Combinatorial Search

- permutations
- backtracking
- counting
- subsets
- paths in a graph

Overview

Exhaustive search. Iterate through all elements of a search space.

Backtracking. Systematic method for examining feasible solutions to a problem, by systematically eliminating infeasible solutions.

Applicability. Huge range of problems (include NP-hard ones).

Caveat. Search space is typically exponential in size ⇒ effectiveness may be limited to relatively small instances.

Caveat to the caveat. Backtracking may prune search space to reasonable size, even for relatively large instances

Warmup: enumerate N-bit strings

Goal. Process all 2^N bit strings of length N.

- Maintain `a[i]` where `a[i]` represents bit `i`.
- Initialize all bits to zero.
- Simple recursive method does the job.

```
private void enumerate(int n)
{
  if (n == N)
  { process(); return; }
  enumerate(n+1);
  a[n] = 1;
  enumerate(n+1);
  a[n] = 0;
}
```

clean up

N = 3
enumerate(0)

```
0 0 0
0 0 1
0 0 0
0 1 0
0 1 1
0 1 0
0 0 0
1 0 0
1 0 1
1 0 0
1 1 0
1 1 1
1 1 0
1 0 0
0 0 0
```

N = 4
enumerate(0)

```
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
```

Remark. Equivalent to counting in binary from 0 to $2^N - 1$.

Warmup: enumerate N-bit strings

```
public class Counter
{
   private int N;    // number of bits
   private int[] a;  // bits (0 or 1)

   public Counter(int N)
   {
      this.N = N;
      a = new int[N];
      enumerate(0);
   }

   private void process()
   {
      for (int i = 0; i < N; i++)
         StdOut.print(a[i]);
      StdOut.println();
   }

   private void enumerate(int n)
   {
     if (n == N)
     { process(); return; }
     enumerate(n+1);
     a[n] = 1;
     enumerate(n+1);
     a[n] = 0;
   }
}
```

all programs in this lecture are variations on this theme

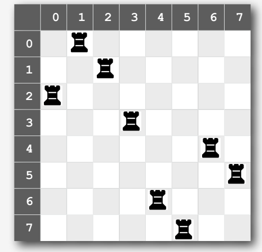
```
public static void main(String[] args)
{
   int N = Integer.parseInt(args[0]);
   Counter counter = new Counter(N);
}
```

```
% java Counter 4
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```

▸ **permutations**
▸ backtracking
▸ counting
▸ subsets
▸ paths in a graph

N-rooks problem

Q. How many ways are there to place N rooks on an N-by-N board so that no rook can attack any other?




```
int[] a = { 1, 2, 0, 3, 6, 7, 4, 5 };
```

Representation. No two rooks in the same row or column ⇒ permutation.

Challenge. Enumerate all N! permutations of 0 to N-1.

Enumerating permutations

Recursive algorithm to enumerate all N! permutations of size N.

- Start with permutation 0 to N-1.
- For each value of `i`
  - swap `i` into position `0`
  - enumerate all (N-1)! arrangements of a[1..N-1]
  - clean up (swap `i` and `0` back into position)

0 followed by perms of 1 2 3 | 1 followed by perms of 0 2 3 | 2 followed by perms of 1 0 3 | 3 followed by perms of 1 2 0

```
0 1     0 1 2     0 1 2 3   1 0 2 3   2 1 0 3   3 1 2 0
1 0     0 2 1     0 1 3 2   1 0 3 2   2 1 3 0   3 1 0 2
        1 0 2     0 2 1 3   1 2 0 3   2 0 1 3   3 2 1 0
        1 2 0     0 2 3 1   1 2 3 0   2 0 3 1   3 2 0 1
        2 1 0     0 3 2 1   1 3 2 0   2 3 0 1   3 0 2 1
        2 0 1     0 3 1 2   1 3 0 2   2 3 1 0   3 0 1 2
                            1 3 2 0
                            1 0 2 3
                            0 1 2 3
```

example showing cleanup swaps that bring perm back to original

Enumerating permutations

Recursive algorithm to enumerate all N! permutations of size N.

- Start with permutation 0 to N-1.
- For each value of `i`
  - swap `i` into position `0`
  - enumerate all (N-1)! arrangements of a[1..N-1]
  - clean up (swap `i` and `0` back into position)

