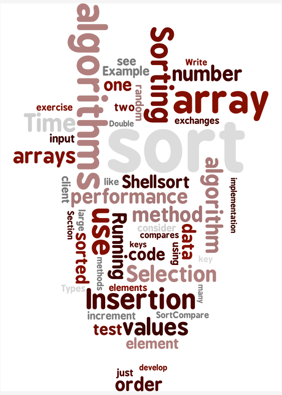
# Elementary Sorts




- rules of the game
- selection sort
- insertion sort
- sorting challenges
- shellsort

*Reference: Algorithms in Java, 4th edition, Section 3.1*



## Sorting problem

Ex. Student record in a University.

file →

| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |

record → (Andrews row)

key → (Aaron)

Sort. Rearrange array of N objects into ascending order.

| | | | | |
|---|---|---|---|---|
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
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| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |



## Sample sort client

Goal. Sort any type of data.

Ex 1. Sort random numbers in ascending order.

```
public class Experiment
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Double[] a = new Double[N];
      for (int i = 0; i < N; i++)
         a[i] = StdRandom.uniform();
      Insertion.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

```
% java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.363292849257276
0.460954145685913
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```



## Sample sort client

Goal. Sort any type of data.

Ex 2. Sort strings from standard input in alphabetical order.

```
public class StringSort
{
   public static void main(String[] args)
   {
      String[] a = StdIn.readAll().split("\\s+");
      Insertion.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

```
% more words3.txt
bed bug dad dot zoo ... all bad bin

% java StringSort < words.txt
all bad bed bug dad ... yes yet zoo
```

## Sample sort client

Goal. Sort any type of data.
Ex 3. Sort the files in a given directory by filename.

```
import java.io.File;
public class FileSort
{
   public static void main(String[] args)
   {
      File directory = new File(args[0]);
      File[] files = directory.listFiles();
      Insertion.sort(files);
      for (int i = 0; i < files.length; i++)
         StdOut.println(files[i]);
   }
}
```

```
% java FileSort .
Insertion.class
Insertion.java
InsertionX.class
InsertionX.java
Selection.class
Selection.java
Shell.class
Shell.java
ShellX.class
ShellX.java
```



## Callbacks

Goal. Sort any type of data.

Q. How can sort know to compare data of type `String`, `Double`, and `File` without any information about the type of an item?

Callbacks.
- Client passes array of objects to sorting routine.
- Sorting routine calls back object's compare function as needed.

Implementing callbacks.
- Java: interfaces.
- C: function pointers.
- C++: class-type functors.
- ML: first-class functions and functors.



## Callbacks: roadmap

client

```
import java.io.File;
public class FileSort
{
   public static void main(String[] args)
   {
      File directory = new File(args[0]);
      File[] files = directory.listFiles();
      Insertion.sort(files);
      for (int i = 0; i < files.length; i++)
         StdOut.println(files[i]);
   }
}
```

object implementation

```
public class File
implements Comparable<File>
{
   ...
   public int compareTo(File b)
   {
      ...
      return -1;
      ...
      return +1;
      ...
      return 0;
   }
}
```

interface

built in to Java

```
public interface Comparable<Item>
{
   public int compareTo(Item);
}
```

sort implementation

```
public static void sort(Comparable[] a)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0; j--)
         if (a[j].compareTo(a[j-1]) < 0)
              exch(a, j, j-1);
         else break;
}
```

Key point: no reference to `File`



## Comparable interface API

Comparable interface. Implement `compareTo()` so that `v.compareTo(w)`:
- Returns a negative integer if `v` is less than `w`.
- Returns a positive integer if `v` is greater than `w`.
- Returns zero if `v` is equal to `w`.

```
public interface Comparable<Item>
{
   public int compareTo(Item that);
}
```

Total order. Implementation must ensure a total order.
- Reflexive: $(a = a)$.
- Antisymmetric: if $(a < b)$ then $(b < a)$; if $(a = b)$ then $(b = a)$.
- Transitive: if $(a \leq b)$ and $(b \leq c)$ then $(a \leq c)$.

Built-in comparable types. `String`, `Double`, `Integer`, `Date`, `File`, ...
User-defined comparable types. Implement the `Comparable` interface.

## Implementing the Comparable interface: example 1

Date data type. Simplified version of `java.util.Date`.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }

   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

only compare dates to other dates



## Implementing the Comparable interface: example 2

Domain names.

- Subdomain: `bolle.cs.princeton.edu`.
- Reverse subdomain: `edu.princeton.cs.bolle`.
- Sort by reverse subdomain to group by category.

