Mergesort

- mergesort
- sorting complexity
- comparators

Reference:
Algorithms in Java. 4th Edition, Section 3.2
`http://www.cs.princeton.edu/algs4`

Two classic sorting algorithms

Critical components in the world's computational infrastructure.
- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. ← today
- Java sort for objects.
- Perl, Python stable sort.

Quicksort. ← next lecture
- Java sort for primitive types.
- C qsort, Unix, g++, Visual C++, Python.

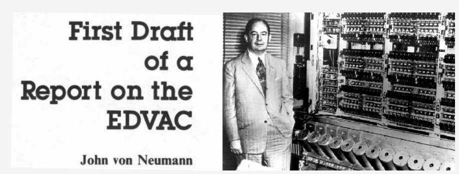
- mergesort
- bottom-up mergesort
- sorting complexity
- comparators

Mergesort

Basic plan.
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview



Mergesort trace

	lo		hi	a[] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
				M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	0,	0,	1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	2,	2,	3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	0,	1,	3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	4,	4,	5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	6,	6,	7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a,	4,	5,	7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a,	0,	3,	7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a,	8,	8,	9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a,	10,	10,	11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a,	8,	9,	11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a,	12,	12,	13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a,	14,	14,	15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a,	12,	13,	15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a,	8,	11,	15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a,	0,	7,	15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Trace of merge results for top-down mergesort

result after recursive call

Merging

Goal. Combine two sorted subarrays into a sorted whole.

Q. How to merge efficiently?

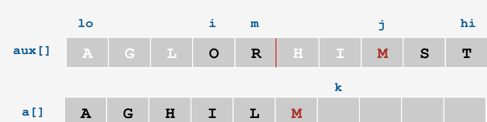
A. Use an auxiliary array.

	k	a[] 0	1	2	3	4	5	6	7	8	9	i	j	aux[] 0	1	2	3	4	5	6	7	8	9
input		E	E	G	M	R	A	C	E	R	T			-	-	-	-	-	-	-	-	-	-
copy		E	E	G	M	R	A	C	E	R	T			E	E	G	M	R	A	C	E	R	T
												0	5										
	0	A										0	6	E	E	G	M	R	A	C	E	R	T
	1	A	C									0	7	E	E	G	M	R		C	E	R	T
	2	A	C	E								1	7	E	E	G	M	R			E	R	T
	3	A	C	E	E							2	7		E	G	M	R			E	R	T
	4	A	C	E	E	E						2	8			G	M	R			E	R	T
	5	A	C	E	E	E	G					3	8			G	M	R				R	T
	6	A	C	E	E	E	G	M				4	8				M	R				R	T
	7	A	C	E	E	E	G	M	R			5	8					R				R	T
	8	A	C	E	E	E	G	M	R	R		5	9									R	T
	9	A	C	E	E	E	G	M	R	R	T	6	10										T
merged result		A	C	E	E	E	G	M	R	R	T												

Abstract in-place merge trace

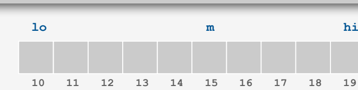
Merging: Java implementation

```
public static void merge(Comparable[] a, int lo, int mid, int hi)
{  // Merge a[lo..m] with a[m+1..hi].

   for (int k = lo; k < hi; k++)                                  copy
      aux[k] = a[k];

   int i = lo, j = mid;
   for (int k = lo; k < hi; k++)
      if      (i == mid)                a[k] = aux[j++];
      else if (j == hi )                a[k] = aux[i++];          merge
      else if (less(aux[j], aux[i]))   a[k] = aux[j++];
      else                              a[k] = aux[i++];

}
```

