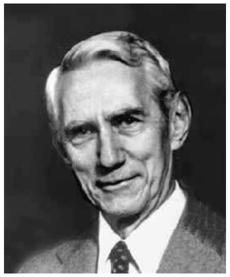
# 6.1 Combinational Circuits

George Boole (1815 – 1864)



Claude Shannon (1916 – 2001)

## Signals and Wires

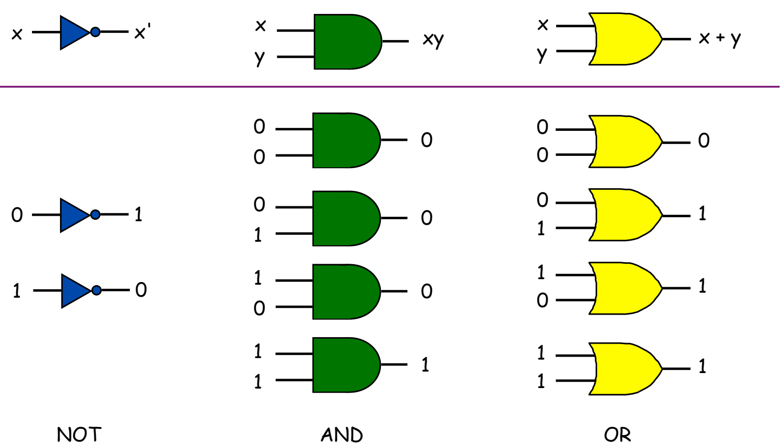
Digital signals
- Binary (or "logical") values: 1 or 0, on or off, high or low voltage

Wires.
- Propagate digital signals from place to place.
- Signals "flow" from left to right.
  - A drawing convention, sometimes violated
  - Actually: flow from producer to consumer(s) of signal



## Logic Gates

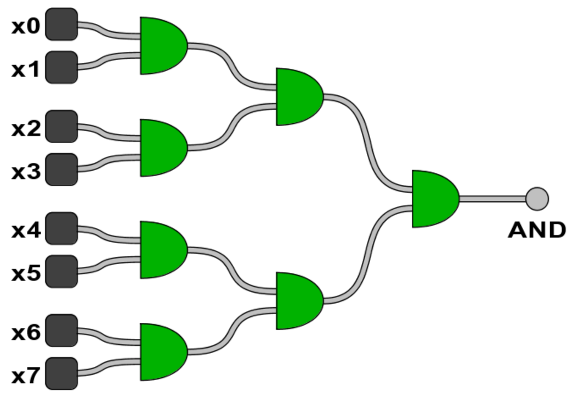
Logical gates.
- Fundamental building blocks.






## Multiway AND Gates

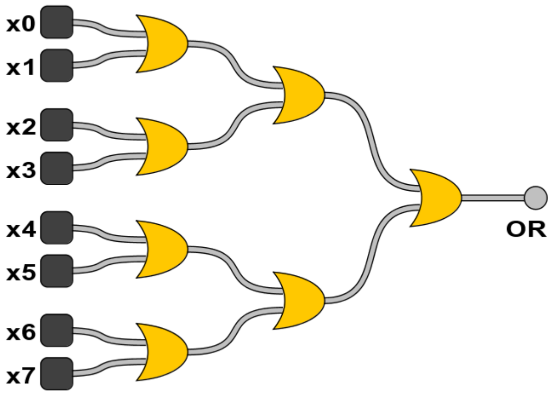
AND($x_0$, $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$).
- 1 if all inputs are 1.
- 0 otherwise.



## Multiway OR Gates

OR($x_0$, $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$).

- 1 if at least one input is 1.
- 0 otherwise.






## Boolean Algebra

History.

- Developed by Boole to solve mathematical logic problems (1847).
- Shannon master's thesis applied it to digital circuits (1937).

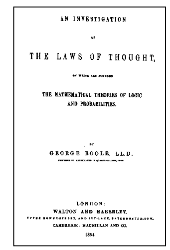
"possibly the most important, and also the most famous, master's thesis of the [20th] century" --Howard Gardner

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

- Boolean variables: signals.
- Boolean functions: circuits.



