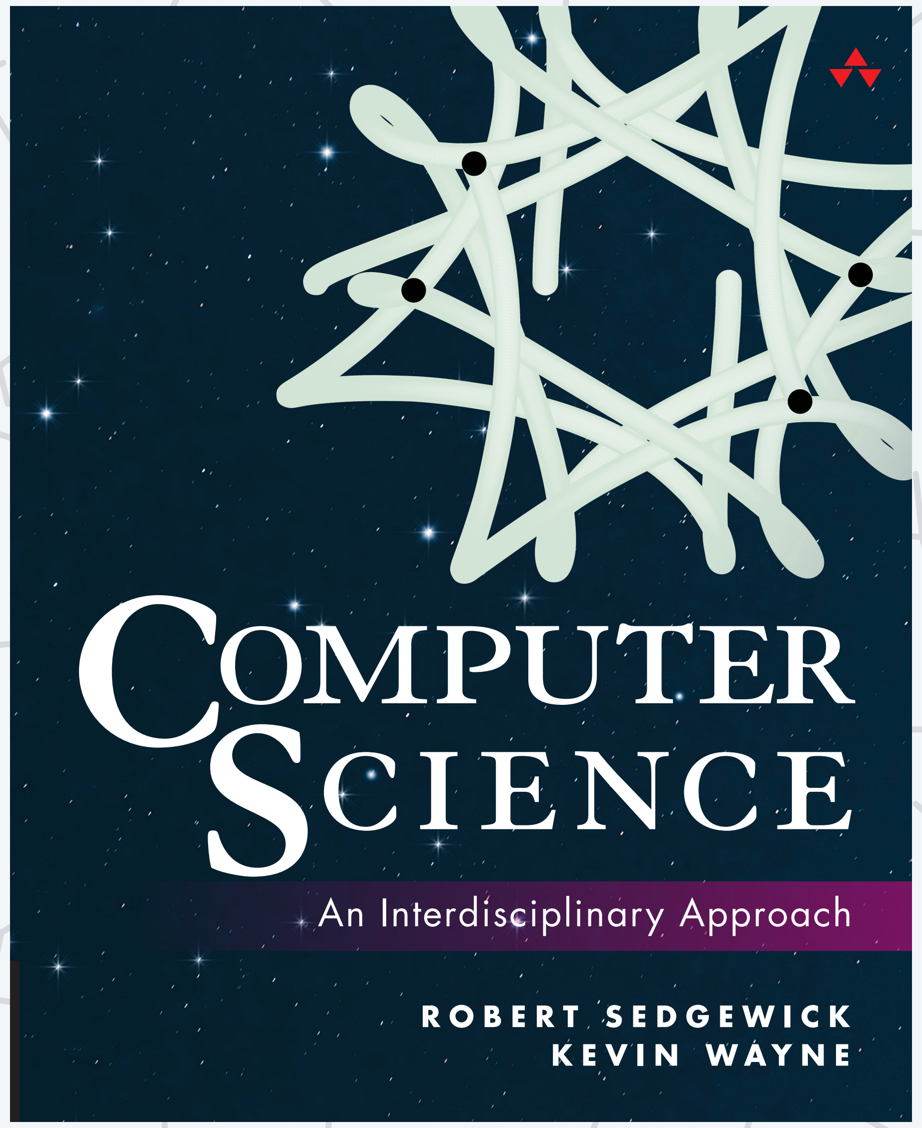

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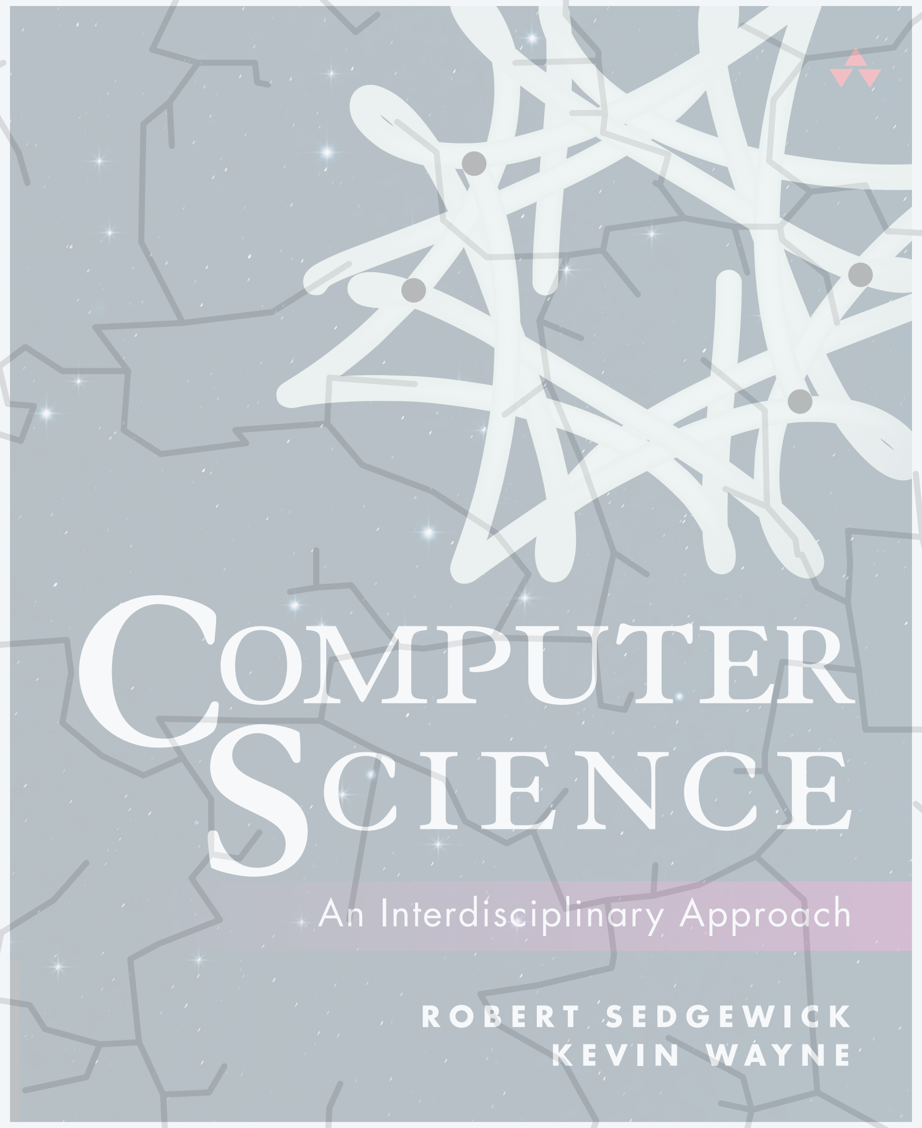
2.3 RECURSION

- *foundations*
- *a classic example*
- *recursive graphics*
- *exponential waste*



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2.3 Recursion

- *foundations*
- *a classic example*
- *recursive graphics*
- *exponential waste*

Overview

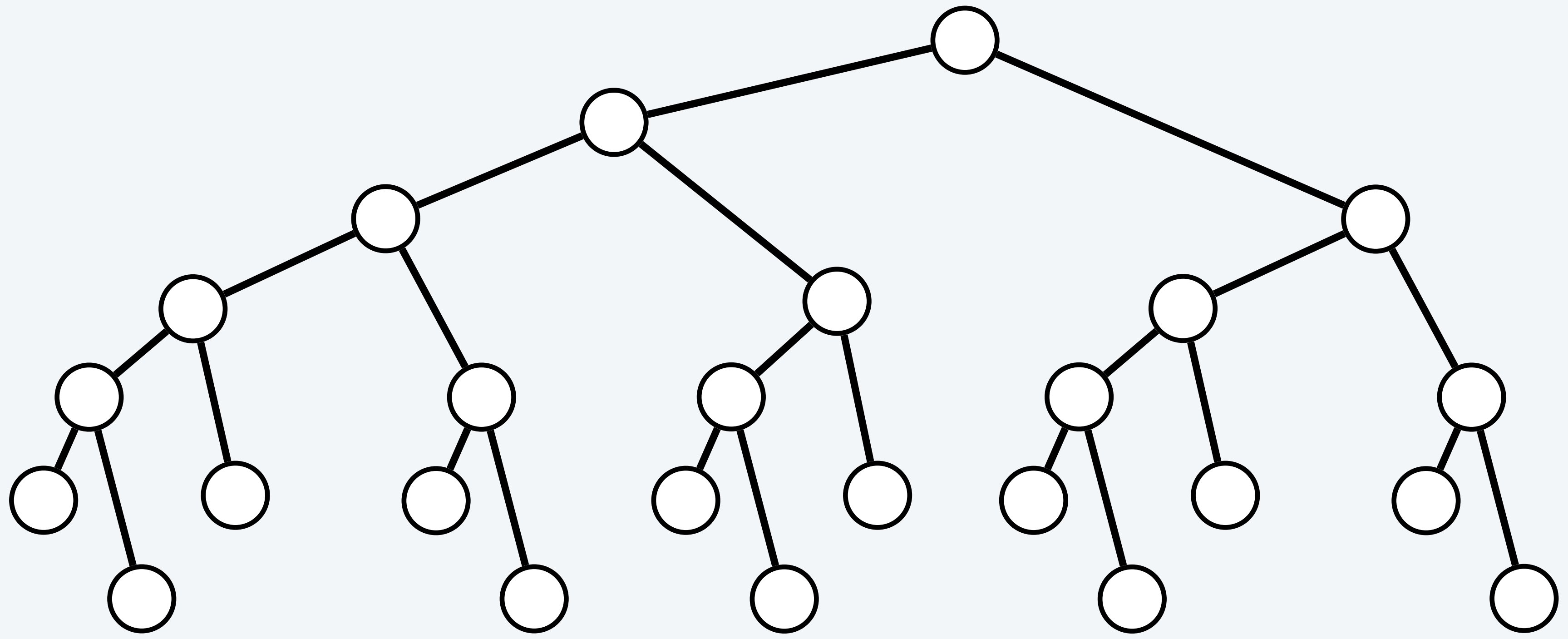
Recursion is when something is specified in terms of itself.

Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Reveals insight into the nature of computation.

Many computational artifacts are naturally self-referential.

- File system with folders containing folders.
- Binary trees.
- Fractal patterns.
- Depth-first search.
- Divide-and-conquer algorithms.
- ...



Recursive functions (in Java)

Recursive function. A function that calls itself.

- **Base case:** If the result can be computed directly, do so.
- **Reduction step:** Otherwise, simplify by calling the function with one (or more) other arguments.

Ex. Factorial function: $n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1.$

- Base case: $1! = 1$
- Reduction step: $n! = n \times (n-1)!$

same function with simpler argument

```
~/cos126/recursion> java-introcs Factorial 3
6

~/cos126/recursion> java-introcs Factorial 4
24

~/cos126/recursion> java-introcs Factorial 5
120
```

recursive function

```java
public class Factorial {
   public static int factorial(int n) {
      if (n == 1) return 1;          // base case
      return n * factorial(n-1);     // reduction step
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      int result = factorial(n);
      StdOut.println(result);
   }
}
```

Review: mechanics of a function call

1. *Evaluate* argument expressions and *assign* values to corresponding parameter variables.
2. *Save environment* (values of all local variables and call location).
3. *Transfer control* to the function.
4. *Restore environment* (with function-call expression evaluating to return value).
5. *Transfer control* back to the calling code.

variable	value
a	100
b	26
max	104

```
public static void main(String[] args) {
    int a = 100;
    int b = 26;

    int max = Math.max(a, 4*b);
    ...

}
```

function-call expression

argument expressions

104

argument values

100 104

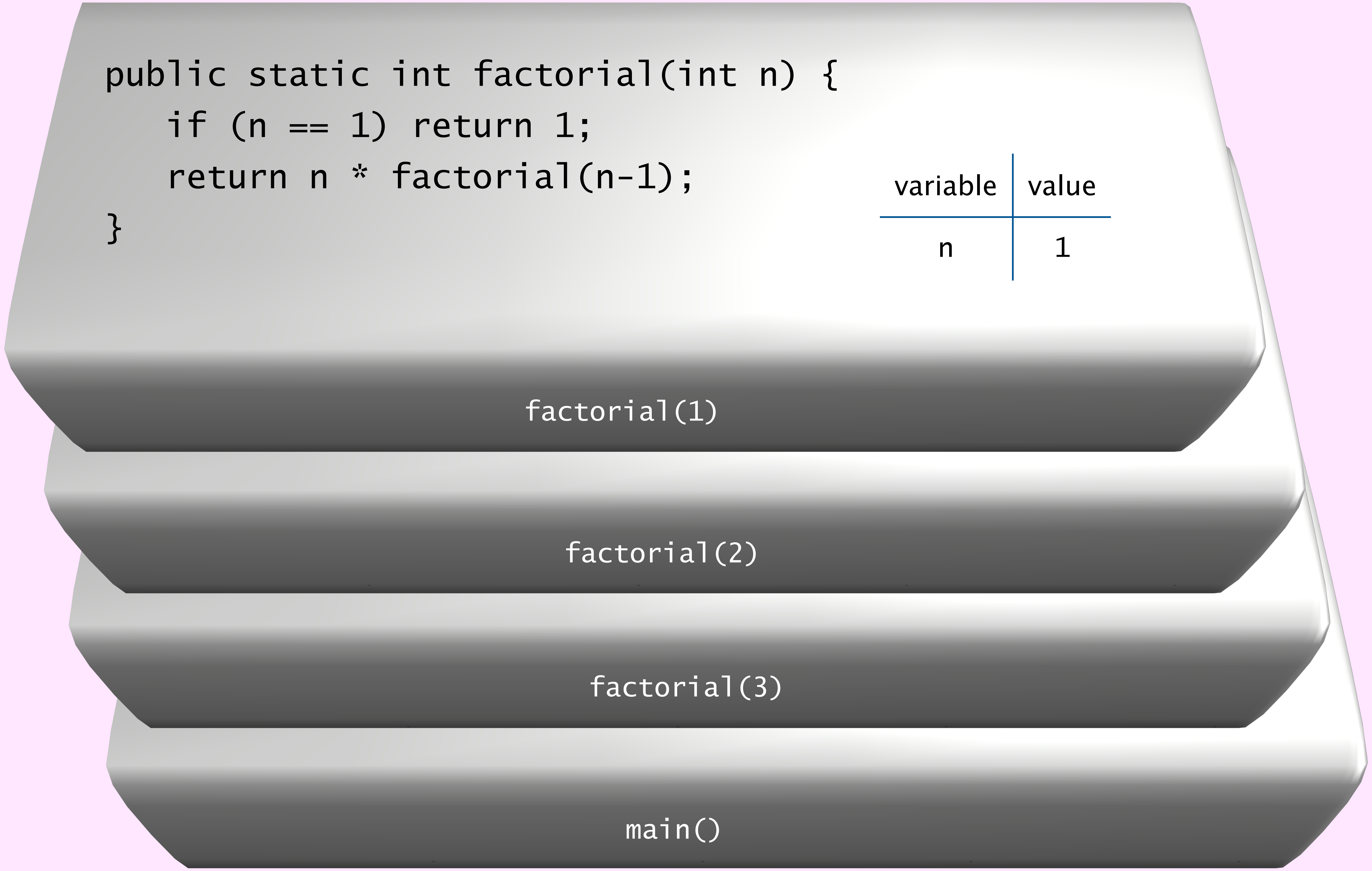
```
public static int max(int x, int y) {
    if (x >= y) return x;
    return y;
}
```

parameter variables

variable	value
x	100
y	104

return value

Factorial function demo



Function-call trace

Function-call trace.

- Print name and arguments when each function is called.
- Print function's return value just before returning.
- Add indentation on function calls and subtract on returns.

```
factorial(5)
    factorial(4)
        factorial(3)
            factorial(2)
                factorial(1)
                    return 1
                return 2 * 1 = 2
            return 3 * 2 = 6
        return 4 * 6 = 24
    return 5 * 24 = 120
```

function-call trace for `factorial(5)`

```
public static int factorial(int n) {
   if (n == 1) return 1;
   return n * factorial(n-1);
}
```

Stack overflow errors

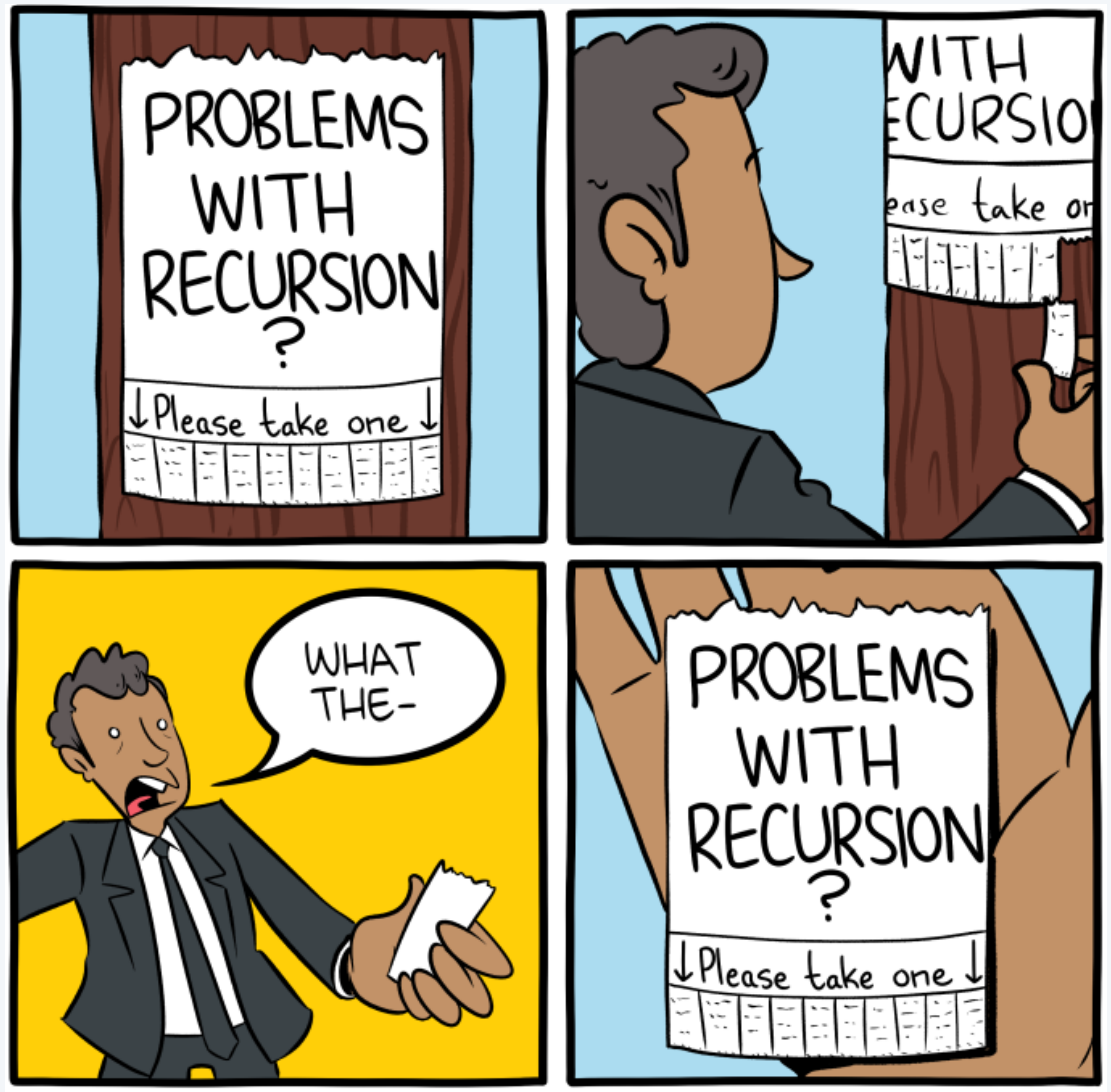
bug	buggy code	error	error message

	`public static int bad1(int n) {`
	`    return n * bad1(n-1);`
	`}`
		missing base case	`~/cos126/recursion> java-introcs Bug1 10`
			`Exception in thread "main"`
			`java.lang.StackOverflowError`
			`   at Bug1.java:4`
			`   at Bug1.java:4`
			`   at Bug1.java:4`
			`   at Bug1.java:4`
			`   ...`