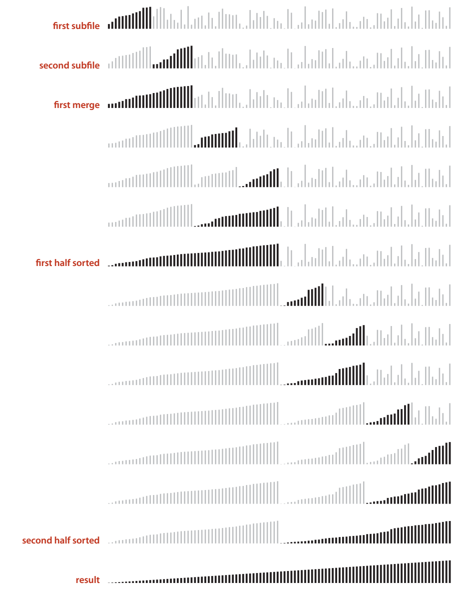


Mergesort: Java implementation

```
public class Merge
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, lo, mid);
      sort(a, mid+1, hi);
      merge(a, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, 0, a.length - 1);
   }
}
```

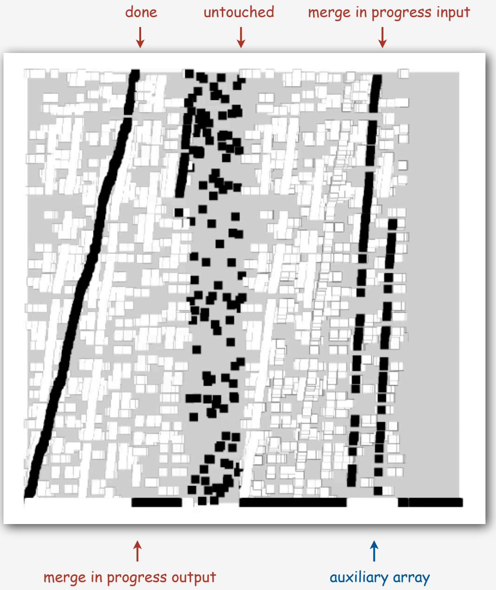


Mergesort visualization



Visual trace of top-down mergesort with cutoff for small subfiles

Mergesort animation



Mergesort animation

Mergesort: empirical analysis

Running time estimates:

- Home pc executes 10^8 comparisons/second.
- Supercomputer executes 10^{12} comparisons/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort: mathematical analysis

Proposition. Mergesort uses $\sim N \lg N$ compares to sort any array of size N.

Def. $T(N)$ = number of compares to mergesort an array of size N.

$$= \underbrace{T(N/2)}_{\text{left half}} + \underbrace{T(N/2)}_{\text{right half}} + \underbrace{N}_{\text{merge}}$$

Mergesort recurrence. $T(N) = 2\,T(N/2) + N$ for $N > 1$, with $T(1) = 0$.

- Not quite right for odd N.
- Same recurrence holds for many divide-and-conquer algorithms.

Solution. $T(N) \sim N \lg N$.

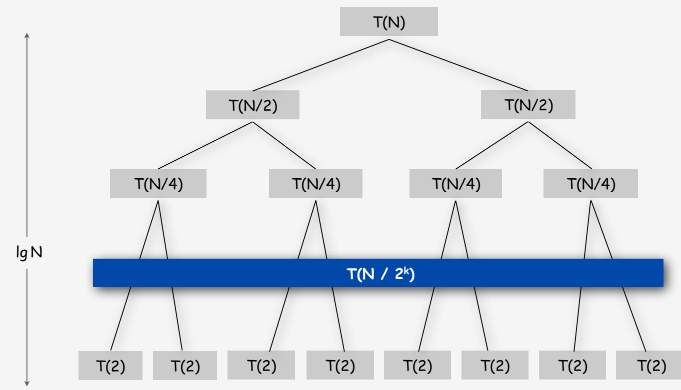
- For simplicity, we'll prove when N is a power of 2.
- True for all N. [see COS 340]

Mergesort recurrence: proof 1

Mergesort recurrence. $T(N) = 2\,T(N/2) + N$ for $N > 1$, with $T(1) = 0$.

Proposition. If N is a power of 2, then $T(N) = N \lg N$.

Pf.



N = N

2 (N/2) = N

4 (N/4) = N

. . .

$2^k (N/2^k)$ = N

. . .

N/2 (2) = N

N lg N

Mergesort recurrence: proof 2

Mergesort recurrence. $T(N) = 2\,T(N/2) + N$ for $N > 1$, with $T(1) = 0$.

Proposition. If N is a power of 2, then $T(N) = N \lg N$.

Pf.