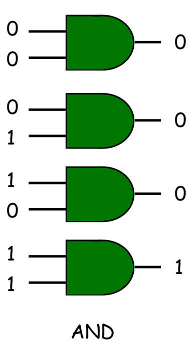



## Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs $\Rightarrow$ $2^N$ rows.

| AND Truth Table | | |
|---|---|---|
| x | y | AND(x, y) |
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |






## Truth Table for Functions of 2 Variables

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one

| Truth Table for All Boolean Functions of 2 Variables | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | y | ZERO | AND | | x | | y | XOR | OR |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

| Truth Table for All Boolean Functions of 2 Variables | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | y | NOR | EQ | y' | | x' | | NAND | ONE |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

## Truth Table for Functions of 3 Variables

Truth table.
- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one
- 256 Boolean functions of 3 variables.
  - every 8-bit value represents one
- 2^(2^N) Boolean functions of N variables!

| Some Functions of 3 Variables | | | | | | |
|---|---|---|---|---|---|---|
| x | y | z | AND | OR | MAJ | ODD |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |



## Universality of AND, OR, NOT

Any Boolean function can be expressed using AND, OR, NOT.
- "Universal"
- Example: XOR(x,y) = xy' + x'y

| Notation | Meaning |
|---|---|
| x' | NOT x |
| x y | x AND y |
| x + y | x OR y |

| Expressing XOR Using AND, OR, NOT | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | y | x' | y' | x'y | xy' | x'y + xy' | XOR |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Exercise. Show {AND, NOT}, {OR, NOT}, {NAND}, {AND, XOR} are universal.
Hint. Use DeMorgan's Law: (xy)' = (x' + y') and (x + y)' = (x'y')



## Sum-of-Products

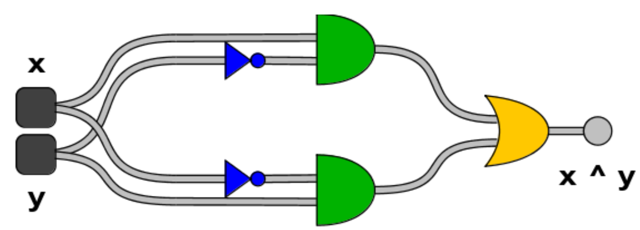
Any Boolean function can be expressed using AND, OR, NOT.
- Sum-of-products is systematic procedure.
  - form AND term for each 1 in truth table of Boolean function
  - OR terms together

| Expressing MAJ Using Sum-of-Products | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | y | z | MAJ | x'yz | xy'z | xyz' | xyz | x'yz + xy'z + xyz' + xyz |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |



## Translate Boolean Formula to Boolean Circuit

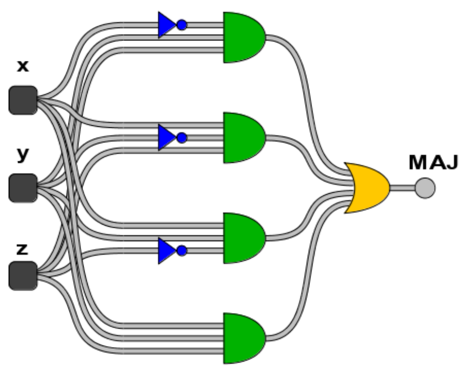
Use sum-of-products form.
- XOR(x, y) = xy' + x'y.



## Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz.



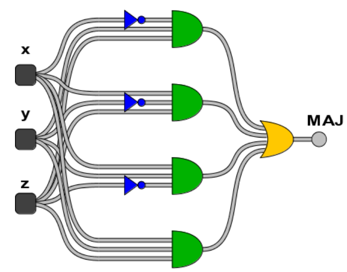



## Simplification Using Boolean Algebra

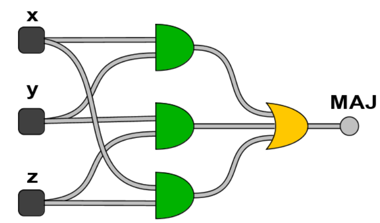
Many possible circuits for each Boolean function.

- Sum-of-products not necessarily optimal in:
  - number of gates (space)
  - depth of circuit (time)

- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.




size = 8, depth = 3




size = 4, depth = 2



## Expressing a Boolean Function Using AND, OR, NOT

Ingredients.

