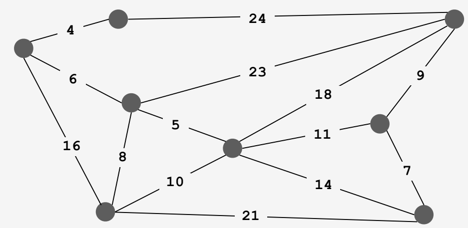
Minimum Spanning Trees

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

References:
Algorithms in Java, Chapter 20
`http://www.cs.princeton.edu/algs4/54mst`

MST Origin

Given. Undirected graph G with positive edge weights (connected).
Goal. Find a min weight set of edges that connects all of the vertices.



G

MST Origin

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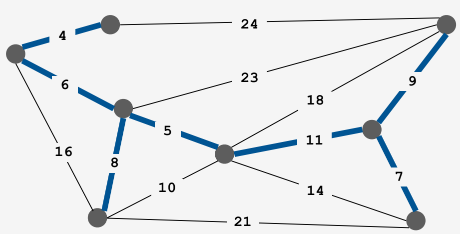
Goal. Find a min weight set of edges that connects all of the vertices.



`weight(T) = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7`

Brute force. Try all possible spanning trees.
- Problem 1: not so easy to implement.
- Problem 2: far too many of them.

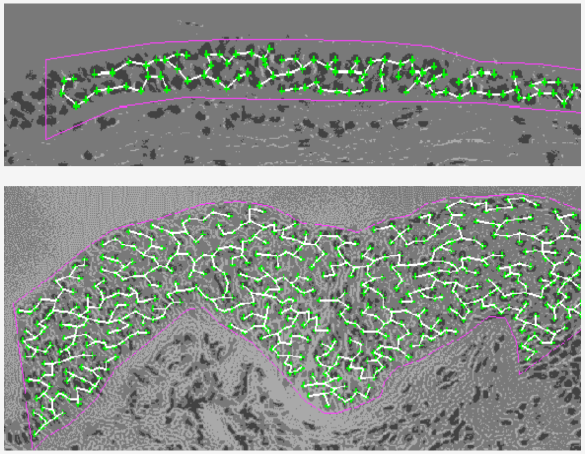
Applications

MST is fundamental problem with diverse applications.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Network design (telephone, electrical, hydraulic, cable, computer, road).
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).

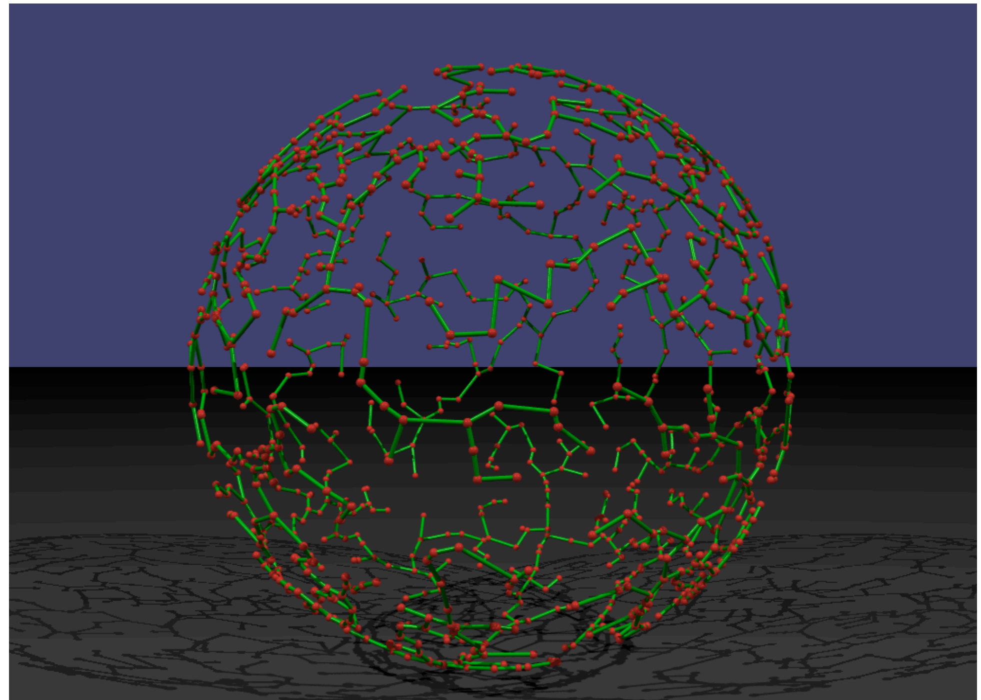
`http://www.ics.uci.edu/~eppstein/gina/mst.html`

