Lecture T2: Turing Machines

Overview

Attempt to understand essential nature of computation by studying properties of simple machine models.

Goal: simplest machine that is "as powerful" as conventional computers.

Surprising Fact 1.

Surprising Fact 2.

Adding Power to FSA

FSA advantages:

- Extremely simple model of computation.
- Cheap to implement in hardware.
- Well suited to certain important tasks.
  - pattern matching, filtering, dishwashers, remote controls, traffic lights, sequential circuits

FSA disadvantages:

- Not sufficiently "powerful" to solve numerous problems of interest.

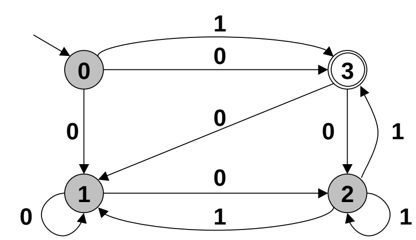
How can we make FSAs more powerful?

- NFSA = FSA + "nondeterminism."
  (ability to guess the right answer!)

Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

- Simple machine with N states.
- Start in state 0.
- Read a bit.
- Depending on current state and input bit
  - move to any of several new states
- Stop when last bit read.
- Accept if ANY choice of new states ends in state X, reject otherwise.



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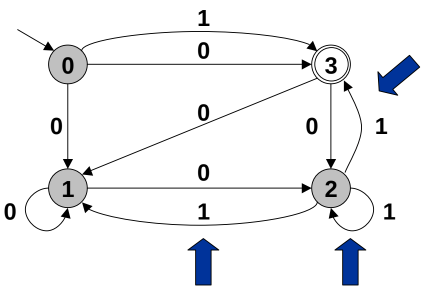

If in state 2, and next bit is 1:
can move to state 1
can move to state 2
can move to state 3

Nondeterministic Finite State Automata

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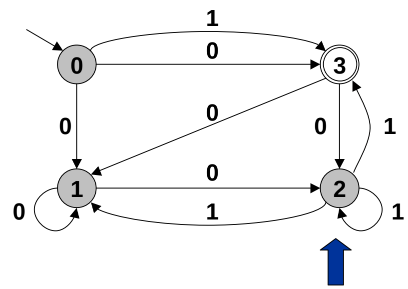

If in state 2, and next bit is 0:
can't move to any state

Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

- Simple machine with N states.
- Start in state 0.
- Read a bit.
- Depending on current state and input bit
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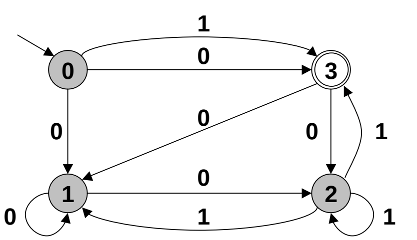


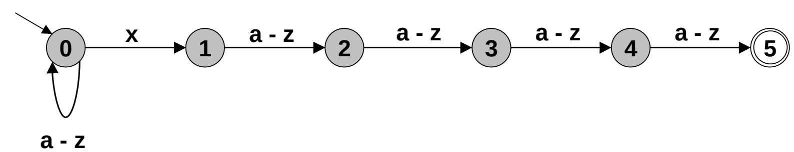
Which strings are accepted?

- 0010001
- 00
- 10000111001100
- 10000111001101

NFSA Example 2

Build an NFSA to match all strings whose 5^{th} to last character is 'x'.

```
% egrep 'x....$' /usr/dict/words
asphyxiate
carboxylic
contextual
inflexible
```



A Systematic Method for NFSA

Harder to determine whether an NFSA accepts a string than an FSA.

- **For FSA, only one possible path to follow.**
- **For NFSA, need to consider many paths.**

Systematic method for NFSA.

- **Keep track of ALL possible states that the NFSA could be in for a given input.**
- **Accept if one of possible ending states is accept state.**

Power of nondeterminism is very useful, but is it essential?

FSA - NFSA Equivalence

Theorem: FSA and NFSA are "equally powerful".

- **Given any NFSA, can construct FSA that accepts same inputs.**

Notation: $X \subseteq Y$.

- **Y is at least as powerful as X.**
- **Machine class Y can be "programmed" to accept all the languages that X can (and maybe more).**

Observation: if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

- **X and Y are equally powerful.**

Proof (Part 1): FSA $\subseteq$ NFSA.

- **A FSA is a special type of NFSA.**

FSA - NFSA Equivalence

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- **X and Y are equally powerful.**

Proof (Part 2): NFSA $\subseteq$ FSA.