```
private void enumerate(int n)
{
   if (n == N)
   {  process(); return;  }

   for (int i = n; i < N; i++)
   {
      exch(n, i);
      enumerate(n+1);
      exch(i, n);       ← clean up
   }
}
```

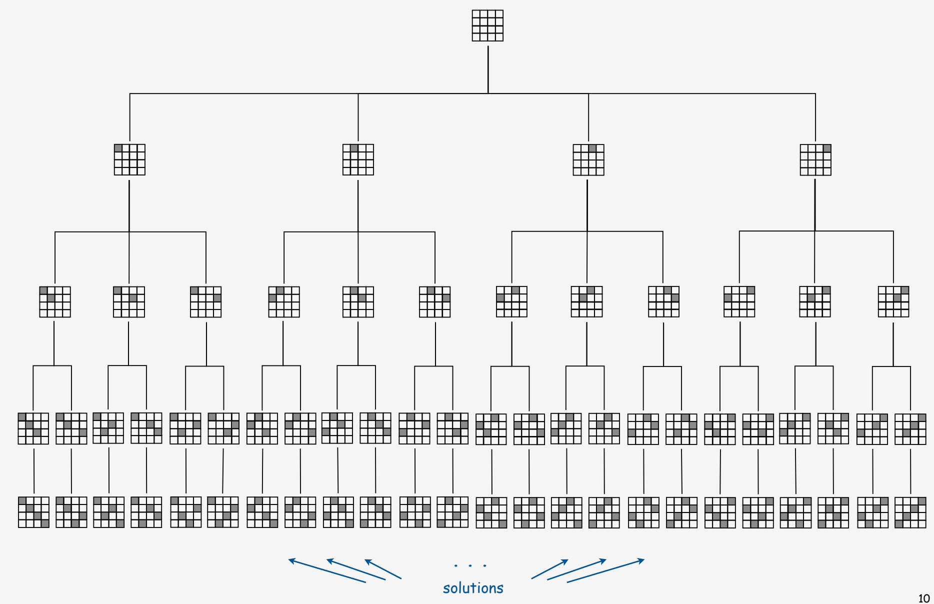
```
% java Rooks 4
0 1 2 3
0 1 3 2
0 2 1 3      0 followed by
0 2 3 1      perms of 1 2 3
0 3 2 1
0 3 1 2
1 0 2 3
1 0 3 2
1 2 0 3      1 followed by
1 2 3 0      perms of 0 2 3
1 3 2 0
1 3 0 2
2 1 0 3
2 1 3 0
2 0 1 3      2 followed by
2 0 3 1      perms of 1 0 3
2 3 0 1
2 3 1 0
3 1 2 0
3 1 0 2
3 2 1 0      3 followed by
3 2 0 1      perms of 1 2 0
3 0 2 1
3 0 1 2
```

Enumerating permutations

```
public class Rooks
{
   private int N;   // number of bits
   private int[] a; // bits (0 or 1)

   public Rooks(int N)
   {
      this.N = N;
      a = new int[N];
      for (int i = 0; i < N; i++)
         a[i] = i;
      enumerate(0);
   }

   private void enumerate(int n)
   { /* as before */  }

   private void enumerate(int n)
   { /* see previous slide */  }

   private void exch(int i, int j)
   {  int t = a[i]; a[i] = a[j]; a[j] = t;  }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Rooks rooks = new Rooks(N);
   }
}
```

```
% java Rooks 2
0 1
1 0

% java Rooks 3
0 1 2
0 2 1
1 0 2
1 2 0
2 1 0
2 0 1
```


4-rooks search tree



N-rooks problem: back-of-envelope running time estimate

Studying slow way to compute N! but good warmup for calculations.