- AND gates.
- OR gates.
- NOT gates.
- Wire.

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.



## ODD Parity Circuit

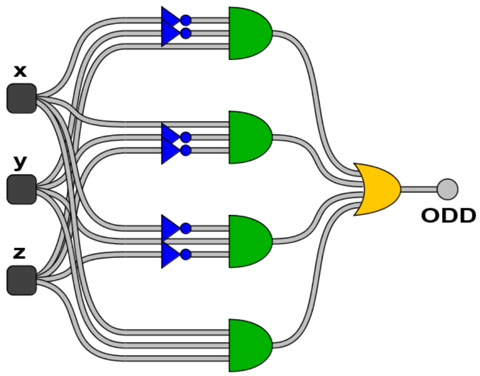
ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.

| Expressing ODD Using Sum-of-Products | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | y | z | ODD | x'y'z | x'yz' | xy'z' | xyz | x'y'z + x'yz' + xy'z' + xyz |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

## ODD Parity Circuit

ODD(x, y, z).
- 1 if odd number of inputs are 1.
- 0 otherwise.






## Let's Make an Adder Circuit

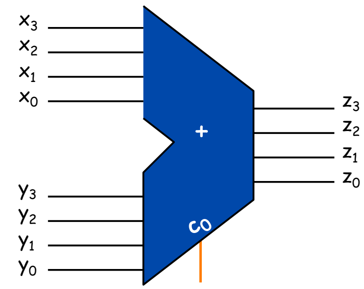
Goal: x + y = z for 4-bit integers.
- We build 4-bit adder: 9 inputs, 4 outputs.
- Same idea scales to 128-bit adder.
- Key computer component.

Step 1.
- Represent input and output in binary.

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | 1 | 0 |
| | 2 | 4 | 8 | 7 |
| + | 3 | 5 | 7 | 9 |
| | 6 | 0 | 6 | 6 |

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | 0 | 0 |
| | 0 | 0 | 1 | 0 |
| + | 0 | 1 | 1 | 1 |
| | 1 | 0 | 0 | 1 |

| | | | | |
|---|---|---|---|---|
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |






## Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.

Step 2. (first attempt)
- Build truth table.
- Why is this a bad idea?
  - 128-bit adder: $2^{256+1}$ rows > # electrons in universe!

| | | | | |
|---|---|---|---|---|
| | | | | $c_0$ |
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

4-Bit Adder Truth Table

| $c_0$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $z_3$ | $z_2$ | $z_1$ | $z_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$2^{8+1}$ = 512 rows!



## Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.

Step 2. (do one bit at a time)
- Build truth table for carry bit.
- Build truth table for summand bit.

| | | | | |
|---|---|---|---|---|
| | $c_3$ | $c_2$ | $c_1$ | $c_0 = 0$ |
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

Carry Bit

| $x_i$ | $y_i$ | $c_i$ | $c_{i+1}$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

Summand Bit

| $x_i$ | $y_i$ | $c_i$ | $z_i$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

## Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 3.

- Derive (simplified) Boolean expression.

| | | | | |
|---|---|---|---|---|
| | $c_3$ | $c_2$ | $c_1$ | $c_0 = 0$ |
| | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
| + | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
| | $z_3$ | $z_2$ | $z_1$ | $z_0$ |

| Carry Bit | | | | |
|---|---|---|---|---|
| $x_i$ | $y_i$ | $c_i$ | $c_{i+1}$ | MAJ |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

| Summand Bit | | | | |
|---|---|---|---|---|
| $x_i$ | $y_i$ | $c_i$ | $z_i$ | ODD |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

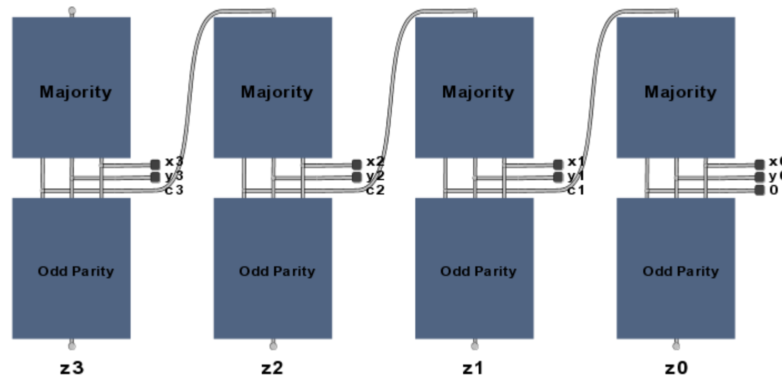


## Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 4.