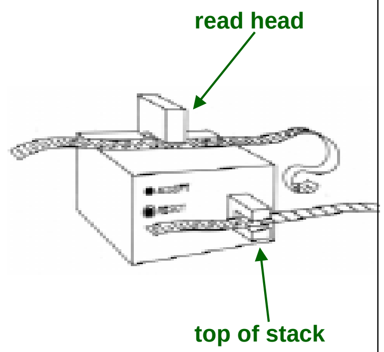
- **Given a nondeterministic FSA, we give recipe to construct a deterministic FSA that recognizes the same language.**
- **One state in FSA for every set of states in the NFSA.**
- **N-state NFSA $\Rightarrow$ 2^N state FSA.**

RE – FSA Equivalence

Theorem: FSA and RE are "equally powerful".

- **We'll spare you the details. ■**
- **Interested students: see supplemental lecture slides.**

Pushdown Automata



How can we make FSAs more powerful?

- Nondeterminism didn't help.
- Instead, add "memory" to the FSA.
- A pushdown stack.
  (amount of memory is arbitrarily large)

Pushdown Automata (PDA).

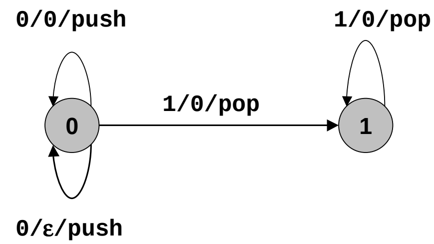
- Simple machine with N states.
- Start in state 0.
- Read a bit, check bit at top of stack.
- Depending on current state/input bit/stack bit:
  - move to new state
  - push the input onto stack, or pop topmost element from stack
- Stop when last bit is read.
- Accept if stack is EMPTY, reject otherwise.

} different accept / reject mechanism

Pushdown Automata

PDA for deciding whether input is of form 0^N1^N .

- N 0's followed by N 1's for some N.
  - `ε, 01, 0011, 000111, 00001111, ...`
- Notation: `x/y/z`
  - if tape input is x and top of stack is y, then do z



Pushdown Automata

How can we make FSA more powerful?

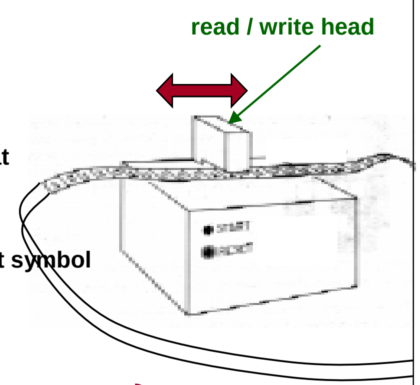
- PDA = FSA + stack.

Did it help?

- More powerful, can recognize:
  - all bit strings with an equal number of 0s and 1s
  - all bit strings of the form 0^N1^N
  - all "balanced" strings in alphabet: `(, {, [, ], }, )`
- Still can't recognize language of all palindromes.
  - `amanaplanacanalpanama`
  - `11*181=1991=181*11`
  - `nolemonsnomelon`

- More powerful machines still needed.

Turing Machine



Turing Machine.

- Simple machine with N states.
- Start in state 0.
- Input on an arbitrarily large TAPE that can be read from *and* written to.
- Read a symbol from tape.
- Depending on current state and input symbol
  - write a symbol to tape
  - move tape right or left
  - move to new state
- Stop if enter yes or no state.
- Accept if yes, reject if no or does not terminate.

} new accept / reject mechanism

Some Examples

Build Turing machines that accepts following languages:

- Equal number of 0s and 1s.
  `#1100#, #0011#, #011101110000#`
- Even length palindromes of 0s and 1s.
  `#0110#, #110011#, #10111000011101#`
- Power of two 1s.
  `#1#, #11#, #1111#, #11111111#`

Notation: `x/y/z`

- if TM head contains character x, then change it to y, and move head in direction z.
- # special character.

C Program to Simulate Turing Machine

Three character alphabet (0 is 'blank').

Position on tape.

- `int head;`

Input: description of machine (9 integers per state s).

- `next[i][s] = t` : if currently in state s and input character read in is i, then transition to state t.
- `out[i][s] = w` : if currently in state s and input character read in is i, then write w to current tape position.
- `move[i][s] = ±1` : if currently in state s and input character is i, then move head one position to left or right.
- `tape[i]` is i^{th} character on tape initially.

Details missing:

- Might run off end of tape.

C Program to Simulate Turing Machine

turing.c

```
#define MAX_TAPE_SIZE   2000
#define STATES           100
#define ACCEPT_STATE      99
. . .
int next[3][STATES], out[3][STATES], move[3][STATES];
char tape[MAX_TAPE_SIZE];
int in, d, state = 0, head = MAX_TAPE_SIZE / 2;

. . . /* read in machine from file */

while (scanf("%1d", &d) != EOF)
   tape[head++] = d;

while (state != ACCEPT_STATE) {
   in = tape[cursor];
   tape[head] =  out[in][state];
   head      += move[in][state];
   state      = next[in][state];
}
```

read in tape (consists of 0, 1, 2)

simulate Turing machine until accept state reached

Nondeterministic Turing Machine

TM with extra ability:

- Choose one of several possible transition states given current tape contents and state.
- No more powerful than deterministic TM.
- Faster than TM? (Stay tuned for NP-Completeness).

