Lecture T5: NP-Completeness

CS Building West Wall, Circa 2001

Overview

Lecture T3:

- What is an algorithm?
  - Turing machine
- Which problems can be solved on a computer?
  - not the halting problem

Lecture T4:

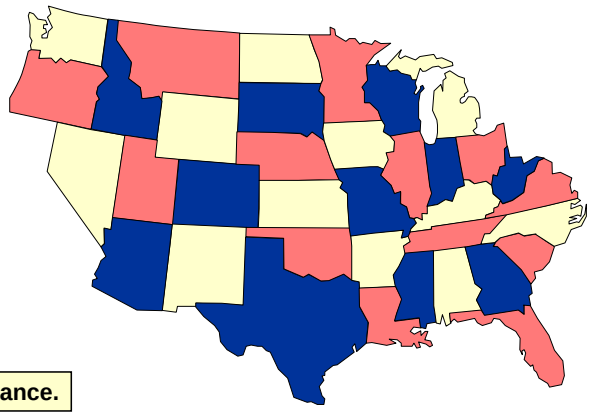
- Which algorithms will be useful in practice?
  - analysis of algorithms

This lecture:

- Which problems can be solved in practice?
  - probably not TSP

Some Hard Problems

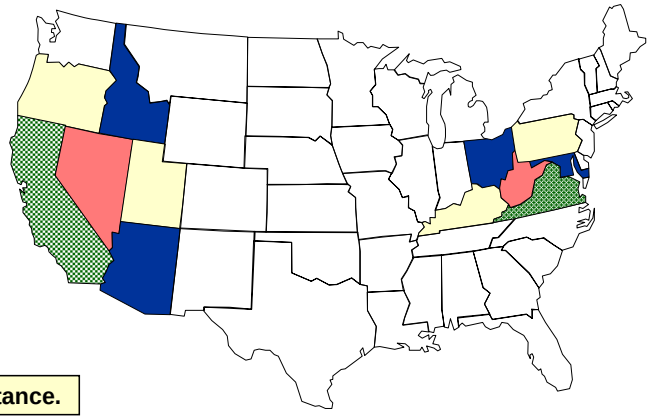
3-COLOR: Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?



YES instance.

Some Hard Problems

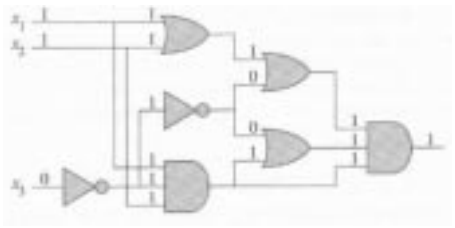
3-COLOR: Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?



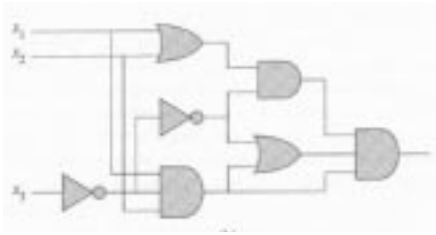
NO instance.

Some Hard Problems

CIRCUIT-SAT: Is there a way to assign inputs to a given Boolean (combinational) circuit that makes it true?



YES instance.



NO instance.

Some Hard Problems

FACTOR: Given two positive integers x and U, is there a nontrivial factor of x that is less than U?

- Factoring is at the heart of RSA encryption.

Example 1: x = 23,536,481,273, U = 110,000.

- YES: x = 104,729 × 224,737.

Example 2: x = 23,536,481,273, U = 100,000.

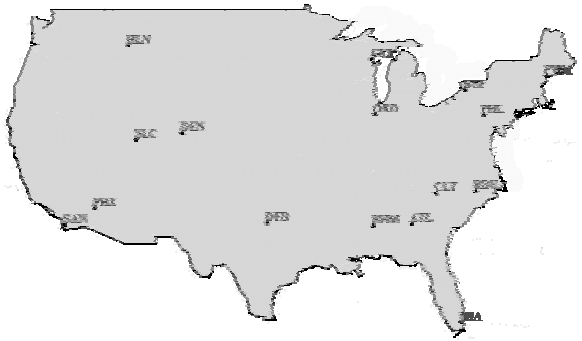
- NO: 104,729 × 224,737 is prime factorization of x.