```
public class Domain implements Comparable<Domain>
{
   private final String[] fields;
   private final int N;

   public Domain(String name)
   {
       fields = name.split("\\.");
       N = fields.length;
   }

   public int compareTo(Domain that)
   {
      for (int i = 0; i < Math.min(this.N, that.N); i++)
      {
         String s = fields[this.N - i - 1];
         String t = fields[that.N - i - 1];
         int cmp = s.compareTo(t);
         if      (cmp < 0) return -1;
         else if (cmp > 0) return +1;
      }
      return this.N - that.N;
   }
}
```

only use this trick when no danger of overflow

subdomains

```
ee.princeton.edu
cs.princeton.edu
princeton.edu
cnn.com
google.com
apple.com
www.cs.princeton.edu
bolle.cs.princeton.edu
```

reverse-sorted subdomains

```
com.apple
com.cnn
com.google
edu.princeton
edu.princeton.cs
edu.princeton.cs.bolle
edu.princeton.cs.www
edu.princeton.ee
```



## Two useful sorting abstractions

Helper functions. Refer to data through compares and exchanges.

Less. Is object `v` less than `w` ?

```
private static boolean less(Comparable v, Comparable w)
{
   return v.compareTo(w) < 0;
}
```

Exchange. Swap object in array `a[]` at index `i` with the one at index `j`.

```
private static void exch(Comparable[] a, int i, int j)
{
   Comparable t = a[i];
   a[i] = a[j];
   a[j] = t;
}
```



## Testing

Q. How to test if an array is sorted?

```
private static boolean isSorted(Comparable[] a)
{
   for (int i = 1; i < a.length; i++)
      if (less(a[i], a[i-1])) return false;
   return true;
}
```

Q. If the sorting algorithm passes the test, did it correctly sort its input?

A. Yes, if data accessed only through `exch()` and `less()`.

- rules of the game
- **selection sort**
- insertion sort
- sorting challenges
- shellsort

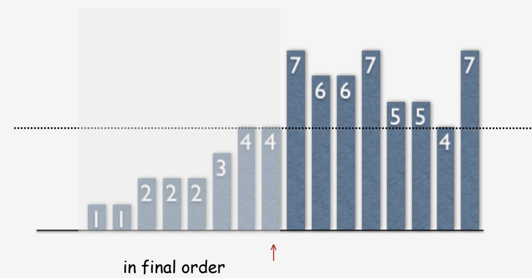


## Selection sort

Algorithm. ↑ scans from left to right.

Invariants.

- Elements to the left of ↑ (including ↑) fixed and in ascending order.
- No element to right of ↑ is smaller than any element to its left.






## Selection sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

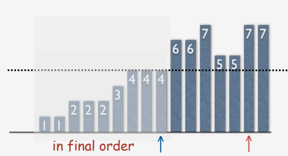
```
i++;
```


- Identify index of minimum item on right.

```
int min = i;
for (int j = i+1; j < N; j++)
   if (less(a[j], a[min]))
      min = j;
```


- Exchange into position.

```
exch(a, i, min);
```






## Selection sort: Java implementation

```
public class Selection {

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }

   private boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private boolean exch(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```

## Selection sort: mathematical analysis

Proposition A. Selection sort uses $(N-1) + (N-2) + \ldots + 1 + 0 \sim N^2/2$ compares and $N$ exchanges.

| i | min | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | S | O | R | T | E | X | A | M | P | L | E |
| 0 | 6 | S | O | R | T | E | X | A | M | P | L | E |
| 1 | 4 | A | O | R | T | E | X | S | M | P | L | E |
| 2 | 10 | A | E | R | T | O | X | S | M | P | L | E |
| 3 | 9 | A | E | E | T | O | X | S | M | P | L | R |
| 4 | 7 | A | E | E | L | O | X | S | M | P | T | R |
| 5 | 7 | A | E | E | L | M | X | S | O | P | T | R |
| 6 | 8 | A | E | E | L | M | O | S | X | P | T | R |
| 7 | 10 | A | E | E | L | M | O | P | X | S | T | R |
| 8 | 8 | A | E | E | L | M | O | P | R | S | T | X |
| 9 | 9 | A | E | E | L | M | O | P | R | S | T | X |
| 10 | 10 | A | E | E | L | M | O | P | R | S | T | X |
| | | A | E | E | L | M | O | P | R | S | T | X |

*entries in black are examined to find the minimum*

*entries in red are* a[min]

*entries in gray are in final position*

Trace of selection sort (array contents just after each exchange)

Running time insensitive to input. Quadratic time, even if array is presorted.