Medical image processing

MST describes arrangement of nuclei in the epithelium for cancer research



http://www.bccrc.ca/ci/ta01_archlevel.html



http://ginger.indstate.edu/ge/gfx

Two Greedy Algorithms

Kruskal's algorithm. Consider edges in ascending order of weight.
Add to T the next edge unless doing so would create a cycle.

Prim's algorithm. Start with any vertex s and greedily grow a tree T from s.
At each step, add to T the edge of min weight that has exactly
one endpoint in T.

> *" Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit. "* — Gordon Gecko

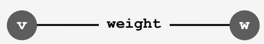
Proposition. Both greedy algorithms compute MST.

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Edge API

Edge abstraction needed for weighted edges.

public class Edge implements Comparable<Edge>		
	`Edge(int v, int w, double weight)`	*create a weighted edge v-w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`double`	`weight()`	*the weight*
`String`	`toString()`	*string representation*



Weighted graph API

public class WeightedGraph		*graph data type*
	`WeightedGraph(int V)`	*create an empty graph with V vertices*
	`WeightedGraph(In in)`	*create a graph from input stream*
`void`	`insert(Edge e)`	*add an edge from v to w*
`Iterable<Edge>`	`adj(int v)`	*return an iterator over edges incident to v*
`int`	`V()`	*return number of vertices*
`String`	`toString()`	*return a string representation*

```
for (int v = 0; v < G.V(); v++)
{
   for (Edge e : G.adj(v))
   {
      int w = e.other(v);
      // process edge v-w
   }
}
```

iterate through all edges
(once in each direction)

Weighted graph: adjacency-set implementation

```
public class WeightedGraph
{
   private final int V;
   private final SET<Edge>[] adj;

   public WeightedGraph(int V)
   {
      this.V = V;
      adj = (SET<Edge>[]) new SET[V];
      for (int v = 0; v < V; v++)
         adj[v] = new SET<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as Graph, but
adjacency sets of Edges
instead of integers

constructor

add edge to both
adjacency sets

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = Math.min(v, w);
      this.w = Math.max(v, w);
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int weight()
   {  return weight;  }

   // See next slide for compare methods.
}
```

constructor

either endpoint

other endpoint

weight of edge

Weighted edge: Java implementation (cont)

```
public static class ByWeight implements Comparator<Edge>
{
   public int compare(Edge e, Edge f)
   {
      if (e.weight < f.weight) return -1;
      if (e.weight > f.weight) return +1;
      return 0;
   }
}
```

order edges by weight
(for sorting in Kruskal)

```
public int compareTo(Edge that)
{
    if (this.v < that.v) return -1;
    if (this.v > that.v) return +1;
    if (this.w < that.w) return -1;
    if (this.w > that.w) return +1;
    return 0;
}
```

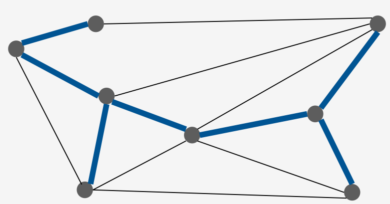
natural order
(for use in a symbol table)

- weighted graph API
- **cycles and cuts**
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Spanning tree

MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.

Def. A spanning tree of a graph G is a subgraph T that is connected and acyclic.



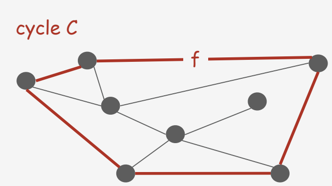
Property. MST of G is always a spanning tree.

