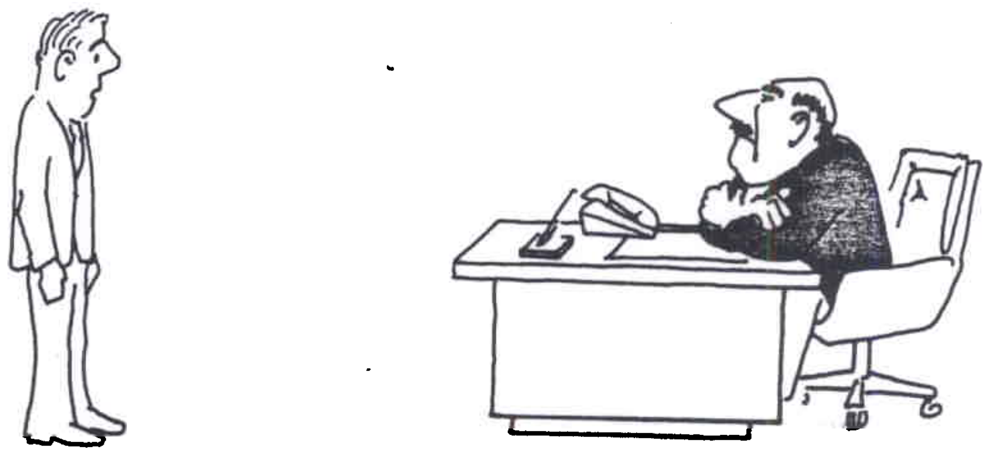
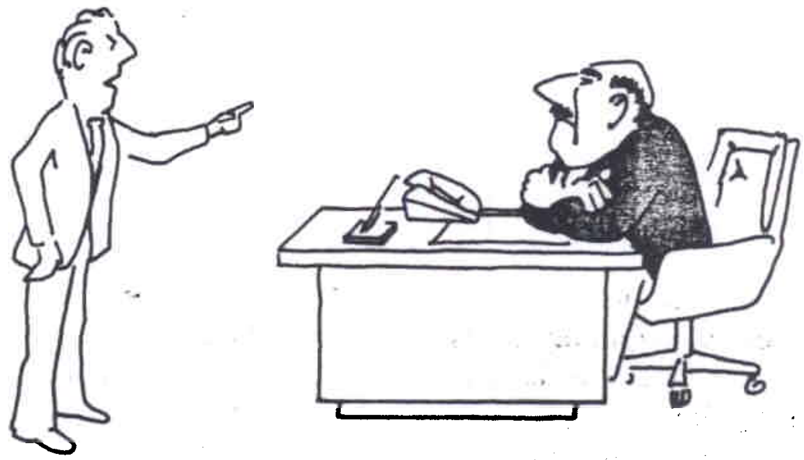


"I can't find an efficient algorithm, I guess I'm just too dumb."



"I can't find an efficient algorithm, because no such algorithm is possible!"

"I can't find an efficient algorithm, but neither can all these famous people."

Reductions

How do we show that a problem is easy?

Reduce each instance to one (or more) instances of a known easy problem.

$I_1 \in P_1 \qquad R(I_1) = I_2 \in P_2$

To solve I_1 in P_1:

1. Apply R to turn I_1 into $I_2 \in P_2$
2. Apply algorithm for P_2 to I_2.

Cost = cost of applying R plus cost of applying P_2 algorithm.

Examples

Reduction to some form of matrix multiplication:

Transitive closure

Context-free language recognition

Reduction to linear programming

Reduction to network flow

etc.

Cost of reduction?

linear time, quadratic time, ...

Positive use of reduction:

$$? \Rightarrow \text{easy}$$

Reduce a problem of unknown complexity

to an easy problem

"Negative" use of reduction:

How do we show that a problem of

unknown complexity is hard?

Reduce a hard problem to it

$$\text{hard} \Rightarrow ?$$

If the questionable problem were easy, so would be

the hard problem XX

Reductions in both directions show

computational equivalence (up to the

cost of the reduction)

$$P_1 \Longleftrightarrow P_2$$

both are easy or both are hard

Transitivity of reductions

$$P_1 \underset{R_1}{\Rightarrow} P_2 \underset{R_2}{\Rightarrow} P_3$$

Gives a reduction from P_1 to P_3

Cost is cost of $R_2(R_1(\,))$

both p-time, overall p-time

linear | linear

R_1 R_2

n an bn

an $(ab)n$

n an^2 bn^2

an^2 $b(an^2)^2 = ba^2n^4$

an^k bn^l

$b(an^k)^l = ba^ln^{kl}$

Satisfiability: Is a Boolean (logical) function true for some choice of variable assignments?

$(x \vee y) \wedge (\bar{x} \vee \bar{y})$ sat: $x=1$, $y=0$

$\wedge$ and

$\vee$ or

$-$ not

x variable

$x, \bar{x}$ literal

$(x \vee \bar{y} \vee z)$ clause: disjuntion ("or") of literals

$(x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$ conjunctive normal form: conjunct ("and") of clauses

$x \vee \bar{x}$ tautology: true for all choices of variables

F is sat iff $\bar{F}$ is not a tautology: can be falsified

Reduction of CNF sat to 3-CNF sat

(at most 3 literals/clause)

$$(x \vee y \vee z \vee w \vee u \vee v) \Rightarrow$$

$$(x \vee y \vee a) \wedge (\bar{a} \vee z \vee b) \wedge (\bar{b} \vee w \vee c)$$

$$\wedge (\bar{c} \vee u \vee v)$$

Needs $k-3$ extra vars per clause of length k.