Example 3: x = 23,536,481,277, U = 23,536,481,277.

- NO: x is prime.

Some Hard Problems

TSP: A travelling salesperson needs to visit N cities. Is there a route of length at most D?



More Hard Problems

More hard computational problems.

- **Biology:** protein folding.
- **Chemistry:** chemical synthesis.
- **Civil engineering:** equilibrium of urban traffic flow.
- **Finance:** find minimum risk portfolio of given return.
- **Electrical engineering:** VLSI layout.
- **Medicine:** reconstructing 3-D shape from biplane angiocardiogram.
- **Operations research:** optimal resource allocation.
- **Physics:** anti-ferromagnetic Potts model.
- **Politics:** Shapley-Shubik voting power.
- **Pop culture:** Minesweeper consistency.
- **Statistics:** optimal experimental design.

Properties of Algorithms

A given problem can be solved by many different algorithms (TMs).

- Which ones are useful in practice?

A working definition: (Jack Edmonds, 1962)

- Efficient: polynomial time for ALL inputs.
  - mergesort requires N $\log_2 N$ steps
- Inefficient: "exponential time" for SOME inputs.
  - brute force TSP takes N! > 2^N steps

Broad and robust definition has led to explosion of useful algorithms for wide spectrum of problems.

Exponential Growth

Exponential growth dwarfs technological change.

- Suppose each electron in the universe had power of today's supercomputers . . .
- And each works for the life of the universe in an effort to solve TSP problem via brute force N! algorithm.

Some Numbers

Quantity	Number
Supercomputer instructions per second	10^{12}
Second per year	10^{9}
Age of universe in years †	10^{13}
Electrons in universe †	10^{79}

† Estimated

- Will not succeed for 1,000 city TSP!

  $1000! >> 10^{1000} >> 10^{79} * 10^{13} * 10^{9} * 10^{12}$

Exponential Growth

"Every year since 1950, the number of American children gunned down has doubled."

- Sociology thesis proposal in 2000.

What do YOU think?

- 2^{50} = 1,125,899,906,842,624
- Joel Best: worst ever social statistic.
  - in Damned lies and statistics: untangling numbers from the media, politicians, and activists
- Thesis cites 1994 Yearbook of the Children's Defense Fund.

Properties of Problems

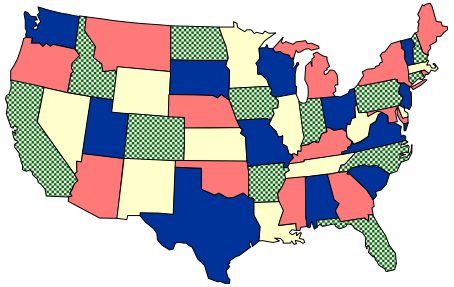
Which PROBLEMS will we be able to solve in practice?

- Those with efficient (polynomial-time) algorithms.

How can I tell if I am trying to solve such a problem?

- 2-COLOR: yes
- 3-COLOR: probably no
- 4-COLOR: yes

- No easy answer! Theory of "NP-completeness" helps.



Theorem (Appel-Haken, 1976).
Every planar map is 4 colorable.

P

Definition of P:

- Set of all decision problems solvable in polynomial time on a deterministic Turing machine.

MULTIPLE: Is the integer y a multiple of x?

- YES: (x, y) = (17, 51).
- NO: (x, y) = (17, 50).

RELPRIME: Are the integers x and y relatively prime?

- YES: (x, y) = (34, 39).
- NO: (x, y) = (34, 51).