Cycle and cut properties

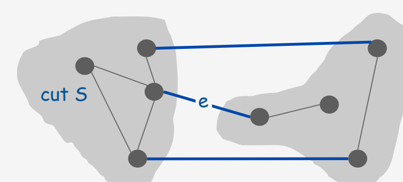
Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max weight edge belonging to C. Then the MST does not contain f.

Cut property. Let S be any subset of vertices, and let e be the min weight edge with exactly one endpoint in S. Then the MST contains e.



f is not in the MST



e is in the MST

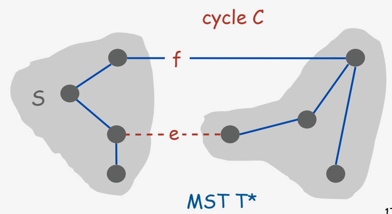
Cycle property: correctness proof

Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max weight edge belonging to C. Then the MST T^* does not contain f.

Pf. [by contradiction]

- Suppose f belongs to T^*. Let's see what happens.
- Deleting f from T^* disconnects T^*. Let S be one side of the cut.
- Some other edge in C, say e, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $w_e < w_f$, weight(T) < weight(T^*).
- Contradicts minimality of T^*. ▪



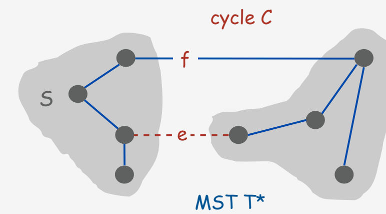
Cut property: correctness proof

Simplifying assumption. All edge weights w_e are distinct.

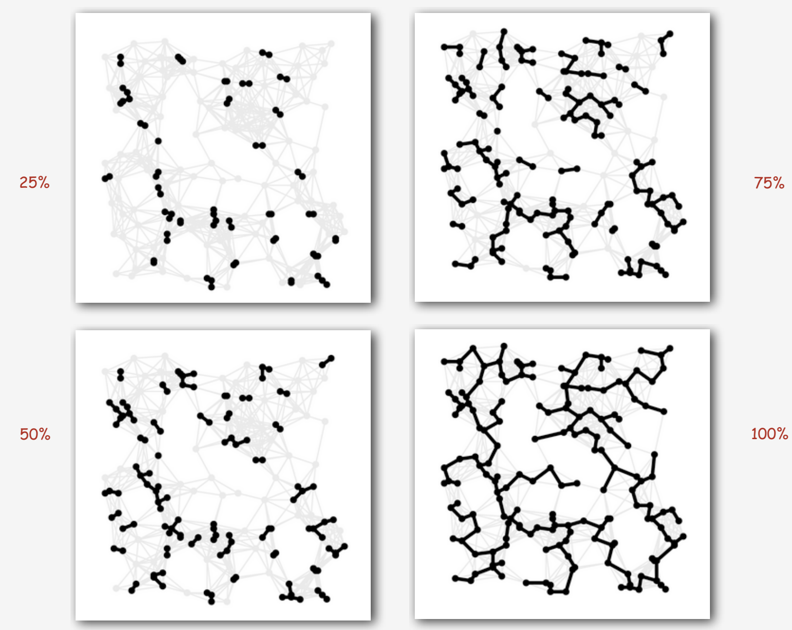
Cut property. Let S be any subset of vertices, and let e be the min weight edge with exactly one endpoint in S. Then the MST T^* contains e.

Pf. [by contradiction]

- Suppose e does not belong to T^*. Let's see what happens.
- Adding e to T^* creates a cycle C in T^*.
- Some other edge in C, say f, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $w_e < w_f$, weight(T) < weight(T^*).
- Contradicts minimality of T^*. ▪