- Transform Boolean expression into circuit.
- Chain together 1-bit adders.






## Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 4.

- Transform Boolean expression into circuit.
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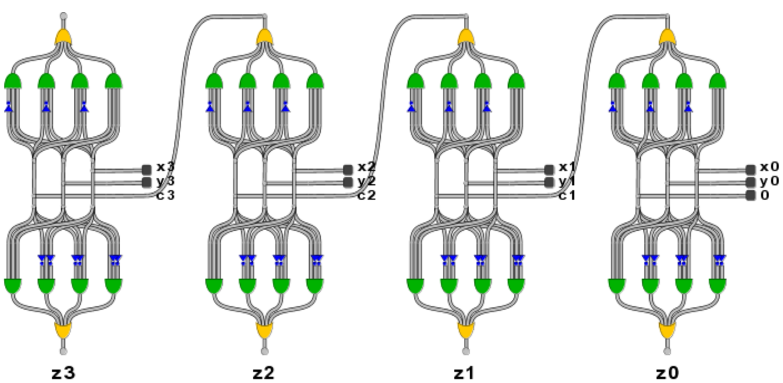
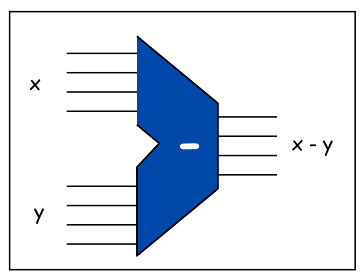




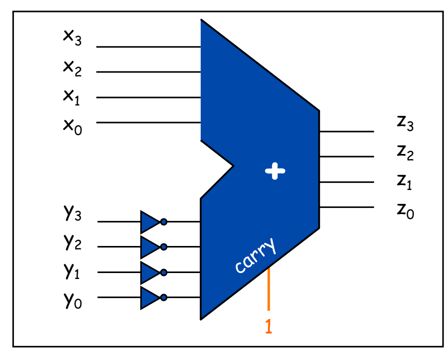

## Subtractor

Subtractor circuit: $z = x - y$.

- One approach: design like adder circuit.
- Better idea: reuse adder circuit.
  - 2's complement: to negate an integer, flip bits, then add 1




4-Bit Subtractor Interface



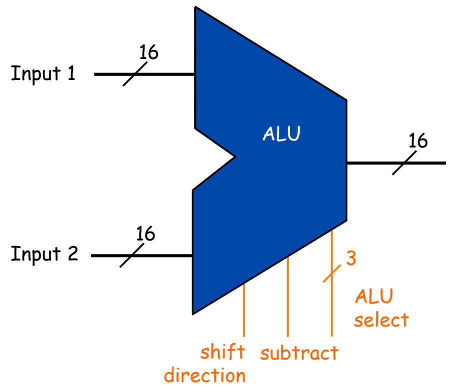

4-Bit Subtractor Implementation

## Arithmetic Logic Unit: Interface

ALU Interface.

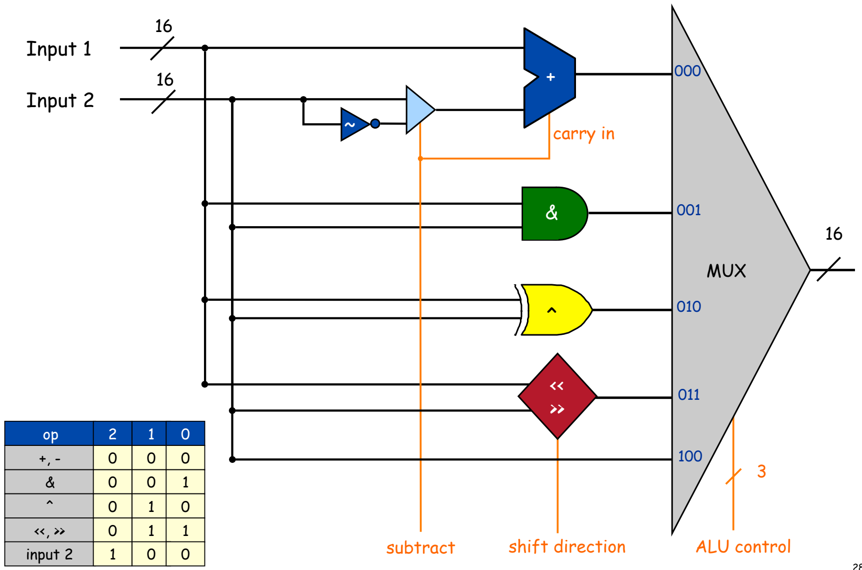
- Add, subtract, bitwise and, bitwise xor, shift left, shift right, copy.
- Associate 3-bit integer with 5 primary ALU operations.
  - ALU performs operations in parallel
  - control wires select which result ALU outputs