Definition important because of Strong Church-Turing thesis.

Strong Church-Turing Thesis

Strong Church-Turing thesis:

- P is the set of all decision problems solvable in polynomial time on REAL computers.

Evidence supporting thesis:

- True for all physical computers.
  - can create deterministic TM that EFFICIENTLY simulates any existing digital computer.

Possible exception?

- Quantum computers – no conventional gates.

NP

EXP: set of all decision problems solvable in exponential time on a deterministic Turing machine.

NP: does NOT mean "not polynomial."

NP: set of all decision problems with efficient certification algorithm.

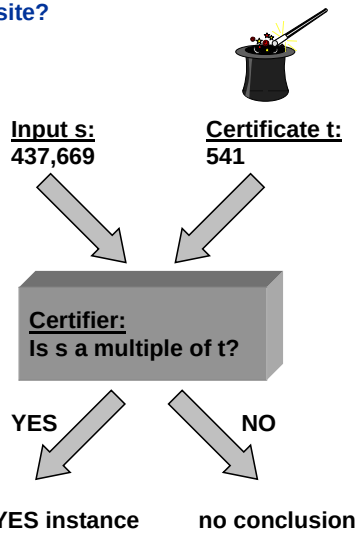
- Efficient: polynomial number of steps on deterministic TM.
- Certifier: check whether a proposed "solution" is correct.
  - proposed solution is called CERTIFICATE (a hint)
  - technical condition: certificate must be of polynomial-size

Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

Observation. s is composite ⇔ there exists an integer 1 < t < s such that s is a multiple of t.

- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)

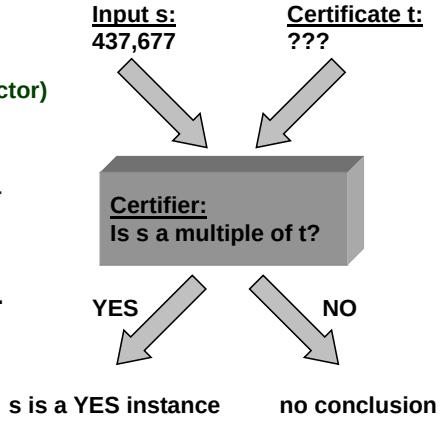


Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

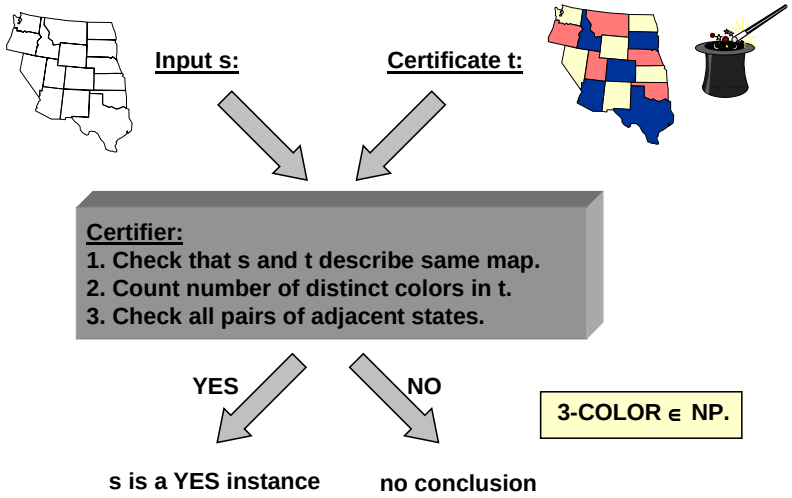
Observation. s is composite ⇔ there exists an integer 1 < t < s such that s is a multiple of t.

- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)
- NO instance: s = 437,677.
  - no certificate can fool verifier into saying YES
- Conclusion: COMPOSITE ∈ NP.



Certifiers and Certificates

3-COLOR: Given planar map, can it be colored with 3 colors?



