Number Systems

- General form of a number in ***base b*** is

$$x = x_n b^n + x_{n-1} b^{n-1} + \ldots + x_1 b^1 + x_0 b^0 + x_{-1} b^{-1} + \ldots + x_{-m} b^{-m}$$

  where x_i are the ***positional coefficients***

- Modern computers use binary arithmetic, i.e., base 2

$$\begin{aligned} 140_{10} &= 1\times10^2 + 4\times10^1 + 0\times10^0 \\ &= 1\times2^7 + 0\times2^6 + 0\times2^5 + 0\times2^4 + 1\times2^3 + 1\times2^2 + 0\times2^1 + 0\times2^0 \\ &= 10001100_2 \\ &= 2\times8^2 + 1\times8^1 + 4\times8^0 = 214_8 \\ &= 8\times16^1 + \mathrm{C}\times16^0 = 8\mathrm{C}_{16} \end{aligned}$$

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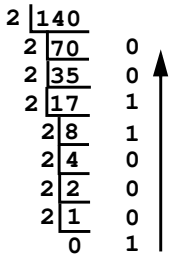
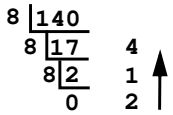
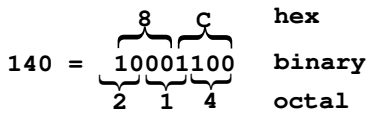
Conversions

- To convert from decimal to binary, divide by 2 repeatedly, read remainders up.





- Easier to convert to octal, then to binary



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Addition

- Addition in base ***b***

$$x_n b^n + x_{n-1} b^{n-1} + x_{n-2} b^{n-2} + \ldots + x_1 b^1 + x_0 b^0$$

$$+ y_n b^n + y_{n-1} b^{n-1} + y_{n-2} b^{n-2} + \ldots + y_1 b^1 + y_0 b^0$$

$$z_{n+1} b^{n+1} + z_n b^n + z_{n-1} b^{n-1} + z_{n-2} b^{n-2} + \ldots + z_1 b^1 + z_0 b^0$$

where $S_i = x_i + y_i + C$, $C = S_{i-1} / b$, and $z_i = S_i \bmod b$ where $S_{-1} = 0$

- Addition in base 2:

```
  00101101
+ 10011001
----------
  11000110
```

- the sum might have ***one*** more digit than the largest operand

Multiplication

- Multiplication in base 2: `00101101 * 10111001`

```
1 00101101
0  00000000
1   00101101
1    00101101
1     00101101
0      00000000
0       00000000
1        00101101
-----------------
  0100000100001O1
```

- The product has about as many digits as the two operands combined, i.e.

$$\log(a \times b) = \log(a) + \log(b)$$

Machine Arithmetic

- Computers usually have a fixed number of binary digits ("bits"), e.g., 32 bits
- For example, using 6 bits, numbered 0 to 5 from the right

  largest number $111111_2 = 63_{10} = 2^6 - 1$

  smallest number $000000_2 = 0$

- What is 50 + 20?

```
  110010
+ 010100
  ------
 1000110
```

- The highest bit doesn't fit, so we get $000110_2 = 6_{10}$
- Spilling over the lefthand side is ***overflow***

Sign Magnitude and One's Complement

- ***Sign-magnitude*** notation:

  bit $n-1$ is the sign; 0 for **+**, 1 for **-**

  bits $n-2$ through 0 hold an unsigned number

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $111111_2 = -31_{10} = -(2^{6-1} - 1)$

- Addition and subtraction are complicated when signs differ
- Sign-magnitude is rarely used
- ***One's-complement*** notation: *-k = (2^n-1) - k = 11111...(n bits) - k*

  bit $n-1$ is the sign; bits $n-2$ through 0 hold an unsigned number

  bits $n-2$ through 0 hold ***complement*** of negative numbers

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $100000_2 = -31_{10} = -(2^{6-1} - 1)$

- Addition and subtraction are easy, but there are ***2*** representations for 0

Two's Complement

- ***Two's-complement*** notation: $-k = 2^n - k = (2^n - 1) - k + 1$

  bit $n-1$ is the sign; bits $n-2$ through 0 hold an unsigned number

  bits $n-2$ through 0 hold the ***complement*** of a negative number ***plus 1***

  largest number $011111_2 = 31_{10} = 2^{6-1} - 1$

  smallest number $100000_2 = -32_{10} = -2^{6-1}$; note ***asymmetry***

- To negate a 2's compl. number: first complement all the bits, then add 1

	start with	complement	increment	
+6	000110	111001	111010	−6
−6	111010	000101	000110	+6
+0	000000	111111	000000	−0
+1	000001	111110	111111	−1
+31	011111	100000	100001	−31
−31	100001	011110	011111	+31
−32	100000	011111	100000	−32

Two's Complement, Cont'd

- Adding 2's-complement numbers: ignore signs, add unsigned bit strings

```
  +20    010100        -20    101100
+ - 7  + 111001      + + 7  + 000111
  ---    ------        ---    ------
  +13    001101        -13    110011

  +20    010100        -20    101100
+ + 7  + 000111      + - 7  + 111001
  ---    ------        ---    ------
  +27    011011        -27    100101
```