	`public static int bad2(int n) {`
	`    if (n == 0) return 1;`
	`    return n * bad2(n + 1);`
	`}`
		reduction step does not converge to base case	`~/cos126/recursion> java-introcs Bug2 10`
			`Exception in thread "main"`
			`java.lang.StackOverflowError`
			`   at Bug2.java:4`
			`   at Bug2.java:4`
			`   at Bug2.java:4`
			`   at Bug2.java:4`
			`   ...`

stackoverflow

Problems with recursion?



https://www.smbc-comics.com

https://www.safelyendangered.com/comic/oh-bother

What is printed by a call to `collatz(6)` ?

A. 6 3 10 5 16 8 4 2 1

B. 1 2 4 8 16 5 10 3 6

C. 2 4 8 16 5 10 3 6

D. 6 3 1

E. stack overflow error

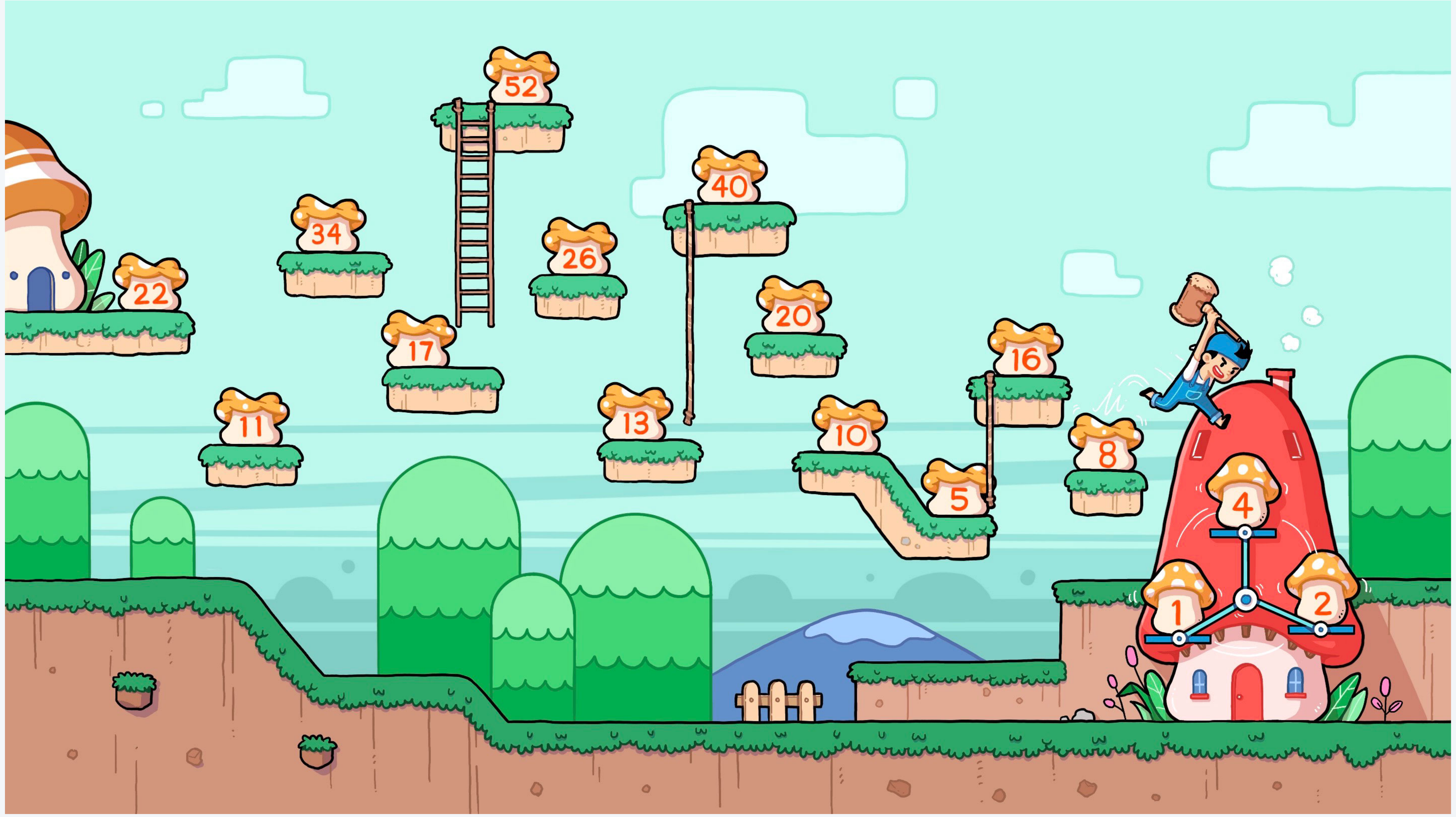
```
public static void collatz(int n) {
   StdOut.print(n + " ");
   if (n == 1) return;
   if (n % 2 == 0) collatz(n / 2);
   else collatz(3*n + 1);
}
```

integer division

Collatz sequence

Famous unsolved problem. Does `collatz(n)` terminate for all $n \geq 1$? ⟵ *assume no arithmetic overflow*

Partial answer. Yes, for all $1 \leq n \leq 2^{68}$.



Saying the digits of a number

Goal. Say the decimal digits in a positive integer n.

- Base case: say nothing when n is zero.
- Reduction step: otherwise,
  - recursively say most significant digits ← `n / 10`
  - then, say the least significant digit ← `n % 10`

```
public static void sayDigits(int n) {
   if (n == 0) return;
   sayDigits(n / 10);
   StdAudio.play((n % 10) + ".wav");
}
```

play WAV file for digit 0–9

```
~/cos126/recursion> java-introcs SayDigits 126
[speaks "1 2 6"]

~/cos126/recursion> java-introcs SayDigits 25000
[speaks "2 5 0 0 0"]
```

```
sayDigits(126)
   sayDigits(12)
      sayDigits(1)
         sayDigits(0)
         play "1.wav"
      play "2.wav"
   play "6.wav"
```

function-call trace of `sayDigits(126)`

Q. How to say digits in binary (instead of decimal)?

A. Replace constant 10 with constant 2.

Recursion: quiz 2

What does `sayDigits(126)` **do with this version of** `sayDigits()` **?**

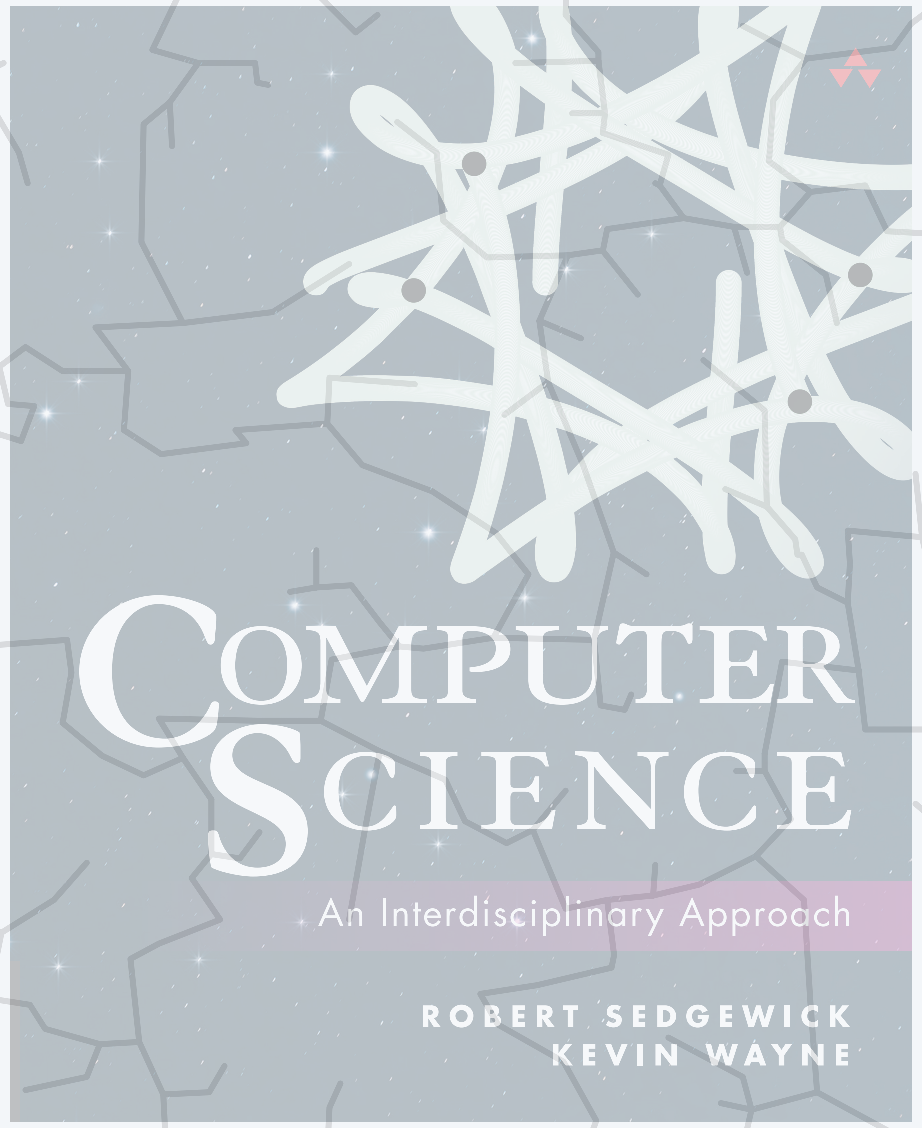
```
public static void sayDigits(int n) {
   if (n == 0) return;
   StdAudio.play((n % 10) + ".wav");
   sayDigits(n / 10);
}
```

