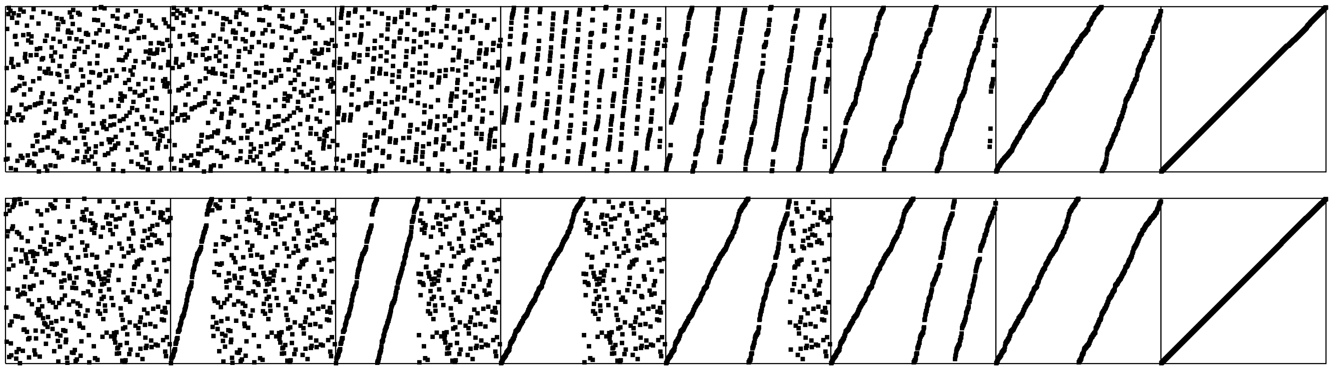
COS 226 Lecture 4: Mergesort

Prototypical divide-and-conquer algorithm



Guaranteed to run in O(N log N) steps

Drawback:

  Linear extra space for the merge

  (can only sort half the memory)

An "optimal" sorting method

Leads us to consider

- recurrence relationships
- computational complexity
- deep hacking
- fractals

Merging two sorted files

```
#define T itemType
merge(T c[], T a[], int N, T b[], int M )
{ int i, j, k;
   for (i = 0, j = 0, k = 0; k < N+M; k++)
      {
         if (i == N) { c[k] = b[j++]; continue; }
         if (j == M) { c[k] = a[i++]; continue; }
         if (less(a[i], b[j]))
            c[k] = a[i++]; else c[k] = b[j++];
      }
}
```

Seems trivial, but try doing it
without using linear extra space

Abstract inplace merge

```
itemType aux[maxN];
merge(itemType a[], int l, int m, int r)
  { int i, j, k;
     for (i = m+1; i > l; i--) aux[i-1] = a[i-1];
     for (j = m; j < r; j++) aux[r+m-j] = a[j+1];
     for (k = l; k <= r; k++)
        if (less(aux[i], aux[j]))
           a[k] = aux[i++]; else a[k] = aux[j--]
  }
```

							A	R	S	T	G	I	N
A	R	S	T	N	I	G							
	R	S	T	N	I	G	A						
	R	S	T	N	I		A	G					
	R	S	T	N			A	G	I				
	R	S	T				A	G	I	N			
		S	T				A	G	I	N	R		
			T				A	G	I	N	R	S	
							A	G	I	N	R	S	T

Tricky way to avoid special tests for
ends of arrays

Still uses extra space, but easier for calling
routine to assume inplace

Mergesort

Sort a file by (recursively) sorting
two halves, then merge the results

A	S	O	R	T	I	N	G	E	X	A	M	P	L	E
A	S													
		O	R											
A	O	R	S											
				I	T									
						G	N							
				G	I	N	T							
A	G	I	N	O	R	S	T							
								E	X					
										A	M			
								A	E	M	X			
												L	P	
												E	L	P
								A	E	E	L	M	P	X
A	A	E	E	G	I	L	M	N	O	P	R	S	T	X