$T(N) = 2\,T(N/2) + N$	given
$T(N)/N = 2\,T(N/2)/N + 1$	divide both sides by N
$= T(N/2)/(N/2) + 1$	algebra
$= T(N/4)/(N/4) + 1 + 1$	apply to first term
$= T(N/8)/(N/8) + 1 + 1 + 1$	apply to first term again
. . .	
$= T(N/N)/(N/N) + 1 + 1 + \ldots + 1$	stop applying, T(1) = 0
$= \lg N$	

Mergesort recurrence: proof 3

Mergesort recurrence. $T(N) = 2\,T(N/2) + N$ for $N > 1$, with $T(1) = 0$.

Proposition. If N is a power of 2, then $T(N) = N \lg N$.

Pf. [by induction on N]

- Base case: $N = 1$.
- Inductive hypothesis: $T(N) = N \lg N$.
- Goal: show that $T(2N) = 2N \lg (2N)$.

$T(2N) = 2\,T(N) + 2N$	given
$= 2 N \lg N + 2 N$	inductive hypothesis
$= 2 N (\lg (2N) - 1) + 2N$	algebra
$= 2 N \lg (2N)$	QED

Mergesort analysis: memory

Proposition G. Mergesort uses extra space proportional to N.

Pf. The array `aux[]` needs to be of size N for the last merge.

two sorted subarrays

A	C	D	G	H	I	M	N	U	V	B	E	F	J	O	P	Q	R	S	T
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

merged result

Def. A sorting algorithm is in-place if it uses O(log N) extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge for the bored. In-place merge. [Kronrud, 1969]

Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 elements.

Stop if already sorted.

- Is biggest element in first half ≤ smallest element in second half?
- Helps for nearly ordered lists.

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

Ex. See `Arrays.sort()`.

- mergesort
- **bottom-up mergesort**
- sorting complexity
- comparators

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ...

lo, m, hi	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Trace of merge results for bottom-up mergesort

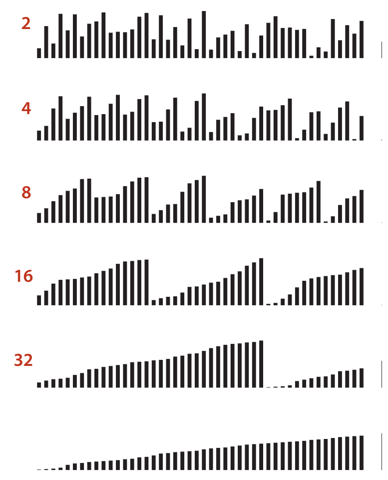
Bottom line. No recursion needed!

Bottom-up mergesort: Java implementation

```
public class MergeBU
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int m, int hi)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      aux = new Comparable[N];
      for (int m = 1; m < N; m = m+m)
         for (int i = 0; i < N-m; i += m+m)
            merge(a, i, i+m, Math.min(i+m+m, N));
   }
}
```

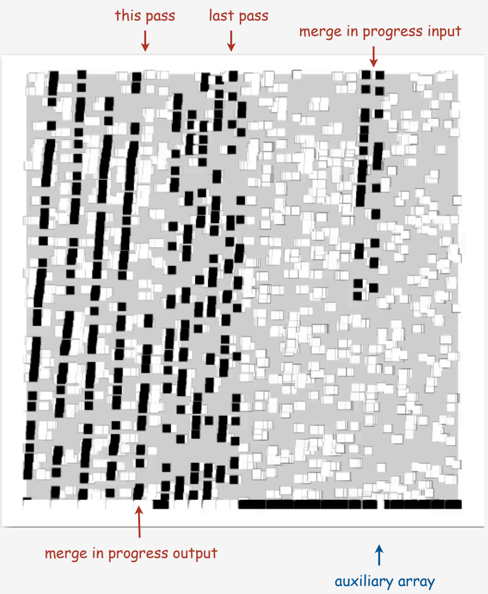
Bottom line. Concise industrial-strength code, if you have the space.