Alternate Definition of NP

NP: set of decision problems with efficient certification algorithms.

NP: set of all decision problems solvable in polynomial time on a NONDETERMINISTIC Turing machine.

- Equivalent definition.
- Intuition: nondeterministic TM can guess and check all possible solutions in parallel.
- Real computer can simulate nondeterministic TM, but takes exponential time unless you get "lucky."

$P \subseteq NP \subseteq EXP$

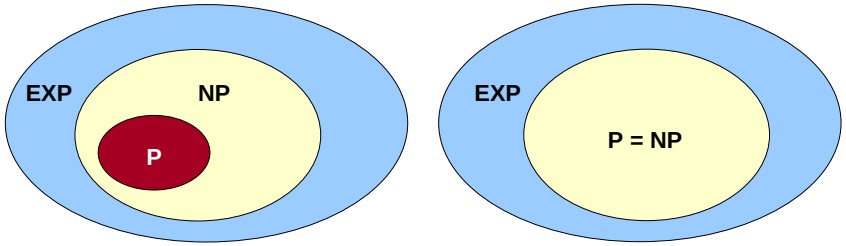
The Main Question

Does P = NP? (Edmonds, 1962)

- Is the original DECISION problem as easy as CERTIFICATION?
- Does nondeterminism help you solve problems faster?

Most important open problem in computer science.

- If yes, staggering practical significance.
- Clay Foundation Millennium $1 million prize.



The Main Question

Does P = NP?

- Is the original DECISION problem as easy as CERTIFICATION?

If yes, then:

- Efficient algorithms for 3-COLOR, TSP, FACTOR.
- Cryptography is theoretically impossible (except for one-time pads) on conventional machines.
- Modern banking system will collapse.
- Harmonial bliss.

If no, then:

- Can't hope to write efficient algorithm for TSP.
  - see NP-completeness

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as CERTIFICATION?

Probably no, since:

- Thousands of researchers have spent four decades in search of polynomial algorithms for many fundamental NP problems without success.
- Consensus opinion: $P \neq NP$.

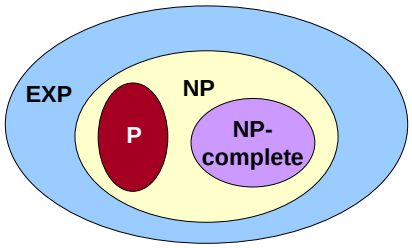
But maybe yes, since:

- No success in proving $P \neq NP$ either.

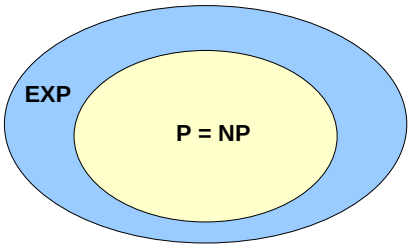
NP-Complete

Definition of NP-complete:

- A problem in NP with the property that if it can be solved efficiently, then it can be used as a subroutine to solve any other problem in NP efficiently.
- "Hardest computational problems" in NP.



If $P \neq NP$



If $P = NP$

NP-Complete

Links together a huge and diverse number of fundamental problems:

- TSP, 3-COLOR, CIRCUIT-SAT, thousands more.
- Given an efficient algorithm for 3-COLOR, can efficiently solve TSP, CIRCUIT-SAT, FACTOR, etc.
- Can implement any program in 3-COLOR.

Note: FACTOR not known to be NP-complete.

Notorious complexity class.

- Only exponential algorithms known for these problems.
- Called intractable - unlikely that they can be solved given limited computing resources.

Reduction

Reduction is a general technique for showing that one problem is harder (easier) than another.