Data movement is minimal. Linear number of exchanges.



- rules of the game
- selection sort
- **insertion sort**
- sorting challenges
- shellsort



## Insertion sort

Algorithm. ↑ scans from left to right.

Invariants.

- Elements to the left of ↑ (including ↑) are in ascending order.
- Elements to the right of ↑ have not yet been seen.




## Insertion sort inner loop

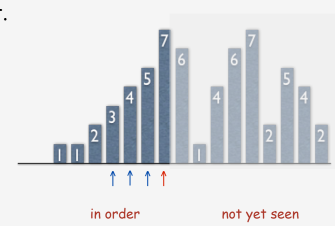
To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```


- Moving from right to left, exchange `a[i]` with each larger element to its left.

```
for (int j = i; j > 0; j--)
   if (less(a[j], a[j-1]))
        exch(a, j, j-1);
   else break;
```



## Insertion sort: Java implementation

```
public class Insertion {

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;
   }

   private boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private boolean exch(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```



## Insertion sort: mathematical analysis

Proposition B. For randomly-ordered data with distinct keys, insertion sort uses ~ $N^2/4$ compares and $N^2/4$ exchanges on the average.

Pf. For randomly data, we expect each element to move halfway back.

| i | j | a[] 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | S | O | R | T | E | X | A | M | P | L | E |
| 1 | 0 | O | S | R | T | E | X | A | M | P | L | E |
| 2 | 1 | O | R | S | T | E | X | A | M | P | L | E |
| 3 | 3 | O | R | S | T | E | X | A | M | P | L | E |
| 4 | 0 | E | O | R | S | T | X | A | M | P | L | E |
| 5 | 5 | E | O | R | S | T | X | A | M | P | L | E |
| 6 | 0 | A | E | O | R | S | T | X | M | P | L | E |
| 7 | 2 | A | E | M | O | R | S | T | X | P | L | E |
| 8 | 4 | A | E | M | O | P | R | S | T | X | L | E |
| 9 | 2 | A | E | L | M | O | P | R | S | T | X | E |
| 10 | 2 | A | E | E | L | M | O | P | R | S | T | X |
| | | A | E | E | L | M | O | P | R | S | T | X |

*entries in gray do not move*

*entry in red is a[j]*

*entries in black moved one position right for insertion*

Trace of insertion sort (array contents just after each insertion)



## Insertion sort: best and worst case

Best case. If the input is in ascending order, insertion sort makes N-1 compares and 0 exchanges.

**A E E L M O P R S T X**

Worst case. If the input is in descending order (and no duplicates), insertion sort makes ~ $N^2/2$ compares and ~ $N^2/2$ exchanges.

**X T S R P O M L E E A**



## Insertion sort: partially sorted inputs

Def. An inversion is a pair of keys that are out of order.

**A E E L M O T R X P S**

**T-R T-P T-S R-P X-P X-S**

(6 inversions)

Def. An array is partially sorted if the number of inversions is O(N).

- Ex 1. A small array appended to a large sorted array.
- Ex 2. An array with only a few elements out of place.

Proposition C. For partially-sorted arrays, insertion sort runs in linear time.