```
% java Rooks 7 | wc -l
5040

% java Rooks 8 | wc -l
40320

% java Rooks 9 | wc -l
362880

% java Rooks 10 | wc -l
3628800

% java Rooks 25 | wc -l
...
```

instant

1.6 seconds

15 seconds

170 seconds

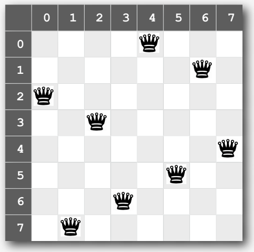
forever

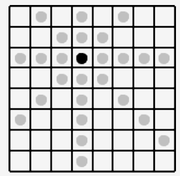
Hypothesis. Running time is about $2(N! / 8!)$ seconds.

- permutations
- **backtracking**
- counting
- subsets
- paths in a graph

N-queens problem

Q. How many ways are there to place N queens on an N-by-N board so that no queen can attack any other?





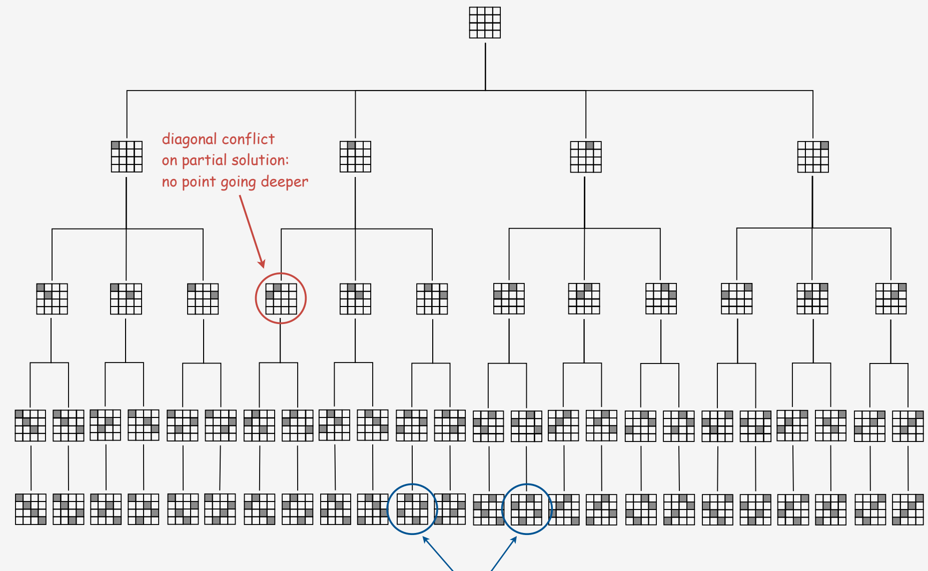
```
int[] a = { 4, 6, 0, 2, 7, 5, 3, 1 };
```

Representation. Solution is a permutation: `a[i]` = column of queen in row `i`.

Additional constraint. No diagonal attack is possible

Challenge. Enumerate (or even count) the solutions.

4-queens search tree



N-queens problem: backtracking solution

Backtracking paradigm. Iterate through elements of search space.

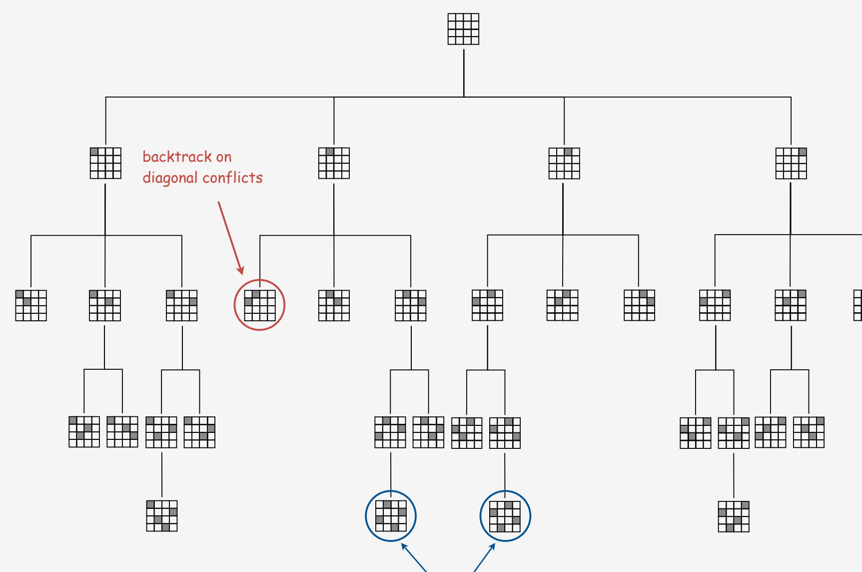
- When there are N possible choices, make one choice and recur.
- If the choice is a dead end, backtrack to previous choice, and make next available choice.

Benefit. Identifying dead ends allows us to prune the search tree.

Ex. [backtracking for N-queens problem]

- Dead end: a diagonal conflict.
- Pruning: backtrack and try next row when diagonal conflict found.

4-queens search tree (pruned)



N-queens problem: backtracking solution