Exercise:

- Nondeterministic TM to recognize language of all bit strings of the form ww for some w.
  - 110110
  - 100011110001111
  - 00110001110000111100110001110000111 1

Abstract Machine Hierarchy

Each machine is strictly more powerful than the previous.

- Power = can recognize more languages.

Are there limits to machine power?

Corresponding hierarchy exists for languages.

- Essential connection between machines and languages.

Machine	Nondeterminism adds power?
Finite state automata	No
Pushdown automata	Yes
Linear bounded automata	Unknown
Turing machine	No

Summary

Abstract machines are foundation of all modern computers.

- Simple computational models are easier to understand.
- Lead to deeper understanding of computation.

Goal: simplest machine "as powerful" as conventional computers.

Abstract machines.

- FSA: simplest machine that is still interesting.
  - pattern matching, sequential circuits (Lecture T1)
  - can't recognize: equal number of 0s and 1s
- PDA: add read/write memory in the form of a stack.
  - compiler design
  - can't recognize: palindromes
- TM: add memory in the form of an arbitrarily large array.
  - general purpose computers (Lecture T3)
  - can't recognize: stay tuned

Lecture T2: Supplemental Notes

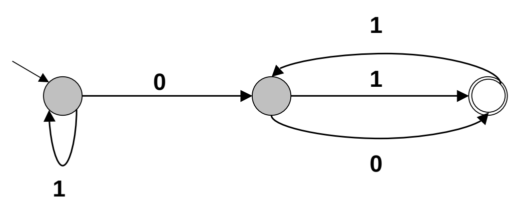
FSA, NFSA, and RE Are Equivalent

Theorem: FSA, NFSA, and RE are "equally powerful".

- NFSA $\subseteq$ FSA

Proof sketch (part 2): FSA $\subseteq$ RE

- Goal: given an FSA, find a RE that matches all strings accepted by the FSA and no other strings.
- Main idea: consider
  - paths from start state(s) to accept state(s): `00 | 01`
  - directed cycles: `(1*)(00 | 01)(11 | 10)*`



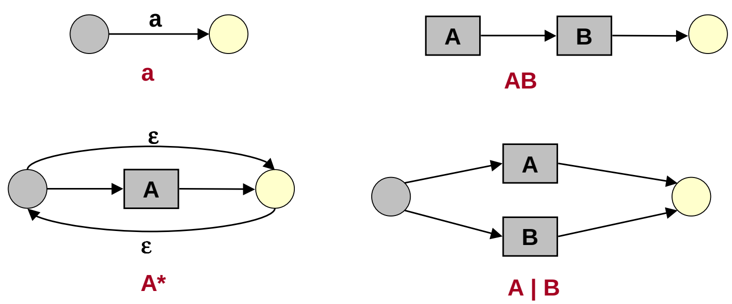
FSA, NFSA, and RE Are Equivalent

Theorem: **FSA, NFSA, and RE are "equally powerful".**

- **NFSA $\subseteq$ FSA $\subseteq$ RE**

Proof sketch (part 3): **RE $\subseteq$ NFSA**

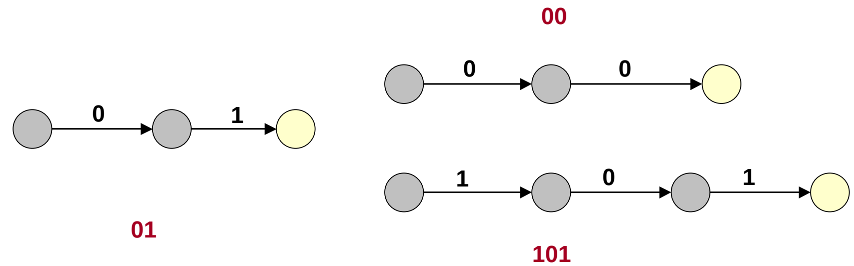
- **Goal: given a RE, construct a NFSA that accepts all strings matched by the RE, and rejects all others.**
- **Use the following rules to construct NFSA:**



FSA, NFSA, and RE Are Equivalent

Example.