the order of these two statements is switched

A. Speaks "1 2 6."

B. Speaks "6 2 1."

C. Stack overflow error.



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2.3 RECURSION

- *foundations*
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Warmup: ruler function (revisited)

Goal. Function `ruler(n)` that returns first $2^n - 1$ values of ruler function.

- Base case: empty for $n = 0$.
- Reduction step: sandwich n between two copies of `ruler(n-1)`.

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

```
public class Ruler {

   public static String ruler(int n) {
      if (n == 0) return " ";           // base case
      return ruler(n-1) + n + ruler(n-1);  // reduction step
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      String result = ruler(n);
      StdOut.println(result);
   }
}
```

base case

reduction step

```
~/cos126/recursion> java-introcs Ruler 1
 1

~/cos126/recursion> java-introcs Ruler 2
 1 2 1

~/cos126/recursion> java-introcs Ruler 3
 1 2 1 3 1 2 1

~/cos126/recursion> java-introcs Ruler 4
 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
```

Tracing a recursive program

Draw the *function-call tree.*

- One node for each function call.
- Label node with return value after children are labeled.

function-call tree for ruler(4)

Which string does `ruler(3)` return for this version of `ruler()` ?

A. `"1 1 2 1 1 2 3 "`

B. `"1 2 1 3 1 2 1 "`

C. `"3 2 1 1 2 1 1 "`

D. `"3 2 2 1 1 1 1 "`

```
public static String ruler(int n) {
   if (n == 0) return "";
   return n + " " + ruler(n-1) + ruler(n-1);
}
```

Towers of Hanoi puzzle

A legend of uncertain origin.

- $n = 64$ disks of differing size; 3 poles; disks on middle pole, from largest to smallest.
- An ancient prophecy has commanded monks to move the disks to another pole.
- When the task is completed, the world will end.

Rules.

- Can move only one disk at a time.
- Cannot put a larger disk on top of a smaller disk.

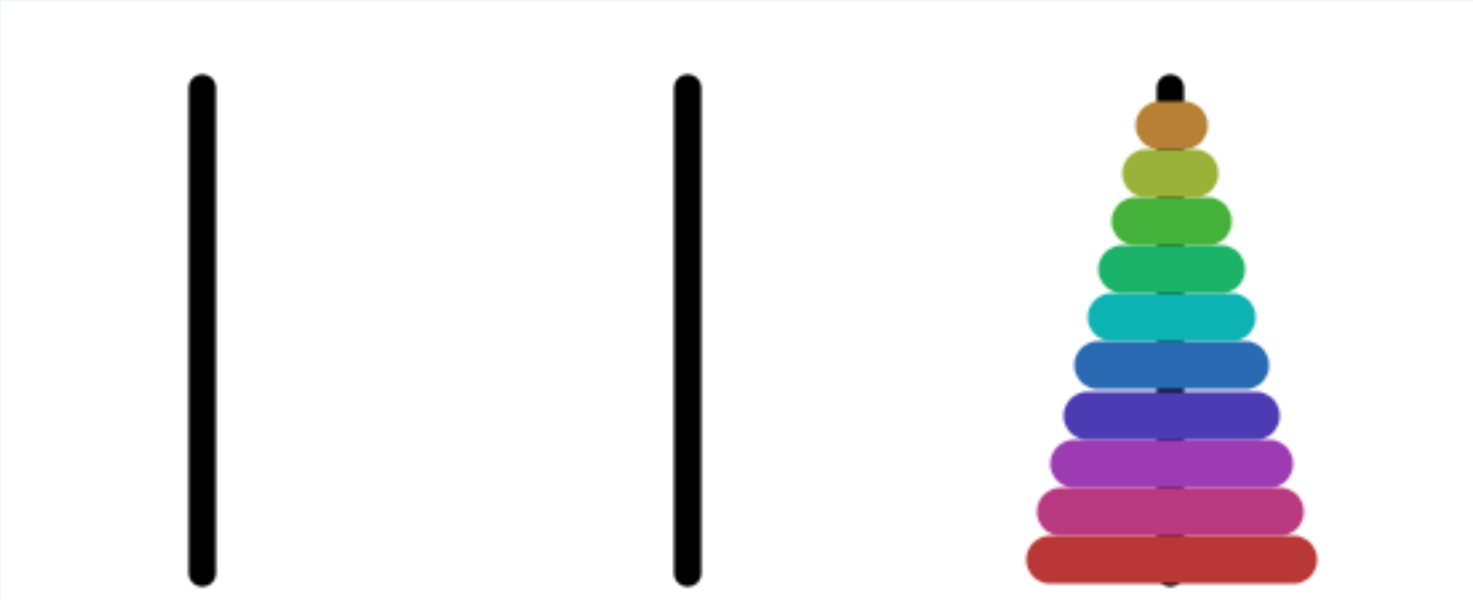
Q1. How to generate a list of instruction for monks.

Q2. When might the world end?

start

n = 10

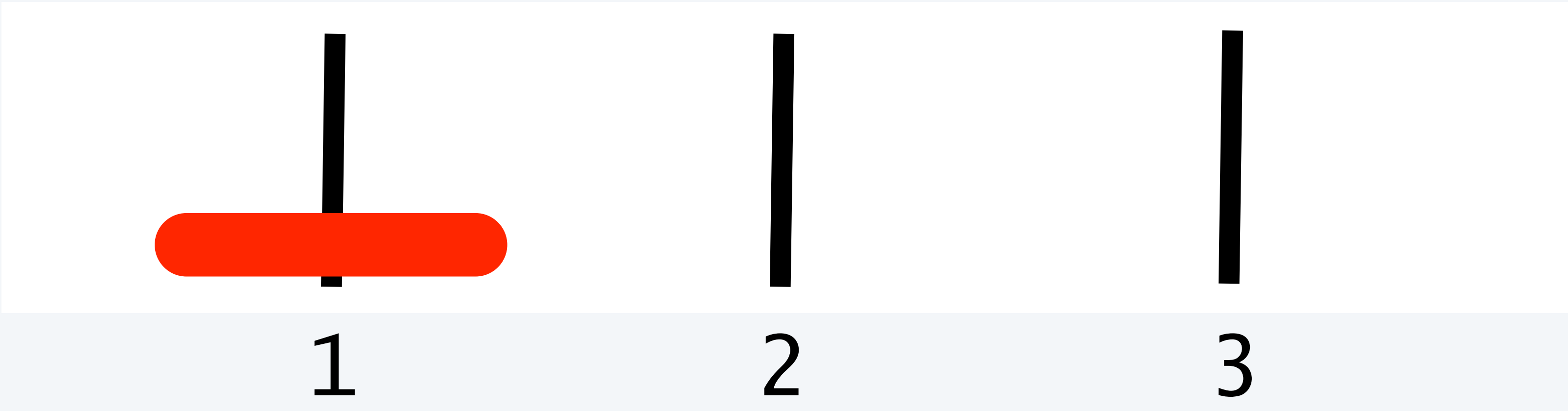
goal



Towers of Hanoi solution

For instructions, use cyclic wraparound.

- Move right means 1 to 2, 2 to 3, or 3 to 1.
- Move left means 1 to 3, 3 to 2, or 2 to 1.