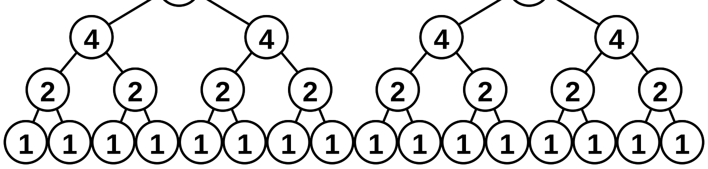
Mergesort implementation

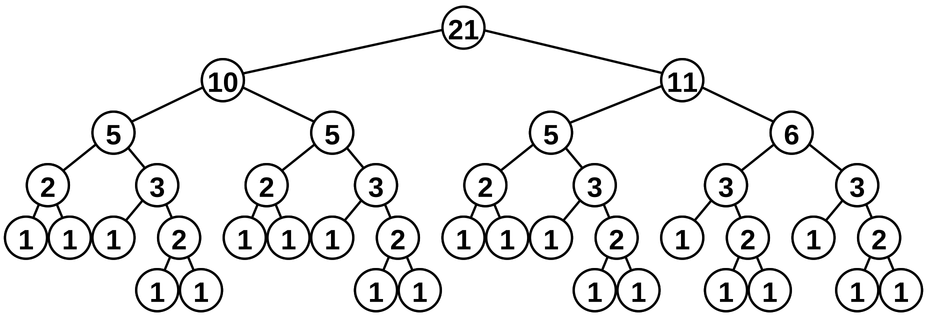
Direct recursive code

Uses abstract inplace merge

```
void mergesort(itemType a[], int l, int r)
  {
    if (r > l)
      { int m = (r+l)/2;
        mergesort(a, l, m);
        mergesort(a, m+1, r);
        merge(a, l, m, r);
      }
  }
```

Tree structures describe merge file sizes





Recurrences

Direct relationship to recursive programs
(most programs are "recursive")

Easy telescoping recurrences

```
T(N) = T(N-1) + 1         T(N) = N
T(2^n) = T(2^(n-1)) + 1  T(N) = lg N if N=2^n
```

Short list of important recurrences

```
T(N) = T(N/2) + 1     T(N) = lg N
T(N) = T(N/2) + N     T(N) = N
T(N) = 2T(N/2) + 1    T(N) = N
T(N) = 2T(N/2) + N    T(N) = N lg N
```

THM: Mergesort uses N lg N comparisons
(guaranteed worst-case bound)

Divide-and-conquer "master theorem" gives
similar results for other algorithms

```
T(N) = aT(N/b) + N^c(lg N)^d
```

Interested in learning more?
- Stay tuned for a few more in CS226
- Take CS 341, CS423
- Read "Introduction to the Analysis of Algs" by Sedgewick and Flajolet

Computational Complexity

Framework to study efficiency of algorithms

Machine model:
    count fundamental operations

Average case:
    predict performance (need input model)

Worst case:
    guarantee preformance (any input)

Upper bound:
    algorithm to solve the problem

Lower bound:
    proof that no algorithm can do better

Caveats:
- worst-case may be unrealistic
- average-case bounds difficult to derive
- costs ignored in analysis may dominate
- machine model may be restrictive

Complexity studies provide
- starting point for practical implementations
- indication of approaches to be avoided

Complexity of sorting

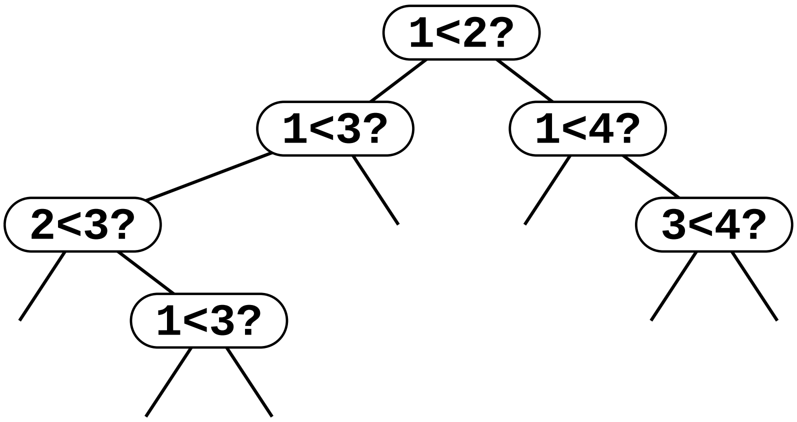
N lg N comparisons necessary and sufficient

Upper bound: N lg N (Mergesort)

Lower bound: N lg N - N/(ln 2) + lg N

THM: All comparison-based sorting methods
must use at least N lg N comparisons

Proof:
DECISION TREE (all sequences of comparisons)



Claim 1: at least N! leaves

Claim 2: height at least lg N!