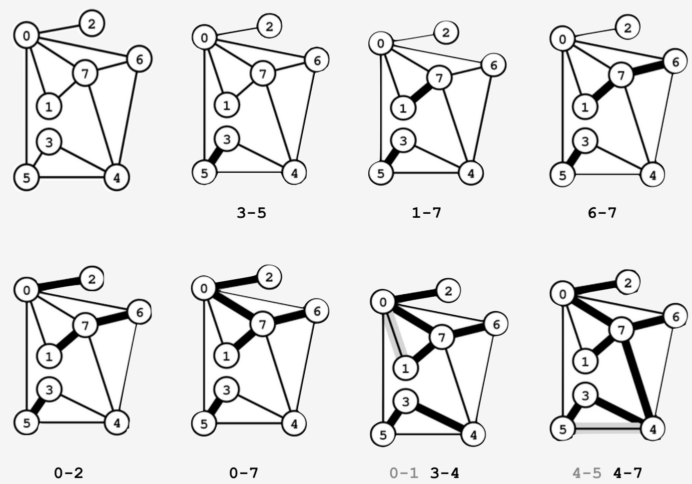


Kruskal's algorithm example



Kruskal's algorithm

Kruskal's algorithm. [Kruskal, 1956] Consider edges in ascending order of weight. Add the next edge to T unless doing so would create a cycle.



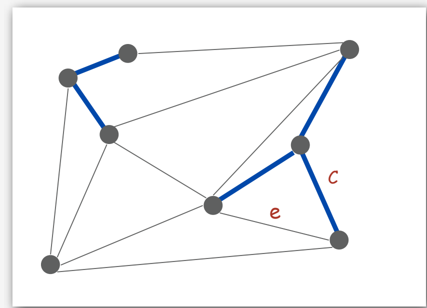
3-5	0.18
1-7	0.21
6-7	0.25
0-2	0.29
0-7	0.31
0-1	0.32
3-4	0.34
4-5	0.40
4-7	0.46
0-6	0.51
4-6	0.51
0-5	0.60

Kruskal's algorithm: correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. [Case 1] Suppose that adding e to T creates a cycle C.

- Edge e is the max weight edge in C.
- Edge e is not in the MST (cycle property).

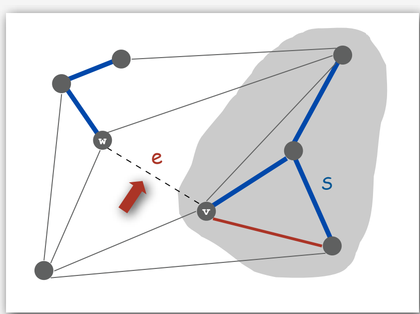


Kruskal's algorithm: correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. [Case 2] Suppose that adding e = (v, w) to T does not create a cycle.

- Let S be the vertices in v's connected component.
- Vertex w is not in S.
- Edge e is the min weight edge with exactly one endpoint in S.
- Edge e is in the MST (cut property). ▪

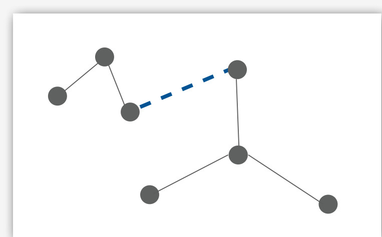


Kruskal implementation challenge

Problem. Check if adding an edge (v, w) to T creates a cycle.

How difficult?

- Intractable.
- O(E + V) time.
- O(V) time. ← run DFS from v, check if w is reachable (T has at most V-1 edges)
- O(log V) time.
- O(log* V) time. ← use the union-find data structure !
- Constant time.

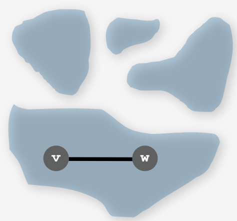


Kruskal's algorithm implementation

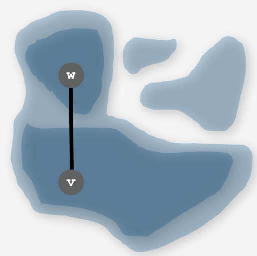
Problem. Check if adding an edge (v, w) to T creates a cycle.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T.
- If v and w are in same component, then adding v-w creates a cycle.
- To add v-w to T, merge sets containing v and w.