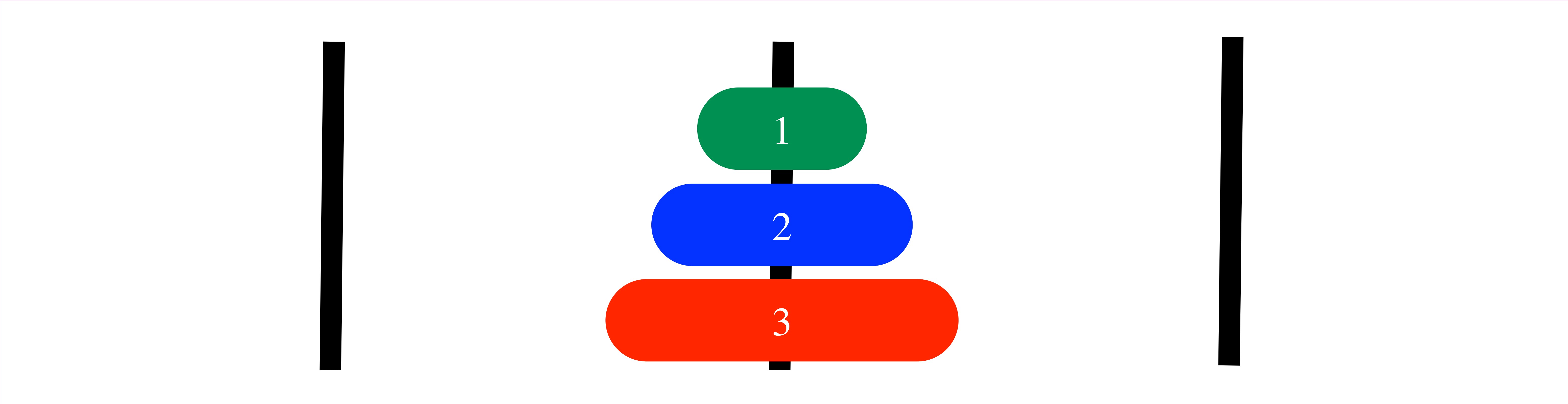
A recursive solution. [to move stack of n disks to the right]

- Base case: if $n = 0$ disks, do nothing.
- Reduction step: otherwise,
  - move $n - 1$ smallest disks to the *left* (recursively)
  - move largest disk to the *right*
  - move $n - 1$ smallest disks to the *left* (recursively)

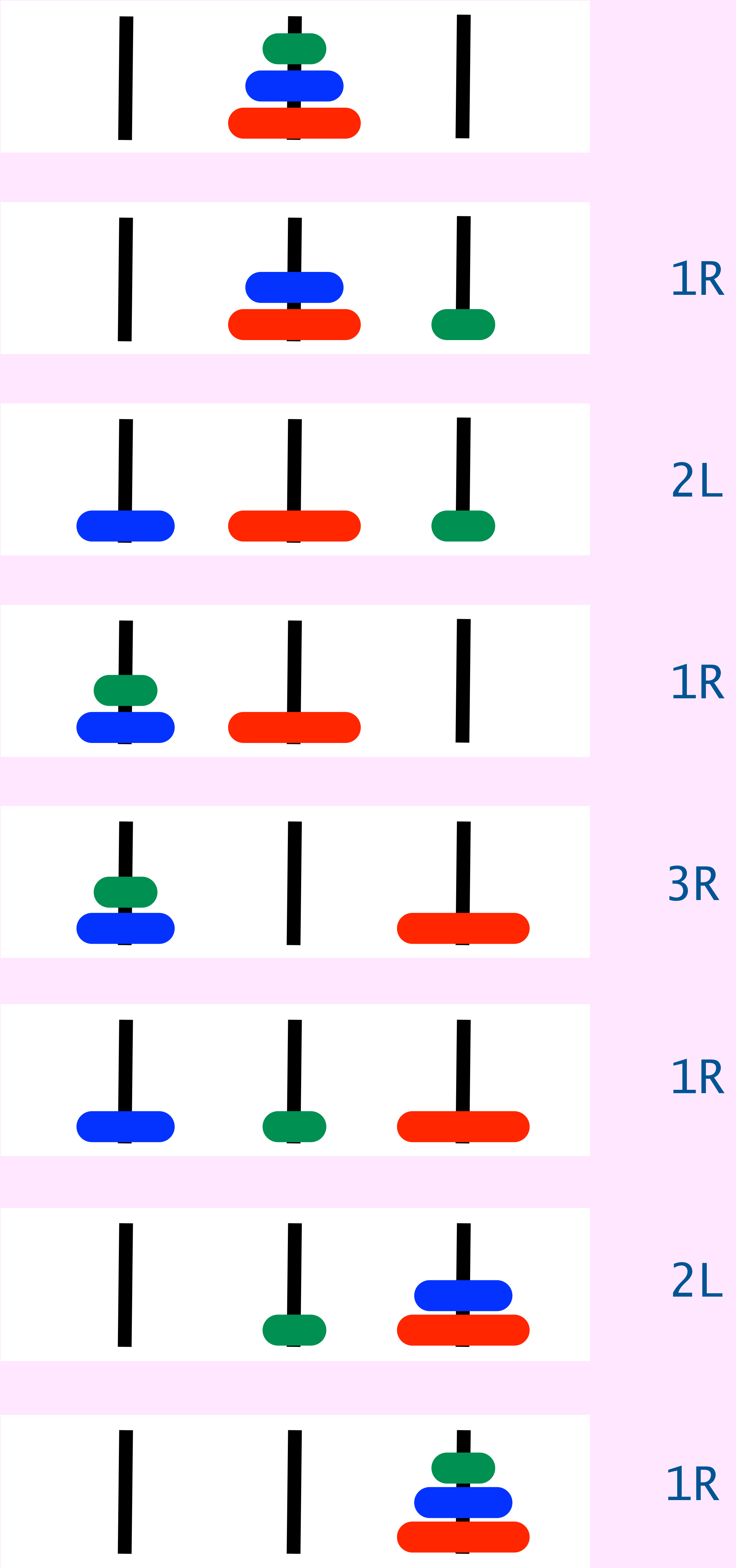
analogous to moving stack of n disks to the right

Towers of Hanoi solution (n = 3)

Notation. Label disks from smallest (1) to largest (n).



1R 2L 1R 3R 1R 2L 1R



Towers of Hanoi: mutually recursive solution

Goal. Function `hanoiRight(n)` that returns instructions for n disk puzzle. ← *and also a similar function* `hanoiLeft(n)`

- Base case: if $n = 0$ disks, do nothing.
- Reduction step: otherwise, sandwich moving disk n right between two calls to `hanoiLeft(n-1)`

```java
public class Hanoi {
   public static String hanoiRight(int n) {
      if (n == 0) return " ";
      return hanoiLeft(n-1) + n + "R" + hanoiLeft(n-1);
   }

   public static String hanoiLeft(int n) {
      if (n == 0) return " ";
      return hanoiRight(n-1) + n + "L" + hanoiRight(n-1);
   }

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      StdOut.println(hanoiRight(n));
   }
}
```

move stack of n disks right (hanoiRight)

move stack of n disks left (hanoiLeft)

⚠ **concise but tricky code; read carefully!**

```
~/cos126/recursion> java-introcs Hanoi 3
 1R 2L 1R 3R 1R 2L 1R

~/cos126/recursion> java-introcs Hanoi 4
 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L
```

Function-call tree for towers of Hanoi

Properties.

- Each disk always moves in the same direction.
- Moving smallest disk always alternates with (unique legal) move not involving smallest disk.
- Solution to puzzle with n disks makes $2^n - 1$ moves.

Answers for towers of Hanoi

Q. How to generate instructions for monks?

A1. [long form] 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L 5L 1L 2R 1L 3L 1L 2R 1L 4R …

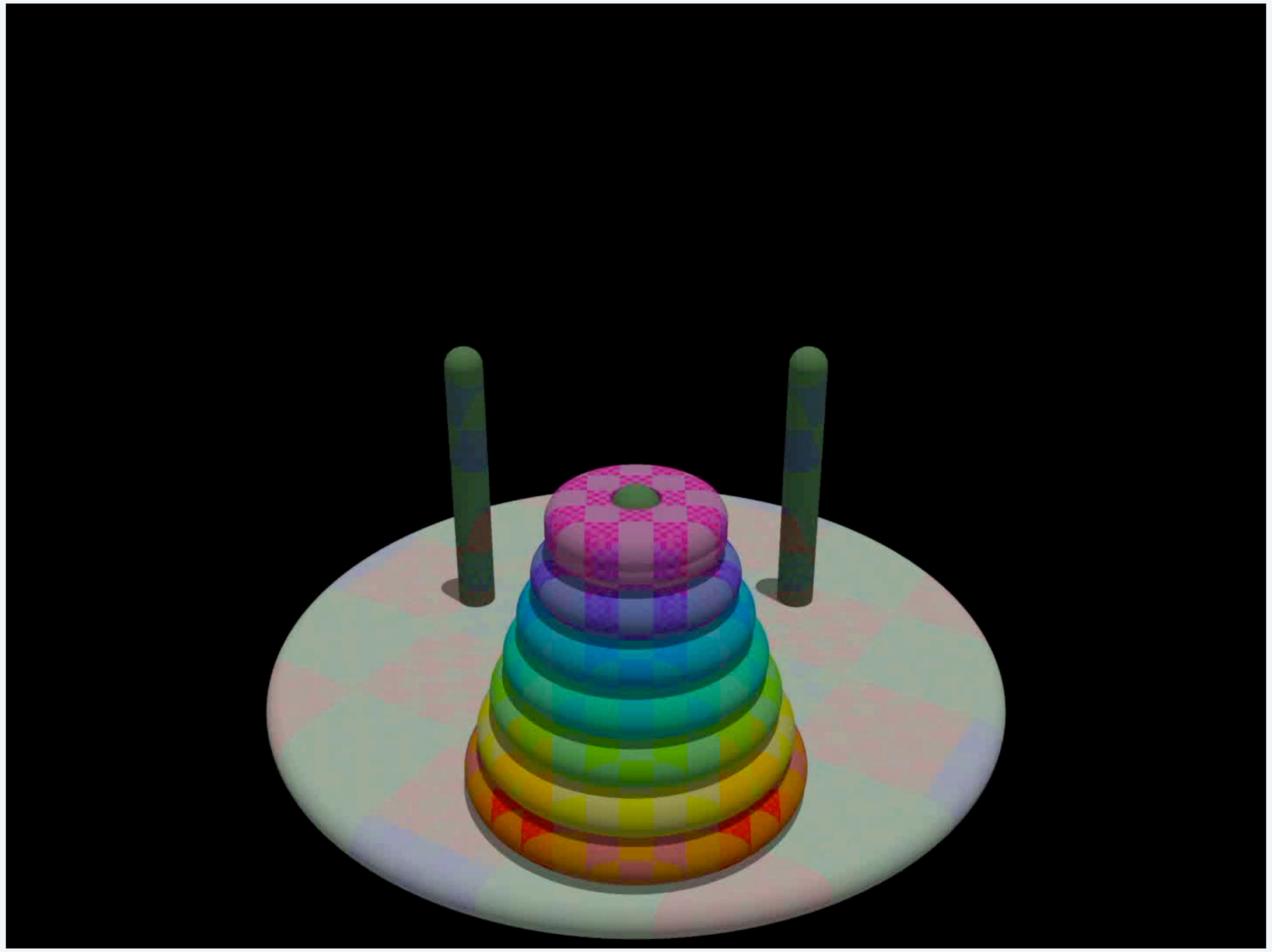
A2. [short form] Alternate 1L with the only legal move not involving disk 1.

if n is odd, alternate 1R

Q. When might the world end?

A. Not soon. Takes $2^{64} - 1$ moves.

recursive solution provably uses fewest moves



Recursion vs. iteration

Fact 1. Any recursive program can be rewritten with loops (and no recursion).