Pf. Number of exchanges equals the number of inversions.

↑ number of compares = exchanges + (N-1)

▸ rules of the game
▸ selection sort
▸ insertion sort
▸ **sorting challenges**
▸ shellsort

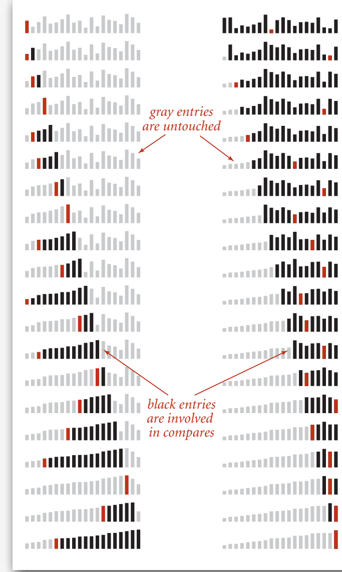


## Sorting challenge 0

Input. Array of doubles.
Plot. Data proportional to length.

Name the sorting method.

- Insertion sort.
- Selection sort.



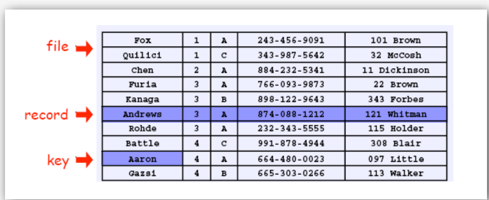



## Sorting challenge 1

Problem. Sort a file of huge records with tiny keys.
Ex. Reorganize your MP3 files.

Which sorting method to use?

- System sort.
- Insertion sort.
- Selection sort.




| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |

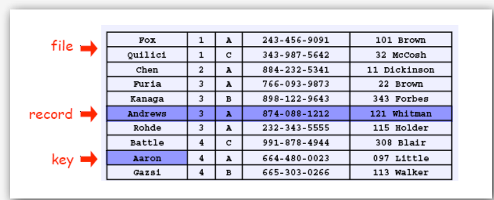


## Sorting challenge 2

Problem. Sort a huge randomly-ordered file of small records.
Ex. Process transaction records for a phone company.

Which sorting method to use?

- System sort.
- Insertion sort.
- Selection sort.




| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
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| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |

## Sorting challenge 3

Problem. Sort a huge number of tiny files (each file is independent)

Ex. Daily customer transaction records.

Which sorting method to use?

- System sort.
- Insertion sort.
- Selection sort.

file →
record →
key →

| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |



## Sorting challenge 4

Problem. Sort a huge file that is already almost in order.

Ex. Resort a huge database after a few changes.

Which sorting method to use?

- System sort.
- Insertion sort.
- Selection sort.

file →
record →
key →

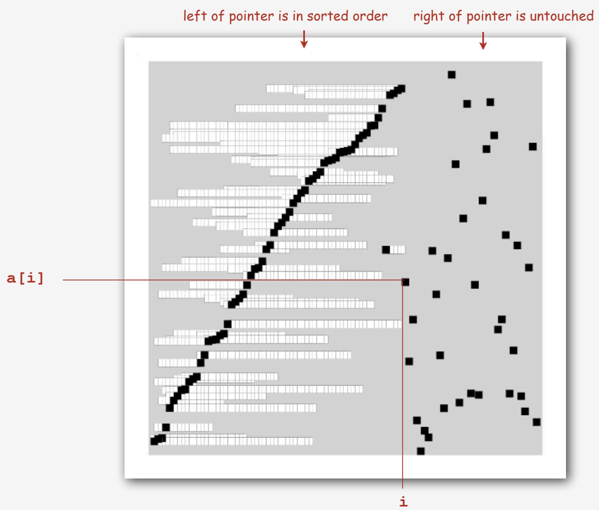
| | | | | |
|---|---|---|---|---|
| Fox | 1 | A | 243-456-9091 | 101 Brown |
| Quilici | 1 | C | 343-987-5642 | 32 McCosh |
| Chen | 2 | A | 884-232-5341 | 11 Dickinson |
| Furia | 3 | A | 766-093-9873 | 22 Brown |
| Kanaga | 3 | B | 898-122-9643 | 343 Forbes |
| Andrews | 3 | A | 874-088-1212 | 121 Whitman |
| Rohde | 3 | A | 232-343-5555 | 115 Holder |
| Battle | 4 | C | 991-878-4944 | 308 Blair |
| Aaron | 4 | A | 664-480-0023 | 097 Little |
| Gazsi | 4 | B | 665-303-0266 | 113 Walker |



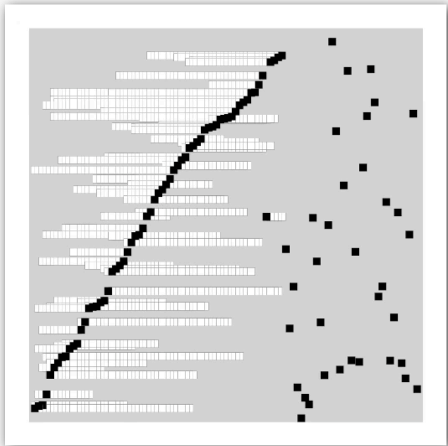




## Insertion sort animation



## Insertion sort animation



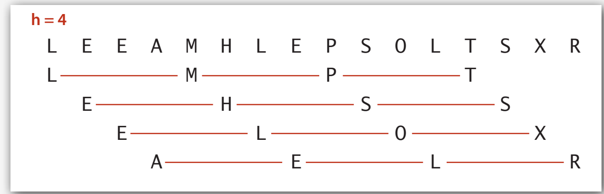
Reason it is slow: excessive data movement.



## Shellsort overview

Idea. Move elements more than one position at a time by h-sorting the file.

an h-sorted file is h interleaved sorted files




Shellsort. h-sort the file for a decreasing sequence of values of h.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | S | H | E | L | L | S | O | R | T | E | X | A | M | P | L | E |
| 13-sort | P | H | E | L | L | S | O | R | T | E | X | A | M | S | L | E |
