parseInt(args[0]);
        moves(N, true);
    }
}
Towers of Hanoi: Recursive Solution

Remarkable properties of recursive solution.
- Takes $2^n - 1$ moves to solve $n$ disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!
- Alternate between two moves:
  - move smallest disc to right if $n$ is even
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.
- Takes 585 billion years for $n = 64$ (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.
- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Many important problems succumb to divide-and-conquer.
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.
- Midpoint displacement method for fractional Brownian motion.
Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

\[ F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases} \]

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

```java
public static long F(int n) {
    if (n == 0) return 0;
    if (n == 1) return 1;
    return F(n-1) + F(n-2);
}
```

A natural for recursion?

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

L. P. Fibonacci (1170 - 1250)

Fibonacci numbers.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci rabbits

Fibonacci Numbers and Nature

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Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

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public static long F(int n) {
    if (n == 0) return 0;
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    return F(n-1) + F(n-2);
}
```

A natural for recursion?
Recursion Challenge 1 (difficult but important)

Q. Is this an efficient way to compute $F(50)$?

A. No, no, no! This code is **spectacularly inefficient**.

```java
public static long F(int n) {
    if (n == 0) return 0;
    if (n == 1) return 1;
    return F(n-1) + F(n-2);
}
```

Recursion tree for naïve Fibonacci function

F(50) is called once.
F(49) is called once.
F(48) is called 2 times.
F(47) is called 3 times.
F(46) is called 5 times.
F(45) is called 8 times.
... F(1) is called 12,586,269,025 times.

Recursion Challenge 2 (easy and also important)

Q. Is this an efficient way to compute $F(50)$?

A. Yes. This code does it with 50 additions.

```java
public static long(int n) {
    long[] F = new long[n+1];
    F[0] = 0; F[1] = 1;
    for (int i = 2; i <= n; i++)
        F[i] = F[i-1] + F[i-2];
    return F[n];
}
```

FYI: classic math

$F(n) = \phi^n - (1-\phi)^n \over \sqrt{5}$

$\phi = golden ratio \approx 1.618$

Context. This is a special case of an important programming technique known as **dynamic programming** (stay tuned).

Summary

**How to write simple recursive programs?**
- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

**Why learn recursion?**
- New mode of thinking.
- Powerful programming tool.

**Divide-and-conquer.** Elegant solution to many important problems.