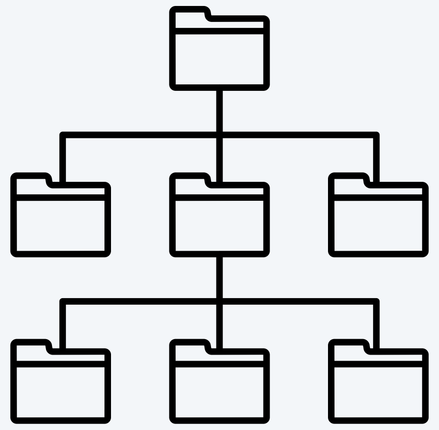
Fact 2. Any program with loops can be rewritten with recursion (and no loops).

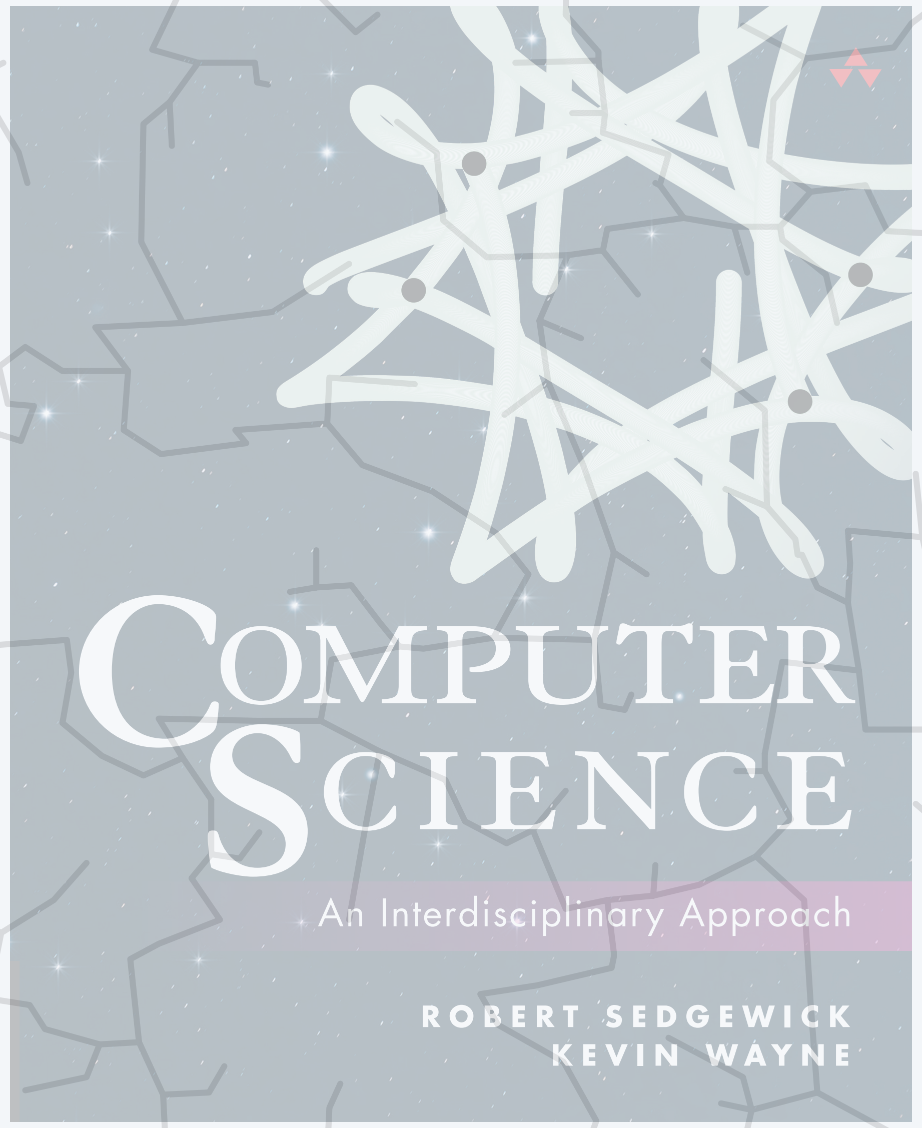
loops	recursion
more memory efficient (no function-call stack)	*concise and elegant code*
easier to trace code (fewer variables)	*easier to reason about code (fewer mutable variables)*

Q. When should I use recursion?

A1. The problem is naturally recursive (e.g., towers of Hanoi).

A2. The data is naturally recursive (e.g., filesystem with folders).