Claim 3: (Stirling's formula for lg N!)
height at least N lg N - N/(ln 2) + lg N

Caveat: what if we don't use comparisons??
(stay tuned for radix sort)

Deep hacking on Mergesort inner loop

Avoid move with recursive argument switch on call to straight merge.

```
void mergesort(T a[], T b[], int l, int r)
  {
    if (r-l <= 8) insertion(a, l, r); else
      { int m = (l+r)/2;
        mergesort(b, a, l, m);
        mergesort(b, a, m+1, r);
        merge(a+l, b+l, m-l+1, b+m+1, r-m);
      }
  }
void sort(itemType a[], int l, int r)
  { int i;
    for (i = l; i <= r; i++) aux[i] = a[i];
    mergesort(a, aux, l, r);
  }
```

Avoid sentinels with "up-down" trick

Combine the two? Doable, but mindbending

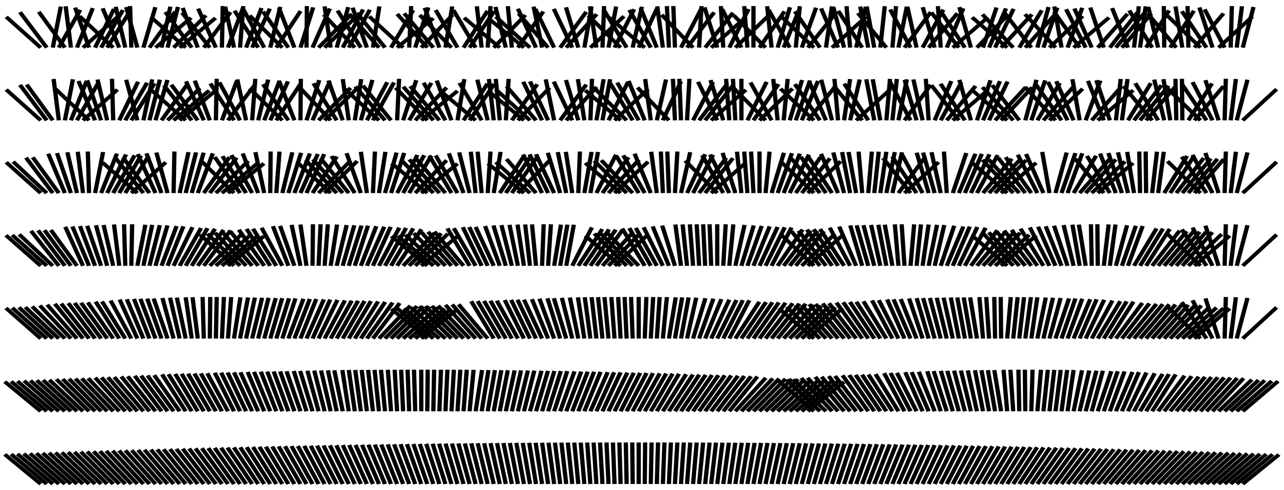
Inner loop: compare, store, two incs
(Quicksort: compare, inc)

Bottom-up mergesort

Pass through the file

- merge adjacent subfiles
- size doubles each time through

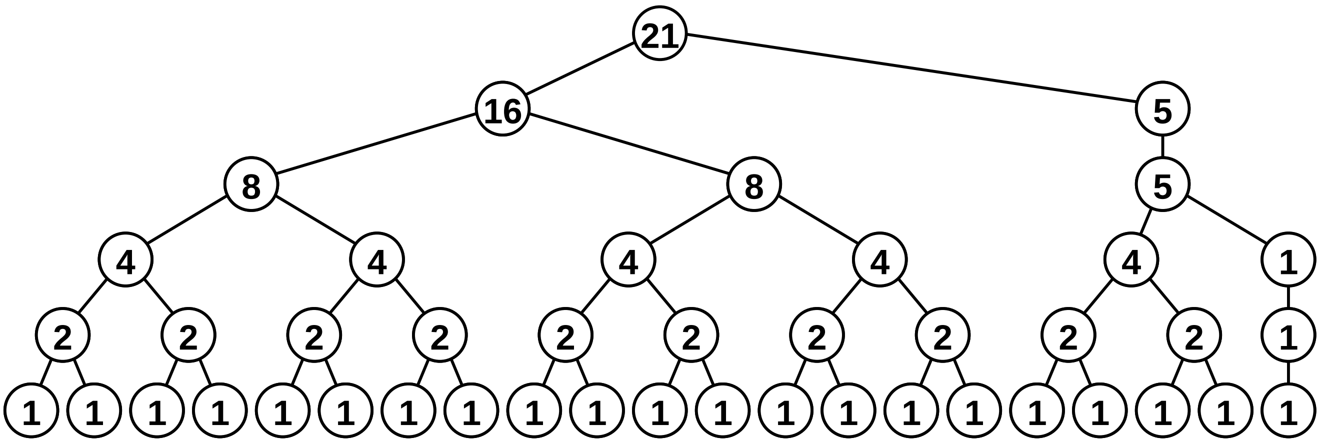
A	S	O	R	T	I	N	G	E	X	A	M	P	L	E
A	S													
		O	R											
				I	T									
						G	N							
								E	X					
										A	M			
												L	P	
A	O	R	S											
				G	I	N	T							
								A	E	M	X			
												E	L	P
A	G	I	N	O	R	S	T							
								A	E	E	L	M	P	X
A	A	E	E	G	I	L	M	N	O	P	R	S	T	X