| op | 2 | 1 | 0 |
|---|---|---|---|
| +, - | 0 | 0 | 0 |
| & | 0 | 0 | 1 |
| ^ | 0 | 1 | 0 |
| <<, >> | 0 | 1 | 1 |
| input 2 | 1 | 0 | 0 |






## Arithmetic Logic Unit: Implementation




| op | 2 | 1 | 0 |
|---|---|---|---|
| +, - | 0 | 0 | 0 |
| & | 0 | 0 | 1 |
| ^ | 0 | 1 | 0 |
| <<, >> | 0 | 1 | 1 |
| input 2 | 1 | 0 | 0 |

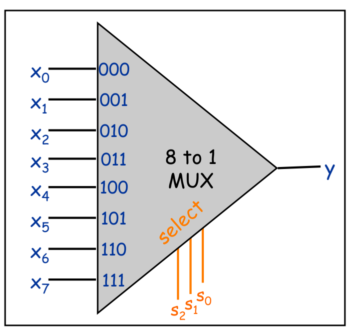


## $2^n$-to-1 Multiplexer

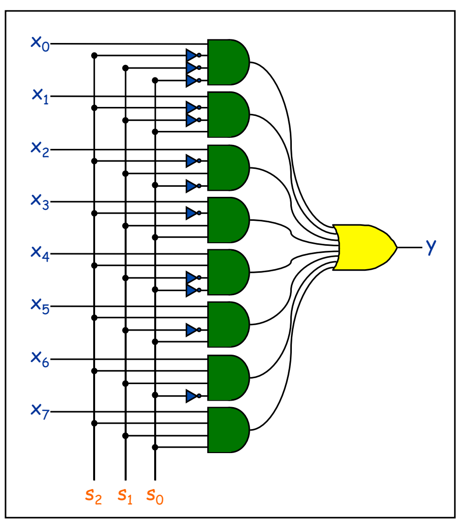
n = 8 for main memory

$2^n$-to-1 multiplexer.

- n select inputs, $2^n$ data inputs, 1 output.
- Copies "selected" data input bit to output.



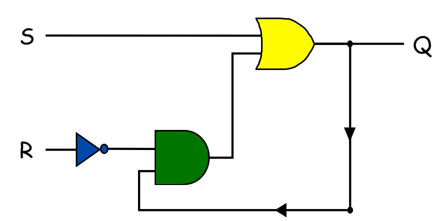

8-to-1 Mux Interface




8-to-1 Mux Implementation



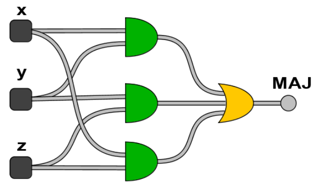
# 6.2: Sequential Circuits



## Sequential vs. Combinational Circuits

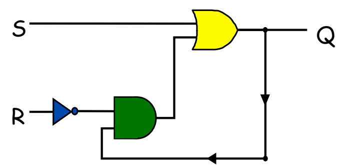
Combinational circuits.

- Output determined solely by inputs.
- Can draw solely with left-to-right signal paths.




Sequential circuits.

- Output determined by inputs AND previous outputs.
- Feedback loop.