Case 1: adding v-w creates a cycle



Case 2: add v-w to T and merge sets

Kruskal's algorithm: Java implementation

```
public class Kruskal
{
   private SET<Edge> mst = new SET<Edge>();

   public Kruskal(WeightedGraph G)
   {
      Edge[] edges = G.edges();
      Arrays.sort(edges, new Edge.ByWeight());

      UnionFind uf = new UnionFind(G.V());
      for (Edge e : edges)
      {
         int v = e.either(), w = e.other(v);
         if (!uf.find(v ,w))
         {
            uf.unite(v, w);
            mst.add(edge);
         }
      }
   }

   public Iterable<Edge> mst()
   {  return mst;  }

}
```

get all edges in graph

sort edges by weight

greedily add edges to MST

Kruskal's algorithm running time

Proposition. Kruskal's algorithm computes MST in O(E log V) time.

Pf.

operation	frequency	time per op
sort	1	E log V
union	V	log* V †
find	E	log* V †

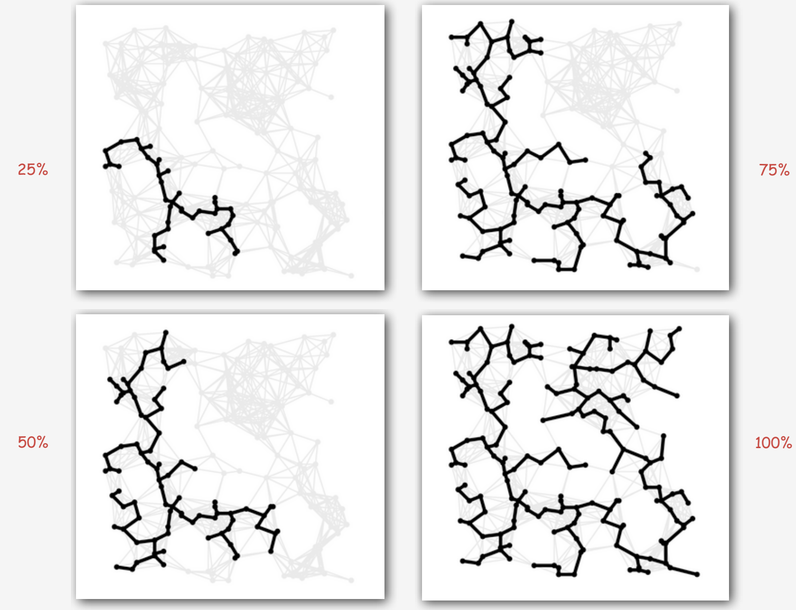
† amortized bound using weighted quick union with path compression

Remark. If edges are already sorted, time is proportional to E log* V.

recall: log* V ≤ 5 in this universe

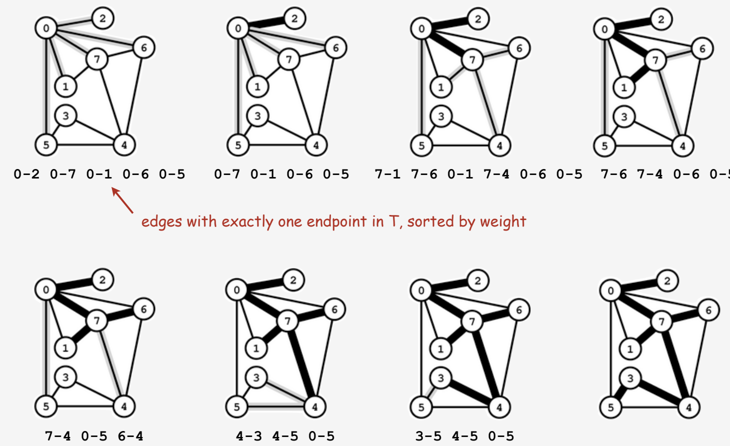
- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- **Prim's algorithm**
- advanced topics

Prim's algorithm example



Prim's algorithm example

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]
Start with vertex 0 and greedily grow tree T. At each step, add edge of min weight that has exactly one endpoint in T.