```
private boolean backtrack(int n)
{
   for (int i = 0; i < n; i++)
   {
      if ((a[i] - a[n]) == (n - i)) return true;
      if ((a[n] - a[i]) == (n - i)) return true;
   }
   return false;
}

private void enumerate(int n)
{
   if (k == N)
   {  process(); return;  }

   for (int i = k; i < N; i++)
   {
      exch(n, i);
      if (!backtrack(n)) enumerate(n+1);
      exch(i, n);
   }
}
```

stop enumerating if adding the n^{th} queen leads to a diagonal violation

```
% java Queens 4
1 3 0 2
2 0 3 1

% java Queens 5
0 2 4 1 3
0 3 1 4 2
1 3 0 2 4
1 4 2 0 3
2 0 3 1 4
2 4 1 3 0
3 1 4 2 0
3 0 2 4 1
4 1 3 0 2
4 2 0 3 1

% java Queens 6
1 3 5 0 2 4
2 5 1 4 0 3
3 0 4 1 5 2
4 2 0 5 3 1
```


N-queens problem: effectiveness of backtracking

Pruning the search tree leads to enormous time savings.

N	Q(N)	N!
2	0	2
3	0	6
4	2	24
5	10	120
6	4	720
7	40	5,040
8	92	40,320
9	352	362,880
10	724	3,628,800
11	2,680	39,916,800
12	14,200	479,001,600
13	73,712	6,227,020,800
14	365,596	87,178,291,200

N-queens problem: How many solutions?

```
% java Queens 13 | wc -l
73712

% java Queens 14 | wc -l
365596

% java Queens 15 | wc -l
2279184

% java Queens 16 | wc -l
2279184

% java Queens 17 | wc -l
...
```

1.1 seconds

5.4 seconds

29 seconds

210 seconds

1352 seconds

Hypothesis. Running time is about $(N! / 2.5^N) / 43{,}000$ seconds.

Conjecture. Q(N) is ~ N! / c^N, where c is about 2.54.

- permutations
- backtracking
- **counting**
- subsets
- paths in a graph

Counting: Java implementation

Goal. Enumerate all N-digit base-R numbers

Solution. Generalize binary counter in lecture warmup.

```
private static void enumerate(int n)
{
   if (n == N)
   {  process(); return;  }

   for (int r = 0; r < R; r++)
   {
      a[n] = r;
      enumerate(n+1);
   }
   a[n] = 0;  <------  clean up not needed, why?
}
```

enumerate N-digit base-R numbers

```
% java Counter 3 2
000
001
010
011
100
101
110
111

% java Count 2 4
00
01
02
03
10
11
12
13
20
21
22
23
30
31
32
33
```


Counting application: Sudoku

Goal. Fill 9-by-9 grid so that every row, column, and box contains each of the digits 1 through 9.

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

Remark. Natural generalization is NP-hard.

Counting application: Sudoku

Goal. Fill 9-by-9 grid so that every row, column, and box contains each of the digits 1 through 9.

7	2	8	9	4	6	3	1	5
9	3	4	2	5	1	6	7	8
5	1	6	7	3	8	2	4	9
1	4	7	5	9	3	8	2	6
3	6	9	4	8	2	1	5	7
8	5	2	1	6	7	4	9	3
2	9	3	6	1	5	7	8	4
4	8	1	3	7	9	5	6	2
6	7	5	8	2	4	9	3	1

Solution. Enumerate all 81-digit base-9 numbers (with backtracking).

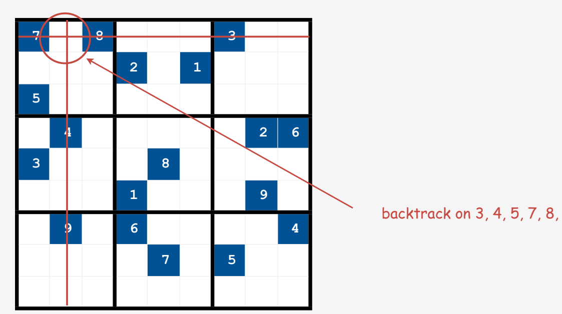
using digits 1 to 9 →

0	1	2	3	4	5	6	7	8		80
7		8					3		...	

Sudoku: backtracking solution

Iterate through elements of search space.

- For each empty cell, there are 9 possible choices.
- Make one choice and recur.
- If you find a conflict in row, column, or box, then backtrack.



Sudoku: Java implementation