Bottom-up mergesort: visual trace

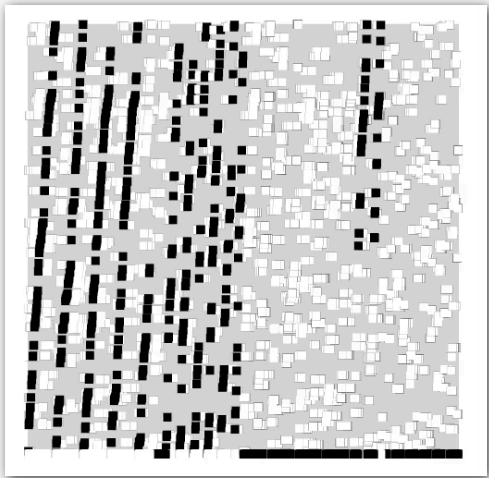


Visual trace of bottom-up mergesort

Botom-up mergesort animation



Botom-up mergesort animation



- mergesort
- bottom-up mergesort
- **sorting complexity**
- comparators

Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X.

Machine model. Focus on fundamental operations.
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best cost guarantee for X.

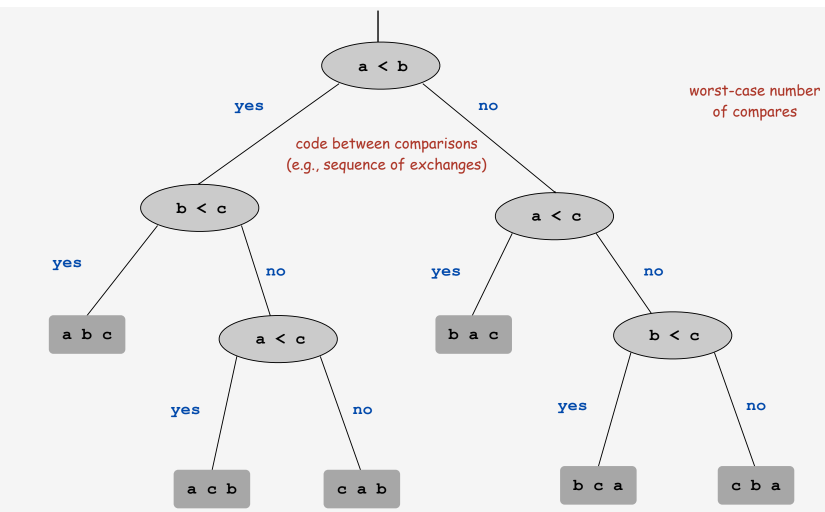
lower bound ~ upper bound

Example: sorting.

access information only through compares

- Machine model = # compares.
- Upper bound = ~ N lg N from mergesort.
- Lower bound = ~ N lg N ?
- Optimal algorithm = mergesort ?

Decision tree (for 3 distinct elements)

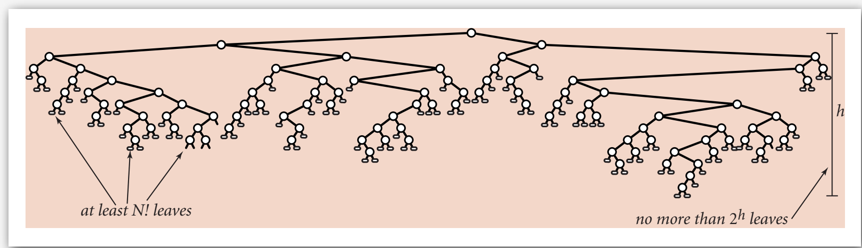


Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use more than $N \lg N - 1.44\,N$ comparisons in the worst-case.

Pf.

- Assume input consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N\,!$ different orderings $\Rightarrow$ at least $N\,!$ leaves.



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- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N\,!$ different orderings $\Rightarrow$ at least $N\,!$ leaves.

$$
\begin{aligned}
2^h &\geq N\,! \\
h &\geq \lg N\,! \\
&\geq \lg (N / e)^N \quad \text{Stirling's formula} \\
&= N \lg N - N \lg e \\
&\geq N \lg N - 1.44\, N
\end{aligned}
$$

Complexity of sorting

Machine model. Focus on fundamental operations.
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best cost guarantee for X.

Example: sorting.

- Machine model = # compares.
- Upper bound = ~ N lg N from mergesort.
- Lower bound = ~ N lg N.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Complexity results in context

Other operations? Mergesort optimality is only about number of compares.