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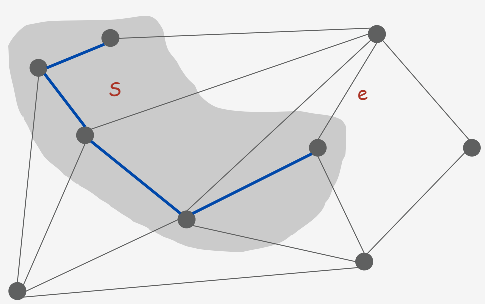
4-7	0.46
6-7	0.25

Prim's algorithm correctness proof

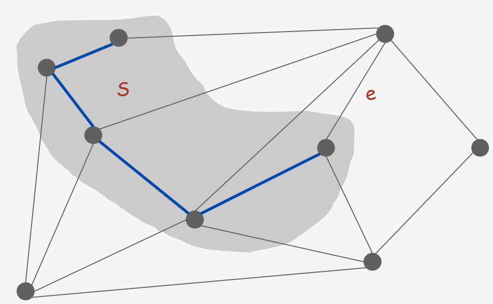
Proposition. Prim's algorithm computes the MST.
Pf.
- Let S be the subset of vertices in current tree T.
- Prim adds the min weight edge e with exactly one endpoint in S.
- Edge e is in the MST (cut property). ▪



Prim implementation challenge

Problem. Find min weight edge with exactly one endpoint in S.

How difficult?
- Intractable.
- O(E) time. ← try all edges
- O(V) time.
- O(log V) time. ← use a priority queue !
- O(log* V) time.
- Constant time.



Prim's algorithm implementation

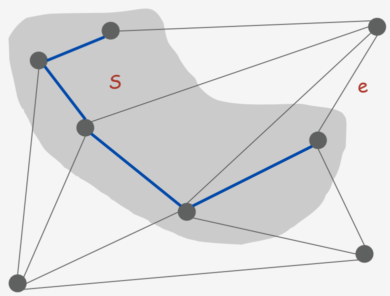
Problem. Find min weight edge with exactly one endpoint in S.

Efficient solution. Maintain a PQ of vertices connected by an edge to S.

- Delete min to determine next vertex v to add to S.
- Disregard v if already in S.
- Add to PQ any vertex brought closer to S by v.

Running time.

- log E steps per edge.
- E log E steps overall.



Key-value priority queue

Associate a value with each key in a priority queue.

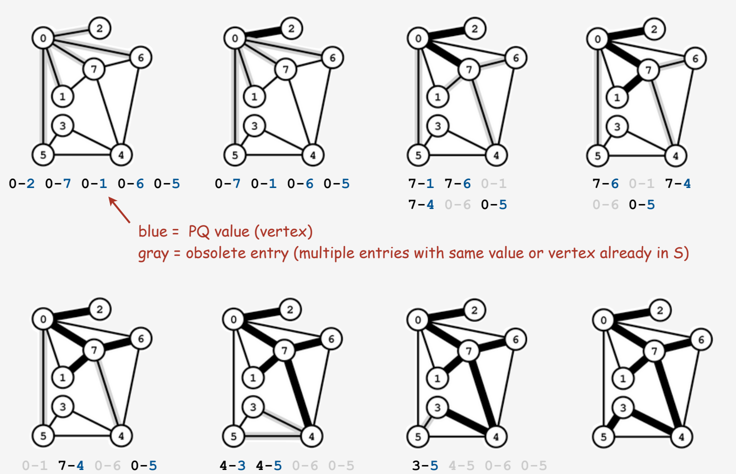
`public class MinPQplus<Key extends Comparable<Key>, Value>`	
`MinPQplus()`	*create key-value priority queue*
`void put(Key key, Value val)`	*put key-value pair into the PQ*
`Value delMin()`	*return value paired with minimal key and delete it*
`boolean isEmpty()`	*is the PQ empty?*

Implementation.

- Start with same code as standard heap-based PQ.
- Use a parallel array `vals[]` (value associated with `keys[i]` is `vals[i]`).
- Modify `exch()` to maintain parallel arrays (do exch in `vals[]`).
- Modify `delMin()` to return `Value`.