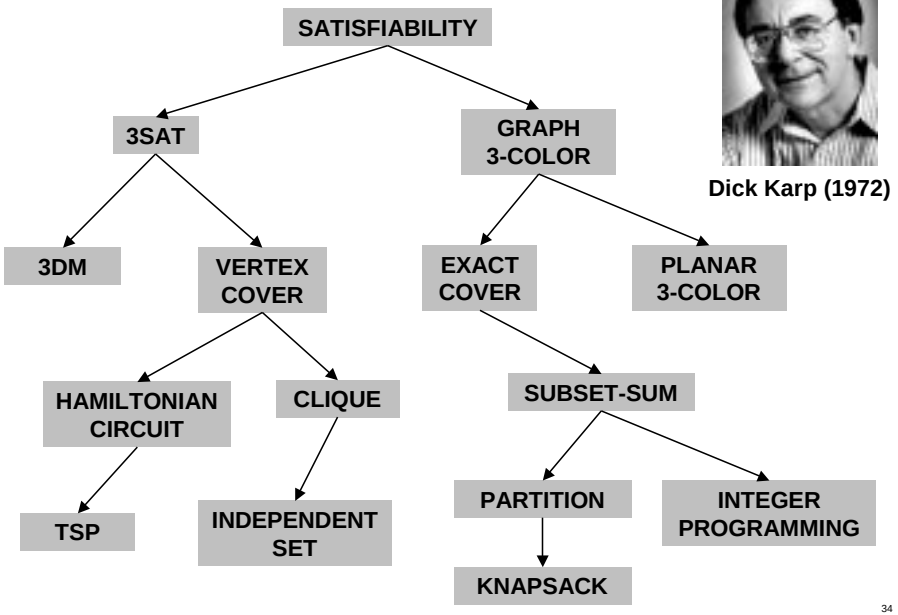
- For problems Y and X, we can often show: if Y can be solved efficiently, then so can X.
- In this case, we say X reduces to Y. (X is "easier" than Y).

Warmup: PRIMALITY reduces to FACTOR.

- Given an efficient algorithm for FACTOR(x, U), want to design an efficient algorithm for PRIMALITY(p).
  - Step 1: Compute FACTOR(p, p).
  - Step 2: If answer = YES, return NO; otherwise return YES.

- Original problem: Is p = 437,669 prime?

Reduction



Dick Karp (1972)

The World's First NP-Complete Problem

SAT is NP-complete. (Cook-Levin, 1960s)

Idea of proof:

- Given problem X ∈ NP, by definition there exists nondeterministic TM M that solves X in polynomial time.
- Use Boolean variables to model which symbol occupies cell i at step t, location of read head at step t, state of finite control at step t, etc.
- Use logic gates to ensure machine makes legal moves, etc.
- SAT instance is satisfiable if and only if TM outputs YES.

Stephen Cook

Coping With NP-Completeness

Hope that worst case doesn't occur.

- Complexity theory deals with worst case behavior. The instance(s) you want to solve may be "easy."
  - TSP where all points are on a line or circle
  - 13,509 US city TSP problem solved

(Cook et. al., 1998)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

- Develop a heuristic, and hope it produces a good solution.
  - TSP assignment
  - Metropolis algorithm, simulating annealing, genetic algorithms
- Design an approximation algorithm: algorithm that is guaranteed to find a high-quality solution in polynomial time.
  - active area of research, but not always possible!
  - Euclidean TSP tour within 1% of optimal

Sanjeev Arora (1997)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

Exploit intractability.

Keep trying to prove P = NP.

Coping With NP-Completeness

A person can be at most two of the following three things:

- Honest.
- Intelligent.
- A politician.

If a problem is NP-complete, you can design an algorithm to do at most two of the following three things:

- Solve the problem to optimality.
- Solve the problem in polynomial time.
- Solve arbitrary instances of the problem.

Summary

Many fundamental problems are NP-complete.

- TSP, CIRCUIT-SAT, 3-COLOR.

Theory says we probably won't be able to design efficient algorithms for NP-complete problems.

- You will surely run into these problems in your scientific life.
- If you know about NP-completeness, you can identify them and avoid wasting time and energy.