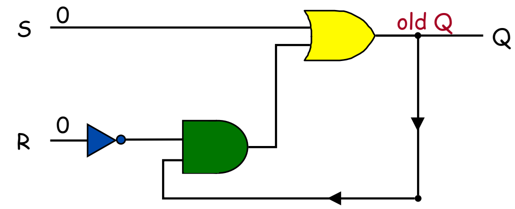





## SR Flip-Flop

What is the value of Q if:

- S = 1 and R = 0 ? ⇒ Q is surely 1.
- S = 0 and R = 1 ? ⇒ Q is surely 0.
- S = 0 and R = 0 ? ⇒ Q is possibly 0 . . . or possibly 1.



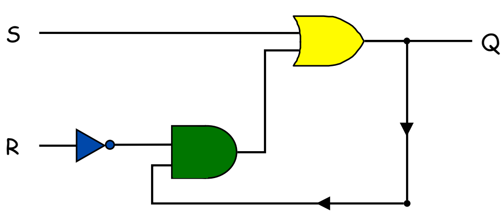

While S = R = 0 , Q *remembers* what it was the last time S or R was 1.



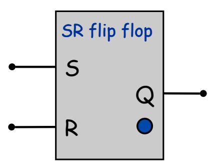
## SR Flip-Flop

SR Flip-Flop.

- S = 1, R = 0 (set) ⇒ "Flips" bit on.
- S = 0, R = 1 (reset) ⇒ "Flops" bit off.
- S = R = 0 ⇒ Status quo.
- S = R = 1 ⇒ Not allowed.




Implementation



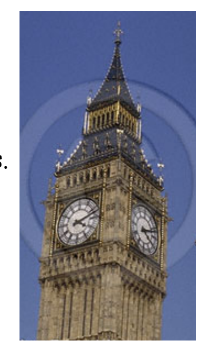

Interface

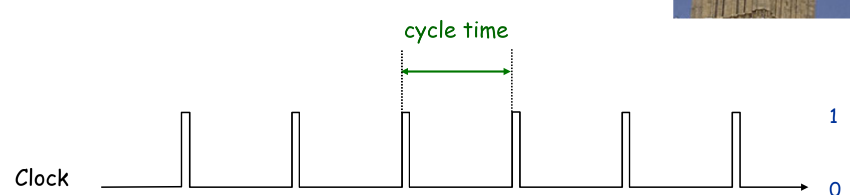


## Clock

Clock.

- Fundamental abstraction.
  - regular on-off pulse
- External analog device.
- Synchronizes operations of different circuit elements.
- 1 GHz clock means 1 billion pulses per second.

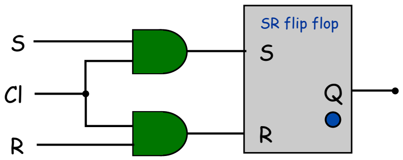




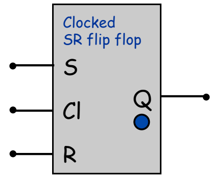
## Clocked SR Flip-Flop

Clocked SR Flip-Flop.

- Same as SR flip-flop except S and R only active when clock is 1.




Implementation



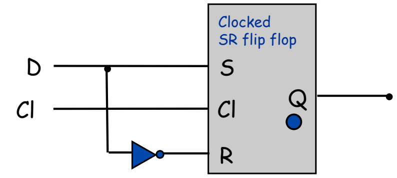

Interface



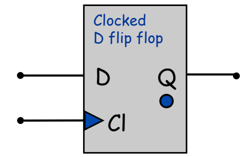
## Clocked D Flip-Flop

Clocked D Flip-Flop.

- Output follows D input while clock is 1.
- Output is remembered while clock is 0.




Implementation




Interface

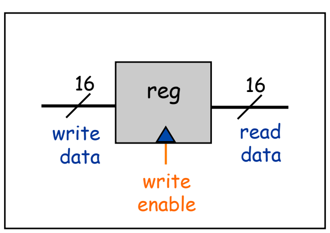


## Stand-Alone Register

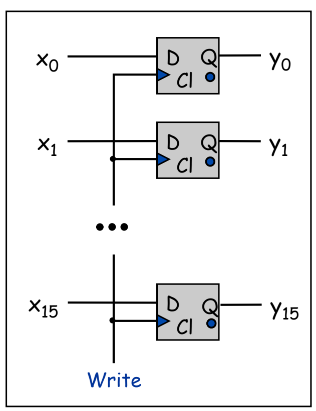
k-bit register.