```
private static void solve(int cell)
{
   if (cell == 81)                              // found a solution
   {  show(board); return;  }

   if (board[cell] != 0)                        // already filled in
   {  solve(cell + 1); return;  }

   for (int n = 1; n <= 9; n++)                 // try all 9 possibilities
   {
      if (! backtrack(cell, n))                 // unless a Sudoku constraint is violated (see booksite for code)
      {
         board[cell] = n;
         solve(cell + 1);
      }
   }

   board[cell] = 0;                             // clean up
}
```

```
% more board.txt
7 0 8 0 0 0 3 0 0
0 0 0 2 0 1 0 0 0
5 0 0 0 0 0 0 0 0
0 4 0 0 0 0 0 2 6
3 0 0 0 8 0 0 0 0
0 0 0 1 0 0 0 9 0
0 9 0 6 0 0 0 0 4
0 0 0 0 7 0 5 0 0
0 0 0 0 0 0 0 0 0

% java Sudoku < board.txt
7 2 8 9 4 6 3 1 5
9 3 4 2 5 1 6 7 8
5 1 6 7 3 8 2 4 9
1 4 7 5 9 3 8 2 6
3 6 9 4 8 2 1 5 7
8 5 2 1 6 7 4 9 3
2 9 3 6 1 5 7 8 4
4 8 1 3 7 9 5 6 2
6 7 5 8 2 4 9 3 1
```


- permutations
- backtracking
- counting
- **subsets**
- paths in a graph

Enumerating subsets: natural binary encoding

Given N items, enumerate all 2^N subsets.
- Count in binary from 0 to 2^N - 1.
- Bit i represents item i.
- If 0, in subset; if 1, not in subset.

i	*binary*	*subset*	*complement*
0	0 0 0 0	empty	4 3 2 1
1	0 0 0 1	1	4 3 2
2	0 0 1 0	2	4 3 1
3	0 0 1 1	2 1	4 3
4	0 1 0 0	3	4 2 1
5	0 1 0 1	3 1	4 2
6	0 1 1 0	3 2	4 1
7	0 1 1 1	3 2 1	4
8	1 0 0 0	4	3 2 1
9	1 0 0 1	4 1	3 2
10	1 0 1 0	4 2	3 1
11	1 0 1 1	4 2 1	3
12	1 1 0 0	4 3	2 1
13	1 1 0 1	4 3 1	2
14	1 1 1 0	4 3 2	1
15	1 1 1 1	4 3 2 1	empty

Enumerating subsets: natural binary encoding

Given N items, enumerate all 2^N subsets.
- Count in binary from 0 to 2^N - 1.
- Maintain `a[i]` where `a[i]` represents item `i`.
- If `0`, `a[i]` in subset; if `1`, `a[i]` not in subset.

Binary counter from warmup does the job.

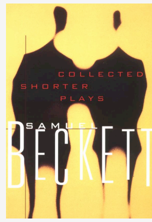
```
private void enumerate(int n)
{
   if (n == N)
   {  process(); return;  }
   enumerate(n+1);
   a[n] = 1;
   enumerate(n+1);
   a[n] = 0;
}
```

Digression: Samuel Beckett play

Quad. Starting with empty stage, 4 characters enter and exit one at a time, such that each subset of actors appears exactly once.

code	*subset*	*move*
0 0 0 0	*empty*	
0 0 0 1	1	enter 1
0 0 1 1	2 1	enter 2
0 0 1 0	2	exit 1
0 1 1 0	3 2	enter 3
0 1 1 1	3 2 1	enter 1
0 1 0 1	3 1	exit 2
0 1 0 0	3	exit 1
1 1 0 0	4 3	enter 4
1 1 0 1	4 3 1	enter 1
1 1 1 1	4 3 2 1	enter 2
1 1 1 0	4 3 2	exit 1
1 0 1 0	4 2	exit 3
1 0 1 1	4 2 1	enter 1
1 0 0 1	4 1	exit 2
1 0 0 0	4	exit 1

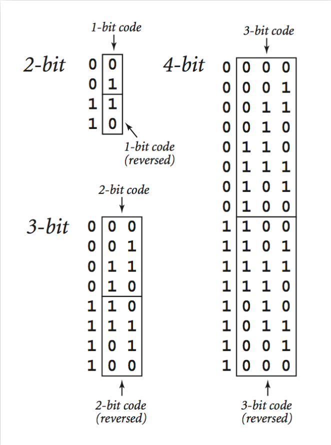
ruler function



Binary reflected gray code

Def. The n-bit binary reflected Gray code is:

- the (n-1) bit code with a 0 prepended to each word, followed by
- the (n-1) bit code in reverse order, with a 1 prepended to each word.



Beckett: Java implementation

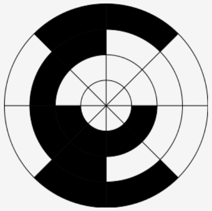
```
public static void moves(int n, boolean enter)
{
   if (n == 0) return;
   moves(n-1, true);
   if (enter) StdOut.println("enter " + n);
   else       StdOut.println("exit  " + n);
   moves(n-1, false);
}
```