Space?

- Mergesort is not optimal with respect to space usage.
- Insertion sort, selection sort, and shellsort are space-optimal.

Challenge. Find an algorithm that is both time- and space-optimal.

Lessons. Use theory as a guide.
Ex. Don't try to design sorting algorithm that uses $\frac{1}{2} N \lg N$ compares.

Complexity results in context (continued)

Lower bound may not hold if the algorithm has information about:

- The key values.
- Their initial arrangement.

Partially ordered arrays. Depending on the initial order of the input, we may not need N lg N compares.

insertion sort requires O(N) compares on an already sorted array

Duplicate keys. Depending on the input distribution of duplicates, we may not need N lg N compares.

stay tuned for 3-way quicksort

Digital properties of keys. We can use digit/character compares instead of key compares for numbers and strings.

stay tuned for radix sorts

- mergesort
- bottom-up mergesort
- sorting complexity
- **comparators**

Natural order

Comparable interface: sort uses type's natural order.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }
   …
   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

Generalized compare

Comparable interface: sort uses type's natural order.

Problem 1. May want to use a non-natural order.

Problem 2. Desired data type may not come with a "natural" order.

Ex. Sort strings by:

- Natural order. `Now is the time`
- Case insensitive. `is Now the time`
- Spanish. `café cafetero cuarto churro nube ñoño`
- British phone book. `McKinley Mackintosh`

pre-1994 order for digraphs ch and ll and rr

```
String[] a;
...
Arrays.sort(a);
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
Arrays.sort(a, Collator.getInstance(Locale.SPANISH));
```

`import java.text.Collator;`

Comparators

Solution. Use Java's `Comparator` interface.

```
public interface Comparator<Key>
{
   public int compare(Key v, Key w);
}
```

Remark. The `compare()` method implements a total order like `compareTo()`.

Advantages. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

- Can add any number of new orders to a data type.
- Can add an order to a library data type with no natural order.

Comparator example

Reverse order. Sort an array of strings in reverse order.

```
public class ReverseOrder implements Comparator<String>
{
   public int compare(String a, String b)
   {
      return b.compareTo(a);
   }
}
```

comparator implementation

```
   ...
   Arrays.sort(a, new ReverseOrder());
   ...
```

client

Sort implementation with comparators

To support comparators in our sort implementations:

- Pass `Comparator` to `sort()` and `less()`.
- Use it in `less()`.

Ex. Insertion sort.

pedantic Java code (see book for simpler version)

```
public static <Key> void sort(Key[] a, Comparator<Key> comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0; j--)
         if (less(comparator, a[j], a[j-1]))
            exch(a, j, j-1);
         else break;
}

private static <Key> boolean less(Comparator<Key> c, Key v, Key w)
{  return c.compare(v, w) < 0;   }

private static <Key> void exch(Key[] a, int i, int j)
{  Key swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```


Generalized compare

Comparators enable multiple sorts of a single file (by different keys).

Ex. Sort students by name or by section.

```
Arrays.sort(students, Student.BY_NAME);
Arrays.sort(students, Student.BY_SECT);
```

sort by name

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	2	A	991-878-4944	308 Blair
Fox	1	A	884-232-5341	11 Dickinson
Furia	3	A	766-093-9873	101 Brown
Gazsi	4	B	665-303-0266	22 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes

then sort by section

Fox	1	A	884-232-5341	11 Dickinson
Chen	2	A	991-878-4944	308 Blair
Andrews	3	A	664-480-0023	097 Little
Furia	3	A	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	665-303-0266	22 Brown

Generalized compare

Ex. Enable sorting students by name or by section.