Bottom-up mergesort implementation

```
void mergesort(itemType a[], int l, int r)
  { int i, m;
      for (m = 1; m < r-l; m = m+m)
         for (i = l; i <= r-m; i += m+m)
            merge(a, i, i+m-1, min(i+m+m-1, r));
  }
```

Different set of merges than for top-down
    unless N is a power of two



Mergesort and numbers

Relationship to binary representation of numbers

THM:
The number of comparisons used by Mergesort
for a file of size N
is the same as
the number of bits in the binary representations
of all the numbers less than N (plus N−1).

0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1

Proof: They satisfy the same recurrence

$$C(2N) = C(N) + C(N) + 2N$$
$$C(2N+1) = C(N) + C(N+1) + 2N+1$$

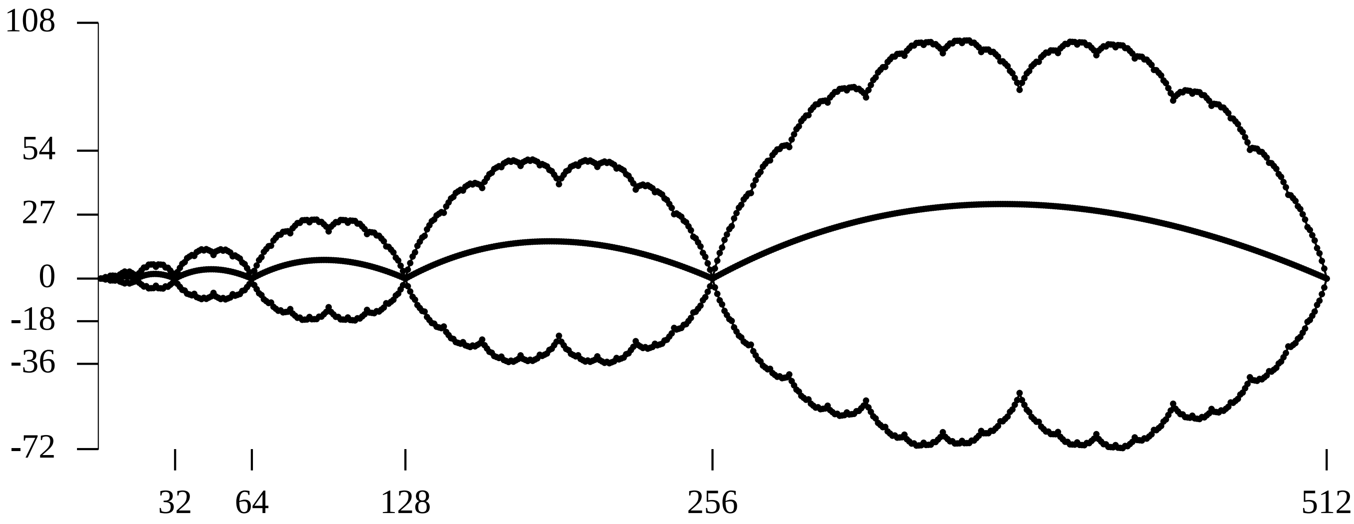
Mergesort and fractals

Divide-and-conquer algorithms
can exhibit erratic periodic behavior

number of bits in numbers less than N
= number of 0 bits
+ number of 1 bits

= (N lg N)/2 + periodic term
+ (N lg N)/2 + periodic term

= N lg N + periodic term



List mergesorts

Top-down: split, sort, and merge

```
struct node *mergesort(struct node *c)
  { struct node *a, *b;
    if (c->next == NULL) return c;

    a = c; b = c->next;
    while ((b != NULL) && (b->next != NULL))
      { c = c->next; b = b->next->next; }
    b = c->next; c->next = NULL;
    return merge(mergesort(a), mergesort(b));

  }
```

Bottom-up: cycle through a circular list

```
struct node *mergesort(struct node *t)
  { struct node *u;
    for (initQ(); t != NULL; t = u)
      { u = t->next; t->next = NULL; putQ(t); }
    t = getQ();
    while (!emptyQ())
      { putQ(t); t = merge(getQ(), getQ()); }
    return t;
  }
```

Mergesort is the method of choice for lists