```
% java Beckett 4
enter 1
enter 2
exit  1
enter 3
enter 1
exit  2
exit  1
enter 4
enter 1
enter 2
exit  1
exit  3
enter 1
exit  2
exit  1
```

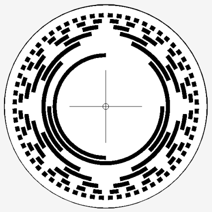
stage directions for 3-actor play `moves(3, true)`

reverse stage directions for 3-actor play `moves(3, false)`

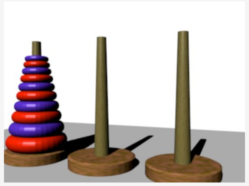
More applications of gray codes



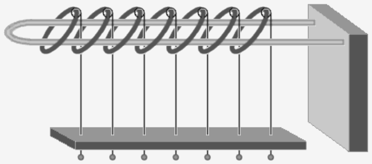
3-bit rotary encoder



8-bit rotary encoder



Towers of Hanoi



Chinese ring puzzle

Enumerating subsets using Gray code

Two simple changes to binary counter from warmup:

- flip `a[k]` instead of setting it to `1`.
- Eliminate cleanup.

Gray code enumeration

```
private void enumerate(int n)
{
  if (n == N)
  {  process(); return;  }
  enumerate(n+1);
  a[n] = 1 - a[n];
  enumerate(n+1);
}
```

```
000
001
011
010
110
111
101
100
```

standard binary (from warmup)

```
private void enumerate(int n)
{
  if (n == N)
  {  process(); return;  }
  enumerate(n+1);
  a[n] = 1;
  enumerate(n+1);
  a[n] = 0;
}
```

```
000
001
010
011
100
101
110
111
```

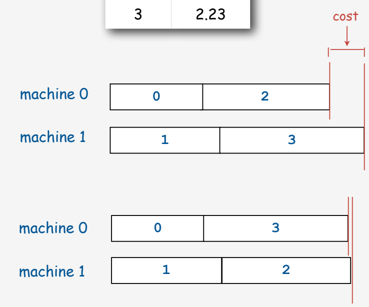
Advantage (same as Beckett). Only one item changes at a time.

Scheduling

Scheduling (set partitioning). Given n jobs of varying length, divide among two machines to minimize the time the last job finishes.

or, equivalently, difference between finish times

job	length
0	1.41
1	1.73
2	2.00
3	2.23




```
public double[] finish(int[] a)
{
    double[] time = new double[2];
    for (int i = 0; i < N; i++)
        time[a[i]] += jobs[i];
    return time;
}

private double cost(int[] a)
{
    double[] time = finish(a);
    return Math.abs(time[0] - time[1]);
}
```

```
i        a[]       time[0]  time[1]

       0 1 1 0      1.41       0

0      0 1 1 0      1.41       0
1      0 1 1 0      1.41     1.73
2      0 1 1 0      1.41     3.73
3      0 1 1 0      3.64     3.73
                    3.64     3.73

               cost: .09
```

Remark. Intractable.

Scheduling (full implementation)

```
public class Scheduler
{
   private int N;           // Number of jobs.
   private int[] a;         // Subset assignments.
   private int[] b;         // Best assignment.
   private double[] jobs;   // Job lengths.