| 4-sort | L | E | E | A | M | H | L | E | P | S | O | L | T | S | X | R |
| 1-sort | A | E | E | E | H | L | L | L | M | O | P | R | S | S | T | X |



## h-sorting

How to h-sort a file? Insertion sort, with stride length h.

3-sorting a file

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| M | O | L | E | E | X | A | S | P | R | T |
| E | O | L | M | E | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |

Why insertion sort?

- Big increments ⇒ small subfiles.
- Small increments ⇒ nearly in order. [stay tuned]



## Shellsort example: increments 7, 3, 1

input

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| S | O | R | T | E | X | A | M | P | L | E |

7-sort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| S | O | R | T | E | X | A | M | P | L | E |
| M | O | R | T | E | X | A | S | P | L | E |
| M | O | R | T | E | X | A | S | P | L | E |
| M | O | L | T | E | X | A | S | P | R | E |
| M | O | L | E | E | X | A | S | P | R | T |

3-sort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| M | O | L | E | E | X | A | S | P | R | T |
| E | O | L | M | E | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |

1-sort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | E | L | O | P | M | S | X | R | T |
| A | E | E | L | O | P | M | S | X | R | T |
| A | E | E | L | O | P | M | S | X | R | T |
| A | E | E | L | M | O | P | S | X | R | T |
| A | E | E | L | M | O | P | S | X | R | T |
| A | E | E | L | M | O | P | S | X | R | T |
| A | E | E | L | M | O | P | R | S | X | T |
| A | E | E | L | M | O | P | R | S | T | X |

result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | E | E | L | M | O | P | R | S | T | X |

## Shellsort: intuition

Proposition. A g-sorted array remains g-sorted after h-sorting it.

Pf. Harder than you'd think!

7-sort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| M | O | R | T | E | X | A | S | P | L | E |
| M | O | R | T | E | X | A | S | P | L | E |
| M | O | L | T | E | X | A | S | P | R | E |
| M | O | L | E | E | X | A | S | P | R | T |
| M | O | L | E | E | X | A | S | P | R | T |

3-sort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| M | O | L | E | E | X | A | S | P | R | T |
| E | O | L | M | E | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| E | E | L | M | O | X | A | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | X | M | S | P | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |
| A | E | L | E | O | P | M | S | X | R | T |

still 7-sorted



## What increments to use?

1, 2, 4, 8, 16, 32 . . .
No.

1, 3, 7, 15, 31, 63, . . .
Maybe.

1, 4, 13, 40, 121, 363, . . .
OK, easy to compute.

1, 5, 19, 41, 109, 209, 505, . . .
Tough to beat in empirical studies.

Interested in learning more?

- See Algs 3 section 6.8 or Knuth volume 3 for details.



## Shellsort: Java implementation