- Signed overflow occurs if

  the carry ***into*** the sign bit differs from the carry ***out*** of the sign bit

```
  +20    010100        -20    101100
+ +17  + 010001      + -17  + 101111
  ---    ------        ---    ------
  -27    100101        +27    011011
```

- Same hardware for ***both*** unsigned and signed, but flags ***two*** conditions

  overflow signed overflow

  carry unsigned overflow

Sign Extension

- To convert from a small signed integer to a larger one, copy the sign bit

	+5	−5
4 bits	0101	1011
8 bits	00000101	11111011

- To convert a large signed integer to a smaller one: check trunced bits

	+5	−5	
8 bits	**00000101**	**11111011**	
4 bits	**0101**	**1011**	**OK!**
	+20	**−20**	
8 bits	00010100	11101100	
4 bits	0100	1100	Bad!

- Hardware does extension, but ***may not*** check for truncation; nor does C

```
short small = -50; long big = small;
printf("%d %d\n, small, big);          -50 -50

long big = 40000; short small = big;
printf("%d %d\n", small, big);         -25536 40000

char c = 255;
printf("%d\n", c);                     -1
```


Floating Point Numbers

- Floating point numbers are like scientific notation

1.386×10^{6}

-3.0083×10^{-14}

4.32×10^{-8}

general form is

$\pm m \times 10^{\pm p}$ — **exponent**

significand

- Significand restricted to range, e.g., $0 \le m < 1$, and fixed number of digits

- Floating point is approx. representation for infinitely many real numbers

$m \times \beta^{k}$ m is an n-bit ***significand*** or ***fraction***

β is the ***base*** (usually 2)

k is the ***exponent***

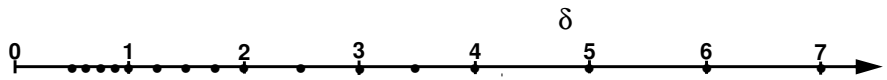
e.g. for base 2

$$0.100011 \times 2^{6} = (1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} + 1 \times 2^{-6}) \times 2^{6}$$

Floating Point Numbers, cont'd

- ***Normalized*** floating point numbers make the representation unique
  most significant digit is nonzero, e.g., $0.00486 \times 10^{1} \Rightarrow 0.486 \times 10^{-1}$
  for floating point numbers, $\beta^{n-1} \le m < \beta^{n}$ or $1/\beta \le |m| < 1$
  i.e., when $\beta = 2$, most significant bit of m is 1
- Example: $n = 3, \beta = 2, -1 \le k \le 2$

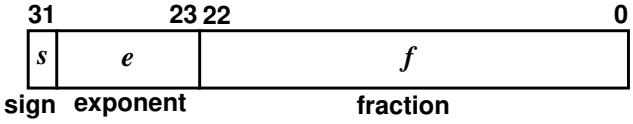
m \ k	-1	0	1	2
1.00	.5	1.	2.	4.
1.01	.625	1.25	2.5	5.
1.10	.75	1.5	3.	6.
1.11	.875	1.75	3.5	7.
	.125	.25	.5	1.



- What about 0.0? Use reserved values of k, e.g.,
  $1.00_2 \times 2^{-2}$ for 0.0, $1.11_2 \times 2^{5}$ for ∞

IEEE Floating Point

- IEEE format uses a ***hidden bit*** to increase precision by 1 bit
  all ***normalized*** floating point numbers have the form $1.f \times 2^{e}$,
  so ***assume*** the leading 1 and omit it
- Single precision (`float`) format



$-126 \le e \le 127$, $bias = 127$, $0 \le f < 2^{23}$

- Values `1.1754943508222875e-38` to `3.4028234663852886000e+38`

$k = e - 127$	f	f. p. number
$-126 \le k \le 127$	$0 \le f < 2^{23}$	$\pm 1.f \times 2^{k}$
128	0	$\pm\infty$
128	$\neq 0$	**NaN (signaling/quiet)**
−127	0	± 0.0
−127	$\neq 0$	$\pm 0.f \times 2^{-126}$ **(denormalized)**

IEEE Floating Point, cont'd

- Double precision (`double`) format

$-1022 \le e \le 1023$, $bias = 1023, 0 \le f < 2^{52}$

- Values: `2.2250738585072014e-308` to `1.7976931348623157e+308`

$k = e - 1023$	f	f. p. number
$-1022 \le k \le 1023$	$0 \le f < 2^{52}$	$\pm 1.f \times 2^k$
1024	0	$\pm\infty$
1024	$\neq 0$	**NaN (signaling/quiet)**
−1023	0	± 0.0
−1023	$\neq 0$	$\pm 0.f \times 2^{-1022}$ **(denormalized)**

- Biased exponents in the most-significant bits are useful because
  - integer compare instructions can be used to compare floating point values
  - a bit string of 0's represents the value 0.0

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