```
public class Student
{
   public static final Comparator<Student> BY_NAME = new ByName();
   public static final Comparator<Student> BY_SECT = new BySect();

   private final String name;
   private final int section;
   ...
   private static class ByName implements Comparator<Student>
   {
      public int compare(Student a, Student b)
      {  return a.name.compareTo(b.name);  }
   }

   private static class BySect implements Comparator<Student>
   {
      public int compare(Student a, Student b)
      {  return a.section - b.section;  }
   }
}
```

only use this trick if no danger of overflow

Generalized compare problem

A typical application. First, sort by name; then sort by section.

`Arrays.sort(students, Student.BY_NAME);`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	2	A	991-878-4944	308 Blair
Fox	1	A	884-232-5341	11 Dickinson
Furia	3	A	766-093-9873	101 Brown
Gazsi	4	B	665-303-0266	22 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes

`Arrays.sort(students, Student.BY_SECT);`

Fox	1	A	884-232-5341	11 Dickinson
Chen	2	A	991-878-4944	308 Blair
Kanaga	3	B	898-122-9643	22 Brown
Andrews	3	A	664-480-0023	097 Little
Furia	3	A	766-093-9873	101 Brown
Rohde	3	A	232-343-5555	343 Forbes
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	665-303-0266	22 Brown

@#%&@!!. Students in section 3 no longer in order by name.

A stable sort preserves the relative order of records with equal keys.

Sorting challenge 5

Which sorts are stable ?

- Insertion sort ?
- Selection sort ?
- Shellsort ?
- Mergesort ?

sorted by time		sorted by location (not stable)		sorted by location (stable)	
Chicago	09:00:00	Chicago	09:25:52	Chicago	09:00:00
Phoenix	09:00:03	Chicago	09:03:13	Chicago	09:00:59
Houston	09:00:13	Chicago	09:21:05	Chicago	09:03:13
Chicago	09:00:59	Chicago	09:19:46	Chicago	09:19:32
Houston	09:01:10	Chicago	09:19:32	Chicago	09:19:46
Chicago	09:03:13	Chicago	09:00:00	Chicago	09:21:05
Seattle	09:10:11	Chicago	09:35:21	Chicago	09:25:52
Seattle	09:10:25	Chicago	09:00:59	Chicago	09:35:21
Phoenix	09:14:25	Houston	09:01:10	Houston	09:00:13
Chicago	09:19:32	Houston	09:00:13	Houston	09:01:10
Chicago	09:19:46	Phoenix	09:37:44	Phoenix	09:00:03
Chicago	09:21:05	Phoenix	09:00:03	Phoenix	09:14:25
Seattle	09:22:43	Phoenix	09:14:25	Phoenix	09:37:44
Seattle	09:22:54	Seattle	09:10:25	Seattle	09:10:11
Chicago	09:25:52	Seattle	09:36:14	Seattle	09:10:25
Chicago	09:35:21	Seattle	09:22:43	Seattle	09:22:43
Seattle	09:36:14	Seattle	09:10:11	Seattle	09:22:54
Phoenix	09:37:44	Seattle	09:22:54	Seattle	09:36:14

no longer sorted by time

still sorted by time

Stability when sorting on a second key

Sorting challenge 5A

Is insertion sort stable ?

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;
   }
}
```

i	j	a[] 0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
1	0	O	S	R	T	E	X	A	M	P	L	E
2	1	O	R	S	T	E	X	A	M	P	L	E
3	3	O	R	S	T	E	X	A	M	P	L	E
4	0	E	O	R	S	T	X	A	M	P	L	E
5	5	E	O	R	S	T	X	A	M	P	L	E
6	0	A	E	O	R	S	T	X	M	P	L	E
7	2	A	E	M	O	R	S	T	X	P	L	E
8	4	A	E	M	O	P	R	S	T	X	L	E
9	2	A	E	L	M	O	P	R	S	T	X	E
10	2	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Sorting challenge 5B

Is selection sort stable ?

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

i	min	a[] 0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Sorting challenge 5C

Is shellsort stable ?

```
public class Shell
{  // Shellsort.
   public static void sort(Comparable[] a)
   {  // Sort a[] into increasing order.
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1; // 1, 4, 13, 40, 121, 364, 1093, ...
      while (h >= 1)
      {  // h-sort the file.
         for (int i = h; i < N; i++)
         {  // Insert a[i] among a[i-h], a[i-2*h], a[i-3*h]... .
            for (int j = i; j > 0 && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```


Sorting challenge 5D

Is mergesort stable ?