- **RE: 01(00 | 101)***



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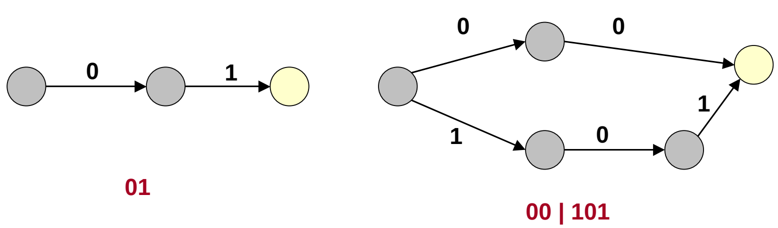
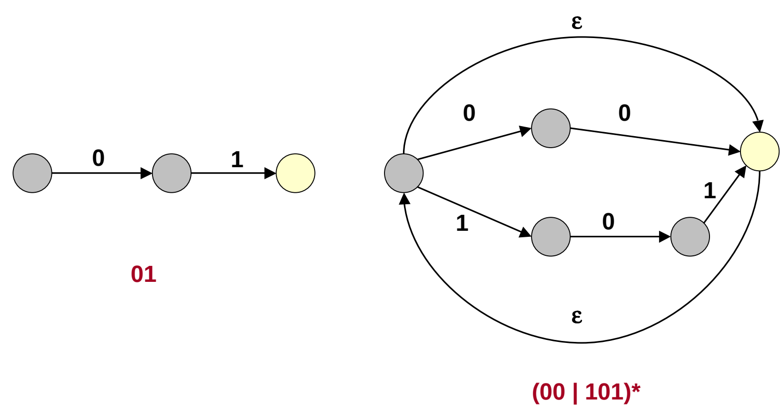


FSA, NFSA, and RE Are Equivalent

Example.

- **RE: 01(00 | 101)***

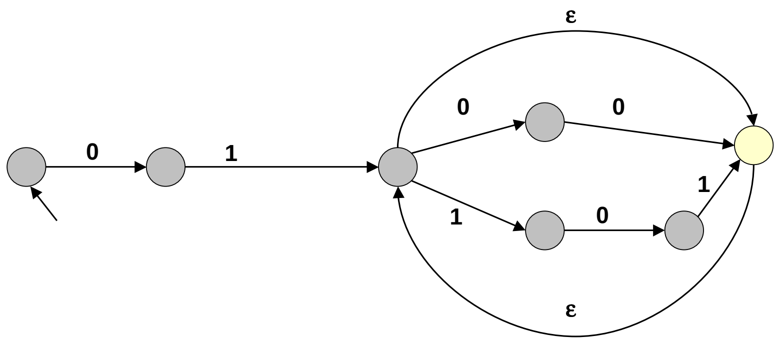
ε - transition: jump states without reading a character to next state



FSA, NFSA, and RE Are Equivalent

Example.

- RE: 01(00 | 101)*



01(00 + 101)*

FSA, NFSA, and RE Are Equivalent

Theorem: FSA, NFSA, and RE are "equally powerful".

- NFSA $\subseteq$ FSA $\subseteq$ RE $\subseteq$ NFSA

Equivalence of languages and machine models is essential in the theory of computation.

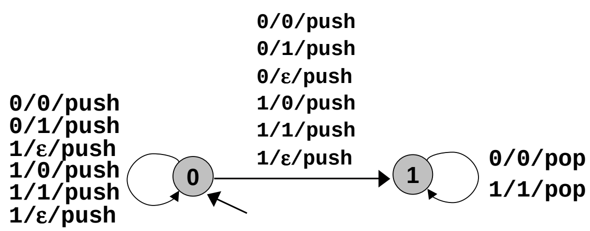
Nondeterminism Does Help PDA's

Nondeterministic pushdown automata (NPDA).

- Same as PDA, except depending on current state/input bit/stack bit
  - move to ANY OF SEVERAL new states
  - push the input onto stack, or pop top-most element from stack

NPDA to recognize all (even length) palindromes.

- Bit string is the same forwards and backwards.



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NPDA to recognize all (even length) palindromes.

- Bit string is the same forwards and backwards.

Nondeterministic PDA more powerful than deterministic PDA.

- PDA $\subseteq$ NPDA trivially.
- PDA cannot recognize language of all (even length) palindromes, but NPDA can.
- Therefore PDA $\subset$ NPDA .

Pushdown Automata

How can we make FSA more powerful?

- NPDA = FSA + stack + nondeterminism.

Did it help?

- Can recognize language of all palindromes.
- Can't recognize some languages:
  - equal number of 0's 1's and 2's
  - $0^N 1^N 2^N$
  - bit strings with a power of two 1's
- Need still more powerful machines.

Linear Bounded Automata

Turing machine.

- No limit on length of tape.

Linear bounded automata (LBA).

- A single tape TM that can only write on the portion of the tape containing the input.
- Note: allowed to increase alphabet size if desired.

LBA is strictly less powerful than TM.

- There are languages that can be recognized by TM but not a LBA.
- We won't dwell on LBA in this course.