Graph coloring reducible to sat,

and vice-versa (p-time reductions)

Must phrase graph coloring as a yes-no question: can graph G be colored with k colors?

G: n vertices, m edges

F: nk variables x_{ij}, one per vertex per color

x_{ij} true iff vertex i colored color j

Clauses:

Each vertex colored

$(x_{i1} \vee x_{i2} \vee \cdots \vee x_{ik})$ $\quad i \in V$ $\quad n$

[No vertex colored twice

$(\bar{x}_{ij} \vee \bar{x}_{il})$ $\quad i \in V,\ j \neq l$ colors $\quad n\binom{k}{2}$]

No adjacent vertices the same color

$(\bar{x}_{il} \vee \bar{x}_{jl})$ $\quad (i,j) \in E,\ l$ a color $\quad mk$

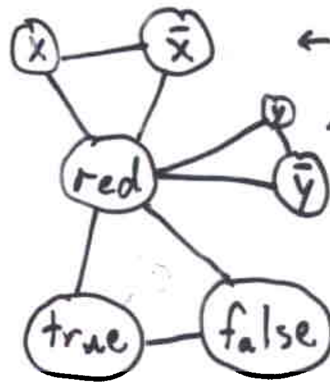
$\#$ literals $= nk + 2n\binom{k}{2} + 2mk$

Vice-versa: (3-sat)

Reduction to 3-coloring

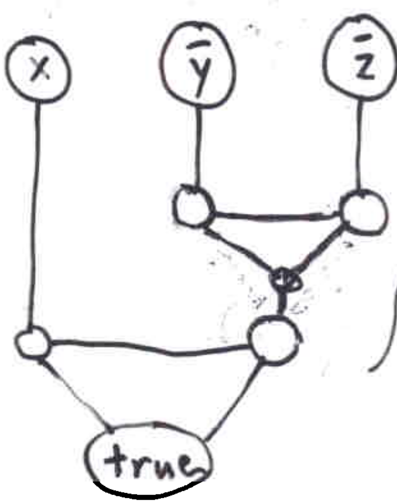
Vertices: x, $\bar{x}$, true, false, red, 5 per clause

x, $\bar{x}$ colored true, false



← every variable

Clause $x \vee \bar{y} \vee \bar{z}$



"gadget" diff. for each clause

Colorable iff formula satisfiable

Sat $\Rightarrow$ Clique

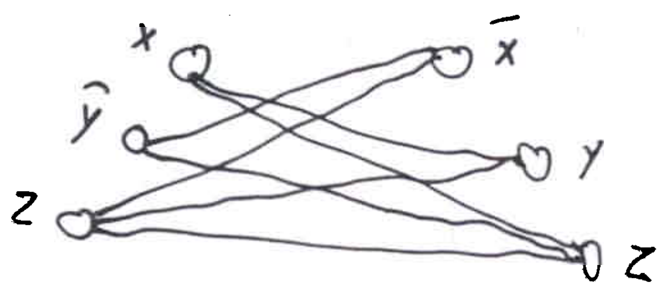
Given a graph, are there k pairwise adjacent vertices?

One vertex per literal occurrence,

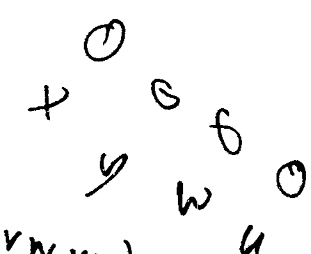
two vertices in different clauses joined by an edge if compatible (not $x, \bar{x}$)

$k =$ # clauses

$(x \vee \bar{y} \vee z)$



$(\bar{x} \vee y \vee z)$



$(x \vee y \vee w \vee u)$

Clique $\Rightarrow$ Sat

x_{ij} $i \in V$, $1 \le j \le k$

x_{ij} true = vertex i is j^{th} in clique

$i, l \in V$ not adj.

$(\bar{x}_{ij} \vee \bar{x}_{lj'})$ $\forall j, j'$

$(\bar{x}_{ij} \vee \bar{x}_{i'j})$ $i \neq i'$ $\forall j$

$(x_{1j} \vee x_{2j} \vee \cdots x_{nj})$ $\forall j$

Clique $\Leftrightarrow$ Independent set

Are there k pairwise nonadjacent vertices?

Complement graph

Clique $\Leftrightarrow$ Vertex cover

Are there k vertices "covering" all edges

S a vertex cover in G iff

$V-S$ is an independent set in G iff

$V-S$ is a clique in $\bar{G}$

P = problems solvable in p-time

NP = yes-no problems s.t. if answer is "yes", can be verified in p-time given a (p-length) "proof" (hint).

p-time on a Turing machine

or random-access machine