```
public class Merge
{
   private static Comparable[] aux;
   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, lo, mid);
      sort(a, mid+1, hi);
      merge(a, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, 0, a.length - 1);
   }
}
```


Sorting challenge 5D (continued)

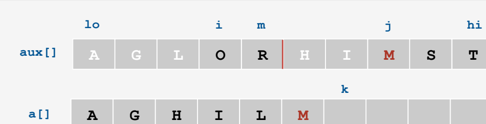
Is merge stable ?

```
public static void merge(Comparable[] a, int lo, int mid, int hi)
{  // Merge a[lo..m] with a[m+1..hi].

   for (int k = lo; k < hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid;
   for (int k = lo; k < hi; k++)
      if       (i == mid)                 a[k] = aux[j++];
      else if  (j == hi )                 a[k] = aux[i++];
      else if  (less(aux[j], aux[i]))     a[k] = aux[j++];
      else                                a[k] = aux[i++];

}
```


Postscript: Optimizing mergesort (a short history)

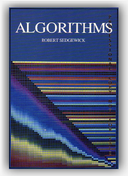
Goal: remove instructions from the inner loop.

```
public static void merge(Comparable[] a, int lo, int mid, int hi)
{  // Merge a[lo..m] with a[m+1..hi].

   for (int k = lo; k < hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid;
   for (int k = lo; k < hi; k++)
      if       (i == mid)                 a[k] = aux[j++];
      else if  (j == hi )                 a[k] = aux[i++];
      else if  (less(aux[j], aux[i]))     a[k] = aux[j++];
      else                                a[k] = aux[i++];

}
```



Postscript: Optimizing mergesort (a short history)

Idea 1 (1960s): Use sentinels




```
a[M] := maxint; b[N] := maxint;
for (int i = 0, j = 0, k = 0; k < M+1; k++)
   if (less(aux[j], aux[i])) aux[k] = a[i++];
                             aux[k] = b[j++];
```

	0			i		M
a[]	A	G	L	O	R	∞

				j		N
b[]	H	I	M	S	T	∞

							k			
aux[]	A	G	H	I	L	M				

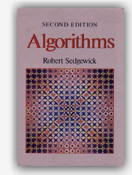
Problem 1: Still need copy

Problem 2: No good place to put sentinels

Problem 3: Complicates data-type interface (what is infinity for your type?)

Postscript: Optimizing mergesort (a short history)

Idea 2 (1980s): Reverse copy




```
public static void merge(Comparable[] a, int lo, int m, int hi)
{
   for (int i = lo; i <= m; i++)                              copy
      aux[i] = a[i];

   for (int j = m+1; j <= hi; j++)                            reverse copy
      aux[j] = a[hi-j+m+1];

   int i = lo, j = hi;
   for (int k = lo; k <= hi; k++)                             merge
      if (less(aux[j], aux[i])) a[k] = aux[j--];
      else                      a[k] = aux[i++];
}
```

	lo			i	m	j			hi
aux[]	A	G	L	O	R	T	S	M	I	H

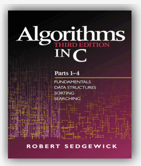
						k				
a[]	A	G	H	I	L	M				

Problem: Copy still in inner loop.

Postscript: Optimizing mergesort (a short history)

Idea 3 (1990s): Eliminate copy with recursive argument switch

```
int m = (r+1)/2;
mergesortABr(b, a, l, m);
mergesortABr(b, a, m+1, r);
mergeAB(a, l, b, l, m, b, m+1, r);
```

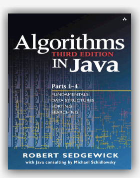


Problem: Complex interactions with reverse copy

Solution: Go back to sentinels.

Java

Arrays.sort()



Sorting challenge 6

Problem: Choose mergesort for Algs 4th edition.

Recursive argument switch is out (recommended only for pros)

Q. Why not use reverse array copy?

```
public static void merge(Comparable[] a, int lo, int m, int hi)
{
   for (int i = lo; i <= m; i++)
      aux[i] = a[i];

   for (int j = m+1; j <= hi; j++)
      aux[j] = a[hi-j+m+1];

   int i = lo, j = hi;
   for (int k = lo; k <= hi; k++)
      if (less(aux[j], aux[i])) a[k] = aux[j--];
      else                      a[k] = aux[i++];
}
```