- Stores k bits.
- Register contents always available on output.
- If write enable is asserted, k input bits get copied into register.

Ex: Program Counter, 16 TOY registers, 256 TOY memory locations.




16-bit Register Interface



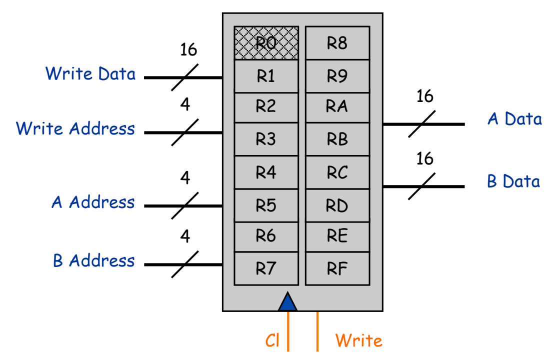

16-bit Register Implementation



## Registers

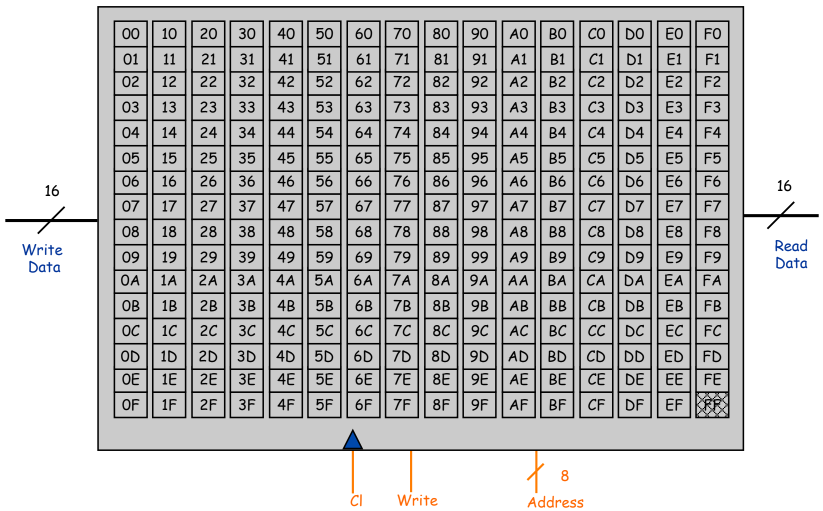
TOY registers: fancy 16 x 16-bit register file.

- Want to be able to read two registers, and write to a third in the same instructions: `R1 ← R2 + R3`.
- 3 address inputs, 1 data input, 2 data outputs.
- Add decoders and muxes for additional ports.



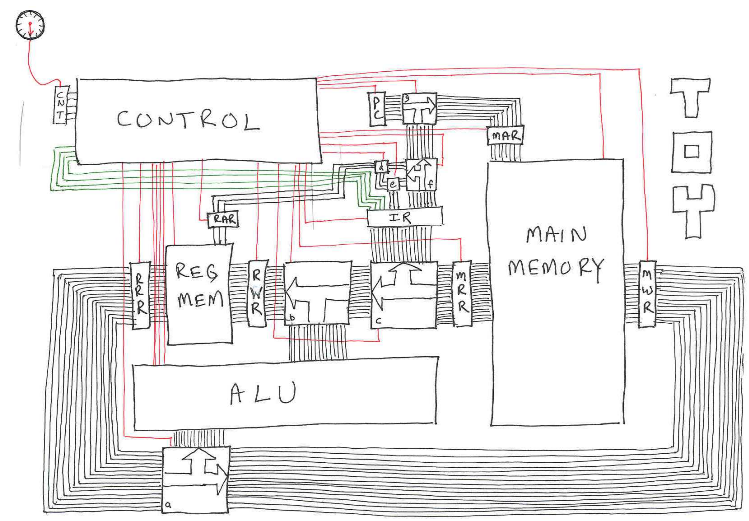
## Main Memory

TOY main memory: 256 x 16-bit register file.