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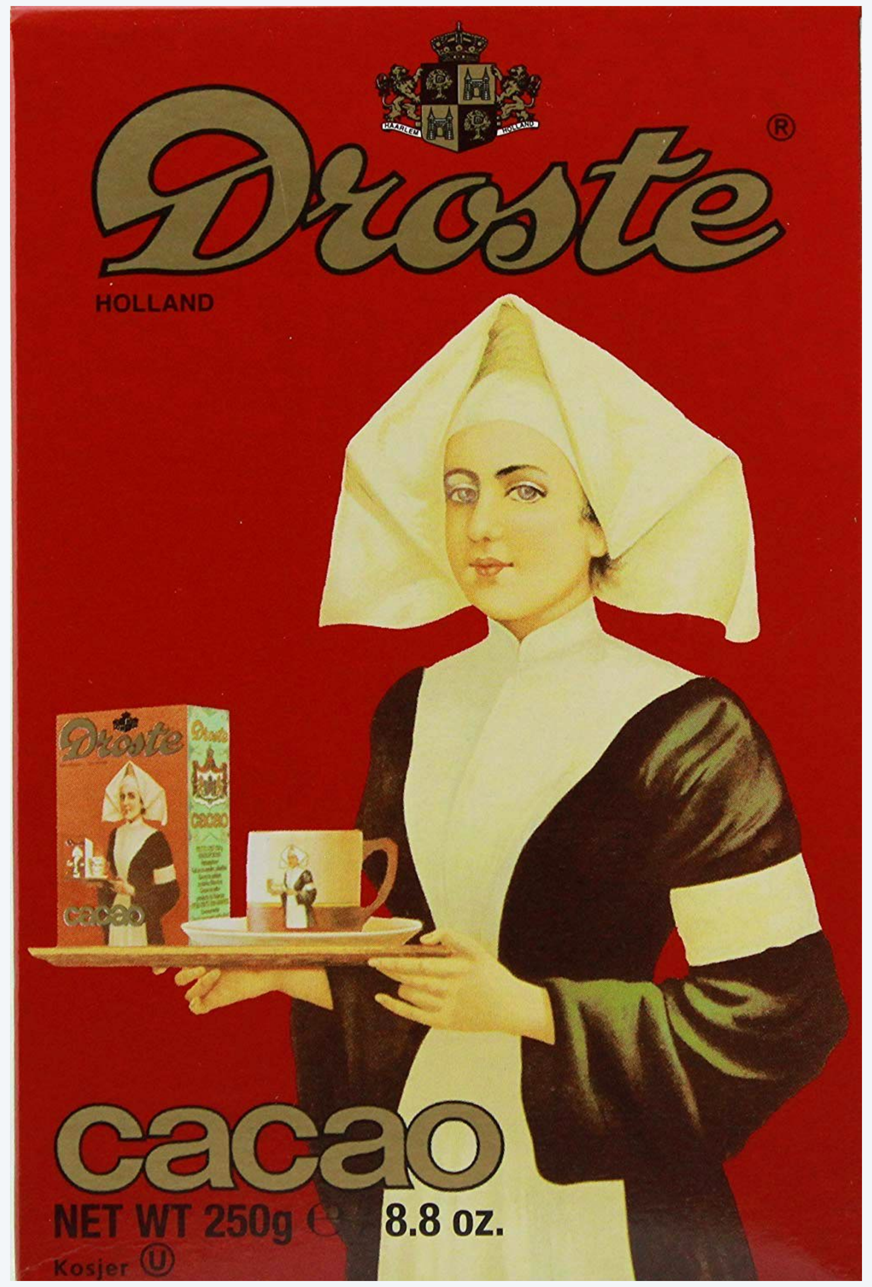
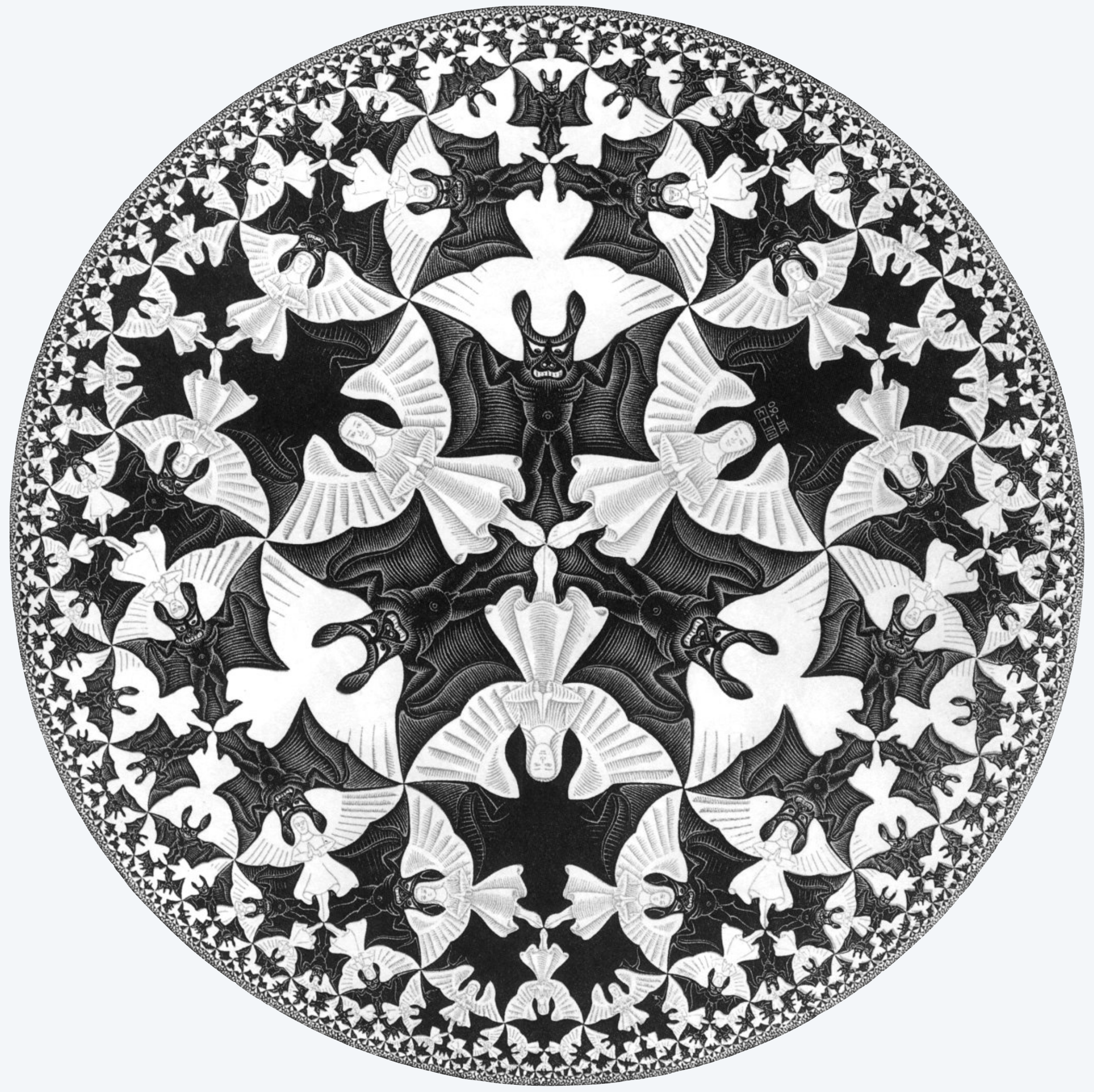
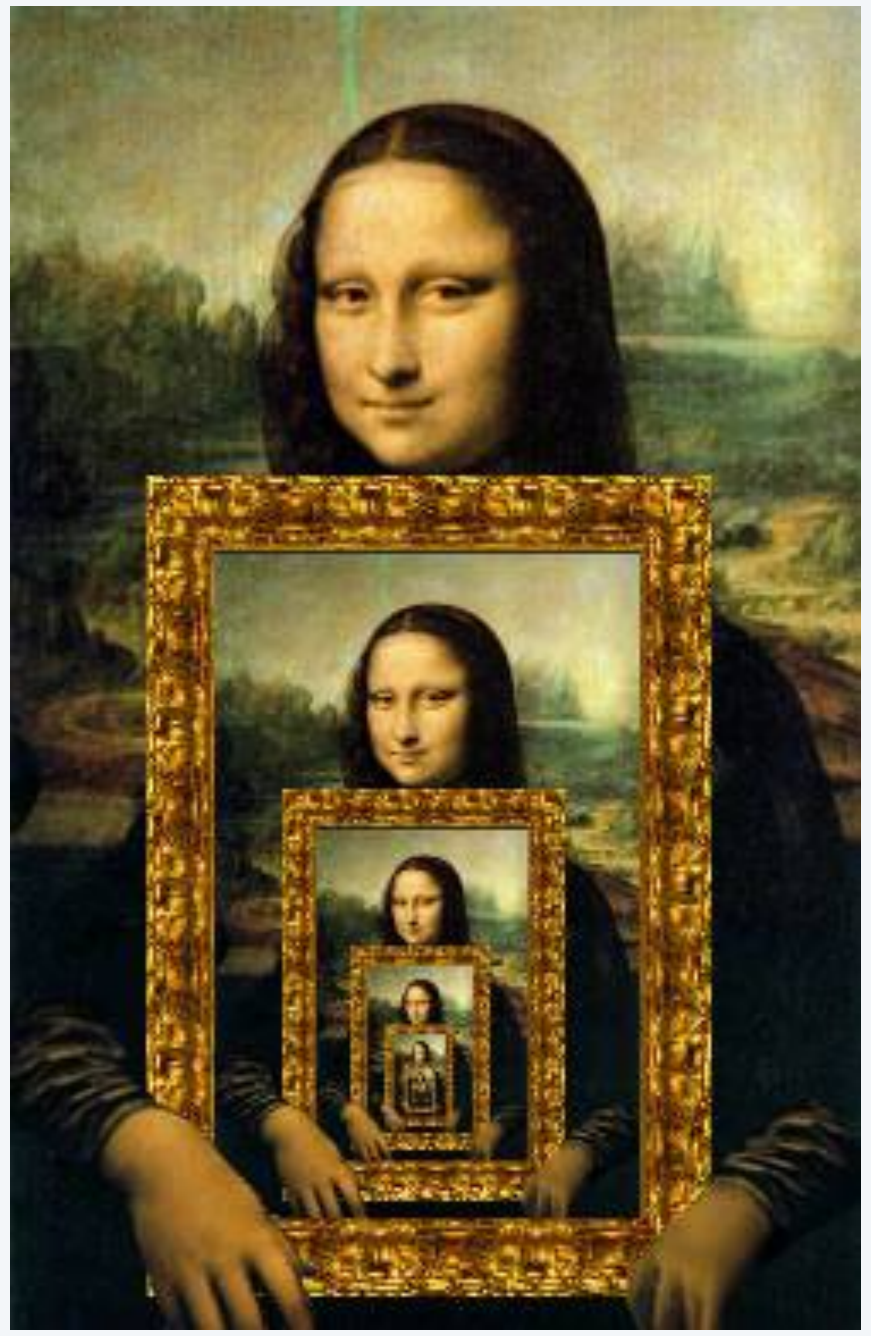
2.3 Recursion

- *foundations*
- *a classic example*
- ***recursive graphics***
- *exponential waste*

Recursive graphics in the wild








WEEKEND Arts FINE ARTS LEISURE

E41

FRIDAY, DECEMBER 15, 2006

The New York Times

Design Life Now at the Cooper-Hewitt National Design Museum includes this toy from the New York company Kidrobot.

Fruits of Design, Certified Organic

ROBERTA SMITH

ART REVIEW

From "The Yale Book of Quotations" to "Postcards From Mars," a selection of the best holiday books.