Prim's algorithm example: lazy implementation

Use PQ: key = edge weight, value = vertex.
(lazy version leaves some obsolete entries on the PQ)



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Lazy implementation of Prim's algorithm

```
public class LazyPrim
{
   private boolean[] marked;  // vertices in MST
   private double[] dist;     // distance to MST
   private Edge[] pred;       // pred[v] is edge attach v to MST

   public LazyPrim(WeightedGraph G)
   {
      marked = new boolean[G.V()];
      pred   = new Edge[G.V()];
      dist   = new double[G.V()];
      for (int v = 0; v < G.V(); v++)
         dist[v] = Double.POSITIVE_INFINITY;
      prim(G, 0);
   }

   // See next slide for prim() implementation.
}
```

Lazy implementation of Prim's algorithm

```
private void prim(WeightedGraph G, int s)
{
   dist[s] = 0.0;
   marked[s] = true;

   MinPQplus<Double, Integer> pq;
   pq = new MinPQplus<Double, Integer>();
   pq.put(dist[s], s);

   while (!pq.isEmpty())
   {
      int v = pq.delMin();
      if (marked[v]) continue;
      marked[v] = true;
      for (Edge e : G.adj(v))
      {
         int w = e.other(v);
         if (!marked[w] && (dist[w] > e.weight()))
         {
            dist[w] = e.weight();
            pred[w] = e;
            pq.insert(dist[w], w);
         }
      }
   }
}
```

key-value PQ

ignore if already in MST

add to PQ any vertices brought closer to S by v

Prim's algorithm running time

Proposition. Prim's algorithm computes MST in O(E log V) time.

operation	frequency	time per op
delmin	V	V log V
insert	E	E log V

Priority queue with decrease-key

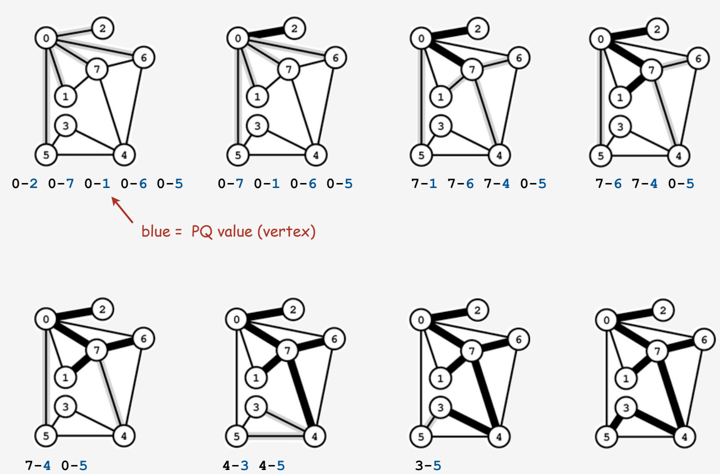
Indexed priority queue.

`public class MinIndexPQ<Key extends Comparable<Key>, Integer>`	
`MinIndexPQ()`	*create key-value indexed priority queue*
`void put(Key key, int v)`	*put key-value pair into the PQ*
`int delMin()`	*return value paired with minimal key and delete it*
`boolean isEmpty()`	*is the PQ empty?*
`boolean contains(int v)`	*is there a key associated with value v?*
`void decreaseKey(Key key, int v)`	*decrease the key associated with v to key*

Implementation. More complicated than `MinPQ`, see text.

Prim's algorithm example: eager implementation

Use IndexMinPQ: key = edge weight, value = vertex.
(eager version has at most one PQ entry per vertex)



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6-7	0.25

Eager implementation of Prim's algorithm

Main benefit. Reduces PQ size guarantee from E to V.

- Not important for the huge sparse graphs found in practice.
- PQ size is far smaller in practice.
- Widely used, but practical utility is debatable.

Removing the distinct edge weight assumption

Simplifying assumption. All edge weights we are distinct.

Approach 1. Introduce tie-breaking rule for `compare()`.

```
public int compare(Edge e, Edge f)
{
   if (e.weight < f.weight) return -1;
   if (e.weight > f.weight) return +1;
   if (e.v < f.v) return -1;
   if (e.v > f.v) return +1;
   if (e.w < f.w) return -1;
   if (e.w > f.w) return +1;
   return 0;
}
```

`return e.compareTo(f);`

Approach 2. Prim and Kruskal still find MST if equal weights!
(only our proof of correctness fails)

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- **advanced topics**

Does a linear-time MST algorithm exist?

deterministic compare-based MST algorithms

year	worst case	discovered by
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

Remark. Linear-time randomized MST algorithm (Karger-Klein-Tarjan 1995).