```
public class Shell
{  // Shellsort.
   public static void sort(Comparable[] a)
   {  // Sort a[] into increasing order.
      int N = a.length;

      int h = 1;
      while (h < N/3) h = 3*h + 1; // 1, 4, 13, 40, 121, 364, 1093, ...

      while (h >= 1)
      {  // h-sort the file.
         for (int i = h; i < N; i++)
         {  // Insert a[i] among a[i-h], a[i-2*h], a[i-3*h]... .
            for (int j = i; j > 0 && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }

         h = h/3;
      }
   }

   private boolean less(Comparable v, Comparable w)
   // As before.
   private boolean exch(Comparable[] a, int i, int j)
   // As before.
}
```

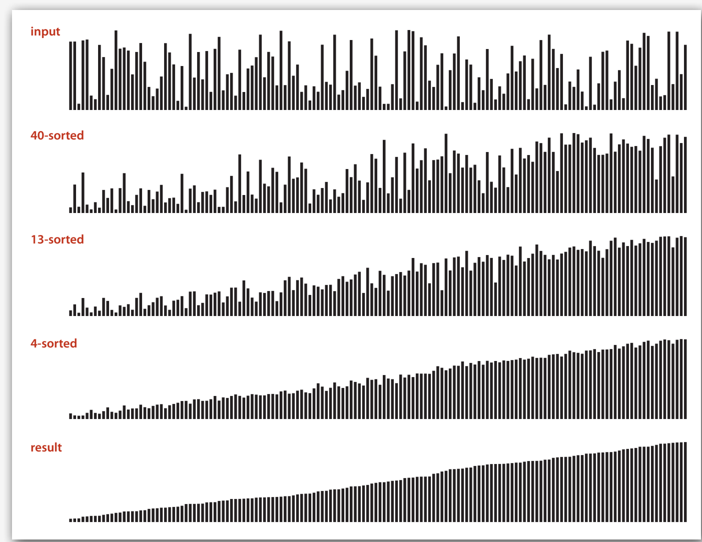
magic increment sequence

insertion sort

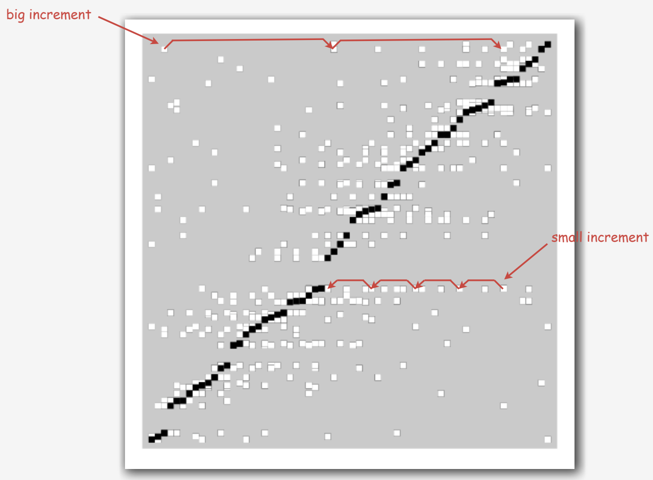
move to next increment



## Visual trace of shellsort

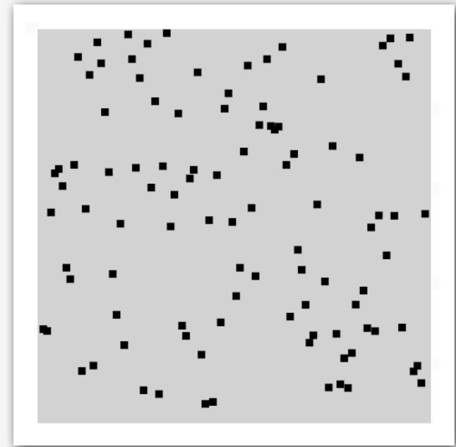


## Shellsort animation






## Shellsort animation



Bottom line: substantially faster than insertion sort!



## Shellsort: analysis

Proposition. The worst-case number of compares for shellsort using the increments 1, 4, 13, 40, ... is $O(N^{3/2})$.

Property. The number of compares used by shellsort with the 3x+1 increments is at most by a small multiple of N times the # of increments used.

| N | compares | $N^{1.289}$ | 2.5 N lg N |
|---|---|---|---|
| 5,000 | 93 | 58 | 106 |
| 10,000 | 209 | 143 | 230 |
| 20,000 | 467 | 349 | 495 |
| 40,000 | 1022 | 855 | 1059 |
| 80,000 | 2266 | 2089 | 2257 |

measured in thousands

Remark. Accurate model has not yet been discovered (!)



## Why are we interested in shellsort?

Example of simple idea leading to substantial performance gains.

Useful in practice.
- Fast unless file size is huge.
- Tiny, fixed footprint for code (used in embedded systems).
- Hardware sort prototype.

Simple algorithm, nontrivial performance, interesting questions
- Asymptotic growth rate?
- Best sequence of increments? ← open problem: find a better increment sequence
- Average case performance?

Lesson. Some good algorithms are still waiting discovery.