   public Scheduler(double[] jobs)
   {
      this.N = jobs.length;
      this.jobs = jobs;
      a = new int[N];
      b = new int[N];
      enumerate(0);
   }

   public int[] best()
   {  return b;  }

   private void enumerate(int n)
   {  /* Gray code enumeration. */   }

   private void process()
   {
     if (cost(a) < cost(b))
       for (int i = 0; i < N; i++)
         b[i] = a[i];
   }

   public static void main(String[] args)
   {  /* create Scheduler, print results */  }
}
```

trace of
`% java Scheduler 4 < jobs.txt`

a[]				finish times		cost
0	0	0	0	7.38	0.00	7.38
0	0	0	1	5.15	2.24	2.91
0	0	1	1	3.15	4.24	1.09
0	0	1	0	5.38	2.00	
0	1	1	0	3.65	3.73	0.08
0	1	1	1	1.41	5.97	
0	1	0	1	3.41	3.97	
0	1	0	0	5.65	1.73	
1	1	0	0	4.24	3.15	
1	1	0	1	2.00	5.38	
1	1	1	1	0.00	7.38	
1	1	1	0	2.24	5.15	
1	0	1	0	3.97	3.41	
1	0	1	1	1.73	5.65	
1	0	0	1	3.73	3.65	
1	0	0	0	5.97	1.41	

```
  MACHINE 0     MACHINE 1
 1.4142135624
                1.7320508076
                2.0000000000
 2.2360679775
-----------------------------
 3.6502815399   3.7320508076
```


Scheduling: improvements

Many opportunities (details omitted).

- Fix last job on machine 0 (quick factor-of-two improvement).
- Backtrack when partial schedule cannot beat best known (check total against goal: half of total job times)

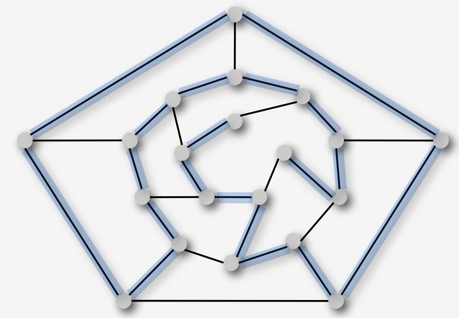
```
private void enumerate(int n)
{
  if (n == N-1)
  {  process(); return;  }
  if (backtrack(n)) return;
  enumerate(n+1);
  a[n] = 1 - a[n];
  enumerate(n+1);
}
```

- Process all 2^k subsets of last k jobs, keep results in memory, (reduces time to 2^{N-k} when 2^k memory available).

- permutations
- backtracking
- counting
- subsets
- **paths in a graph**

Hamilton path

Goal. Find a simple path that visits every vertex exactly once.

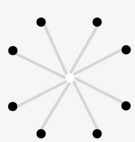


visit every edge exactly once

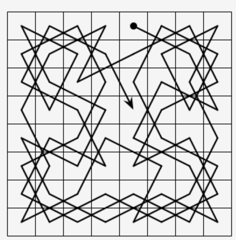
Remark. Euler path easy, but Hamilton path is NP-complete.

Knight's tour

Goal. Find a sequence of moves for a knight so that, starting from any square, it visits every square on a chessboard exactly once.



legal knight moves



a knight's tour

Solution. Find a Hamilton path in knight's graph.

Hamilton path: backtracking solution

Backtracking solution. To find Hamilton path starting at `v`:

- Add `v` to current path.
- For each vertex `w` adjacent to `v`
  - find a simple path starting at `w` using all remaining vertices
- Clean up: remove `v` from current path.

Q. How to implement?

A. Add cleanup to DFS (!!)

Hamilton path: Java implementation

```
public class HamiltonPath
{
   private boolean[] marked;    // vertices on current path
   private int count = 0;       // number of Hamiltonian paths

   public HamiltonPath(Graph G)
   {
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         dfs(G, v, 1);
   }

   private void dfs(Graph G, int v, int depth)
   {
      marked[v] = true;

      if (depth == G.V()) count++;

      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w, depth+1);

      marked[v] = false;
   }
}
```

length of current path (recursion depth)

clean up

Backtracking summary

problem	enumeration	backtracking
N-rooks	permutations	no
N-queens	permutations	yes
Sudoku	base-9 numbers	yes
scheduling	subsets	yes
Hamilton path	paths in a graph	yes

The longest path

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write,
I have a weakness,
I'm addicted to completeness,
And I keep searching for the longest path.

The algorithm I would like to see
Is of polynomial degree,
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done: GPA 2.1
Is more than I hope for.

Garey, Johnson, Karp and other men (and women)
Tried to make it order N log N.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.

Recorded by Dan Barrett in 1988
while a student at Johns Hopkins
during a difficult algorithms final