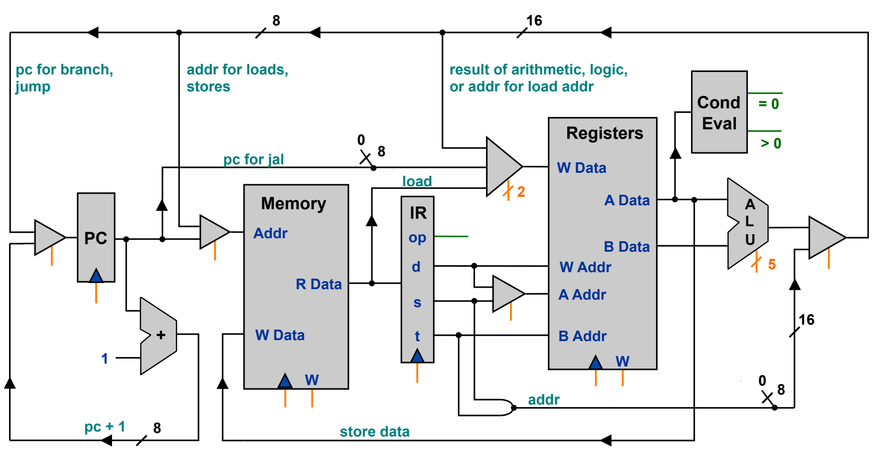





## 6.3: Machine Architecture




## The TOY Datapath

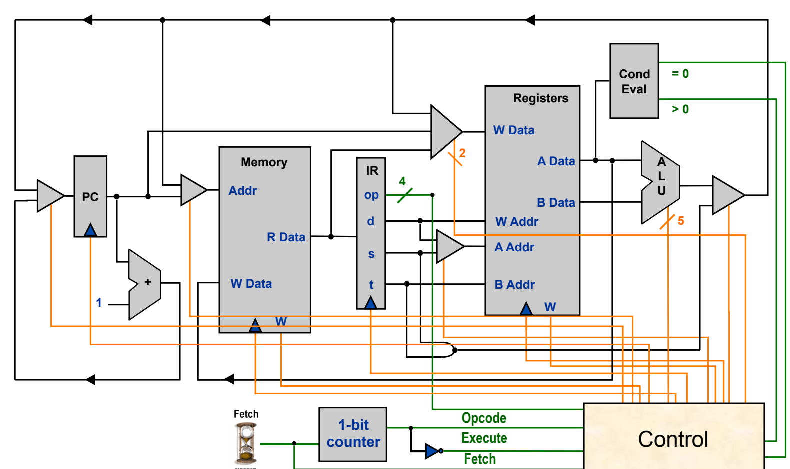





## Control

Control: controls components, enables connections.

- Input: opcode, clock, conditional evaluation. (green)
- Output: control wires. (orange)



## Summary

Combinational circuits: how to compute things

- And, Or, Not primitives sufficient for any Boolean function
- Systematic method: truth tables and sum-of-products
- Examples
  - Majority
  - Binary adder
  - Multiplexor

Sequential circuits: where to put things

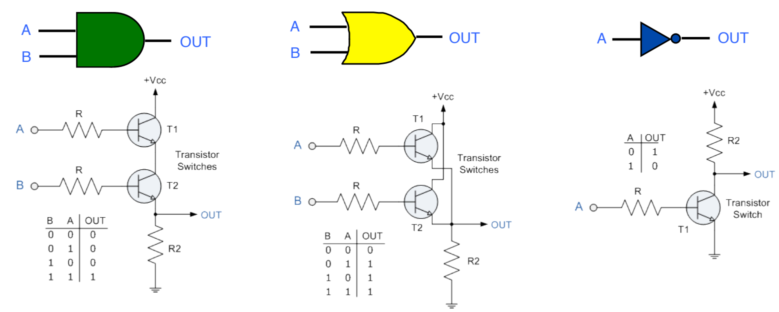
- Flip flop primitive holds one bit
- Many flip flops make a *register* (16 for TOY)
- Many registers make a *register file*
- Lots and lot of registers make a *memory* (256 for TOY)

A whole computer

- Uses *combinational circuits* to perform computations
- Uses *sequential circuits* to store results
- Uses a little of each for control



## The final secret




All three of our logic primitives can be made
using a *single** type of electronic primitive: the *transistor*!

*not counting the passive resistors