The Gifts to Open Again and Again

WILLIAM GRIMES

BOOKS

Black, White and Read All Over Over

By RANDY KENNEDY

Divine and Devotee Meet Across Hinges

HOLLAND COTTER

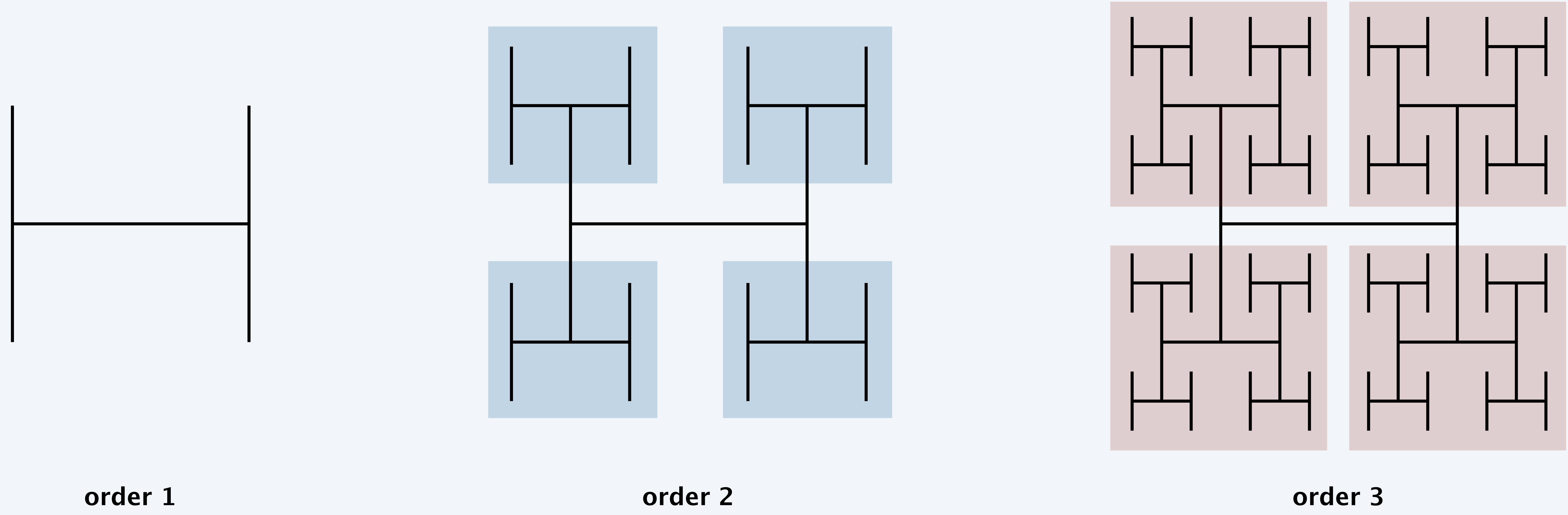
ART REVIEW

Prayers and Portraits: Unfolding the Netherlandish Diptych

"Hello, World" of recursive graphics: H-trees

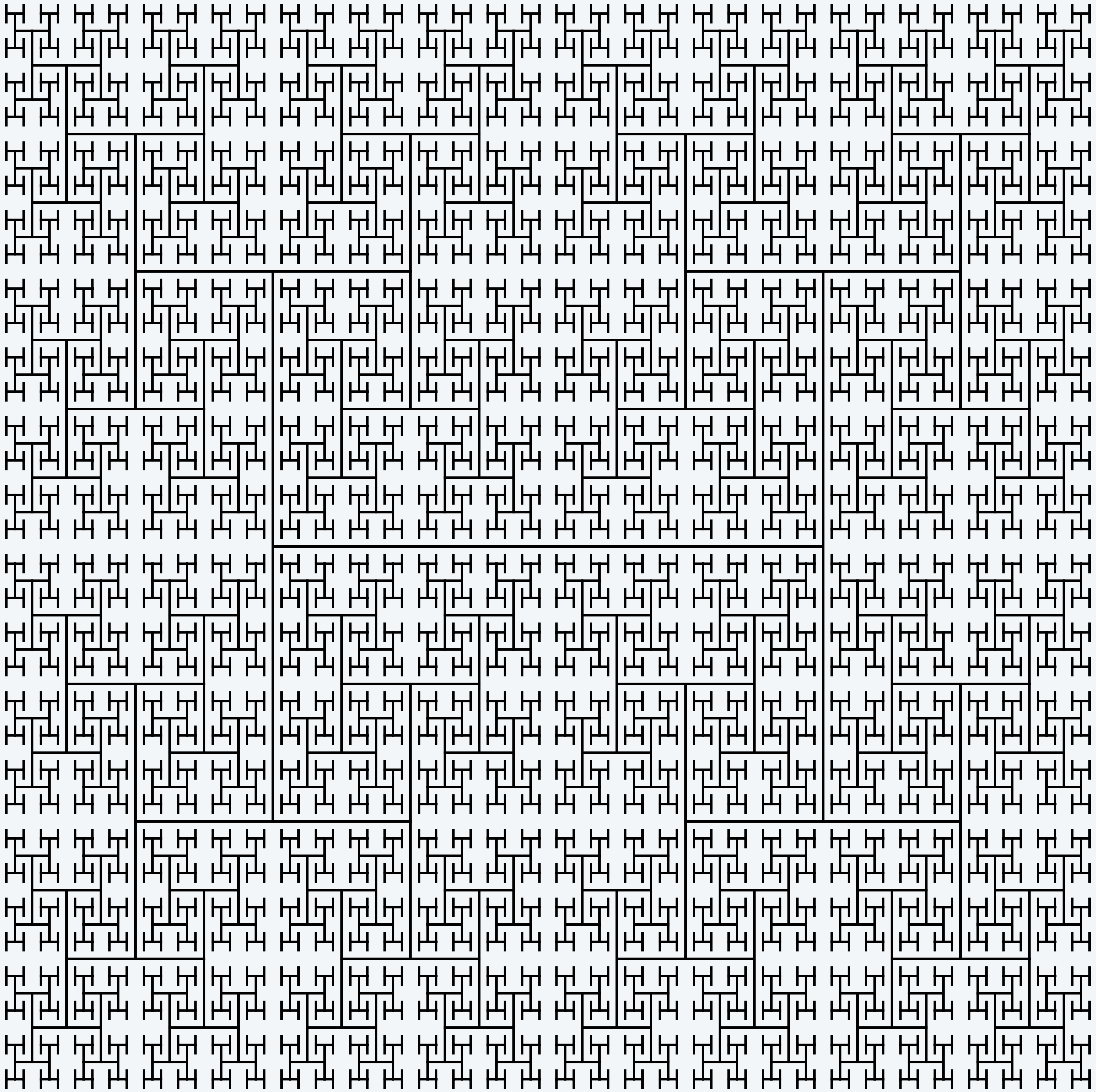
H-tree of order n.

- Base case: if n is 0, draw nothing.
- Reduction step:
  - draw an H
  - draw four H-trees of order $n - 1$ and half the size, centered at the tips of the H



H-trees

Application. Connect a large set of regularly spaced sites to a single source.



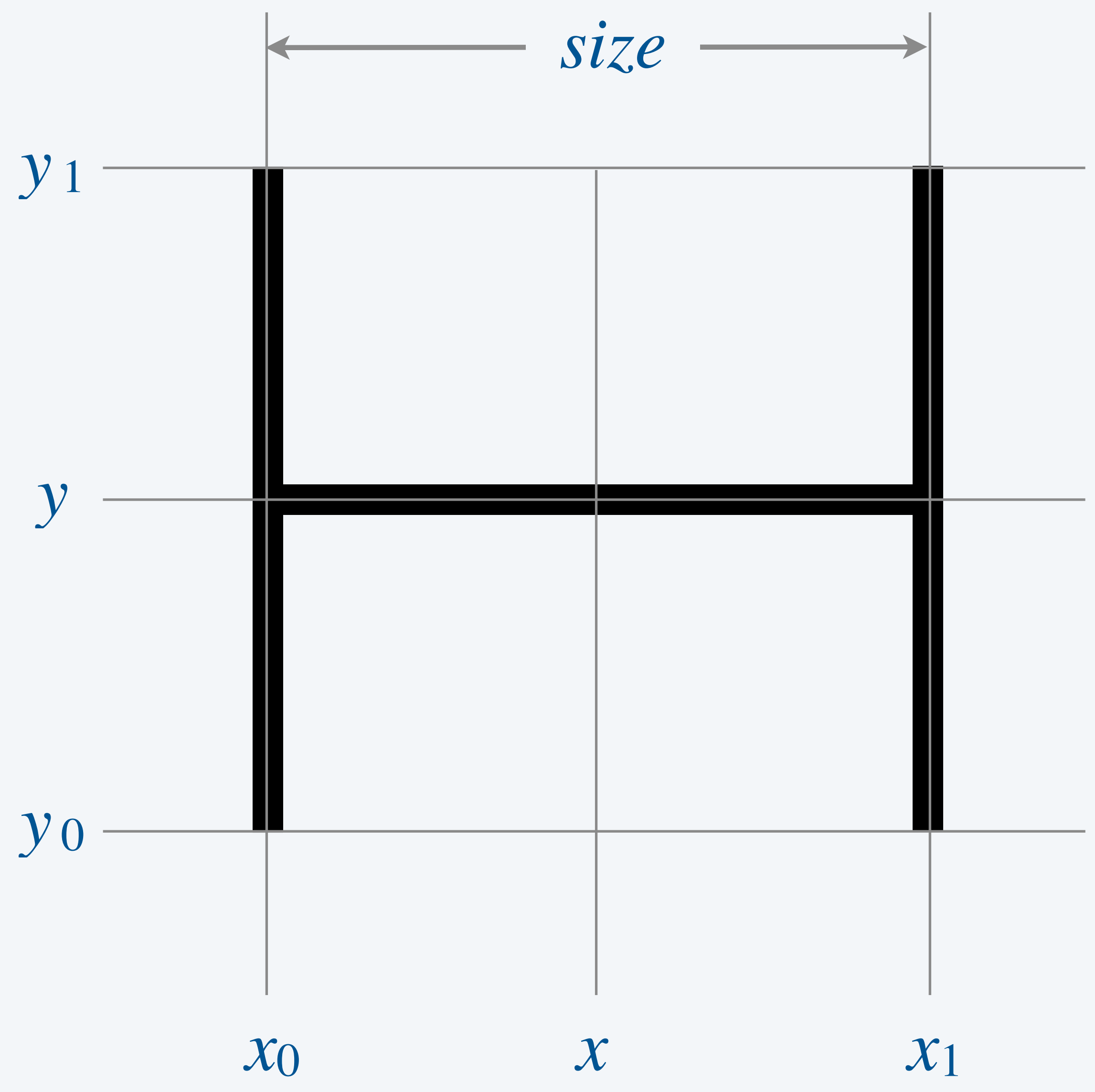
Recursive H-tree implementation

```java
public class Htree {

   public static void draw(int n, double size, double x, double y) {
      if (n == 0) return;
      double x0 = x - size/2, x1 = x + size/2;
      double y0 = y - size/2, y1 = y + size/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      draw(n-1, size/2, x0, y0);   // lower left
      draw(n-1, size/2, x0, y1);   // upper left
      draw(n-1, size/2, x1, y0);   // lower right
      draw(n-1, size/2, x1, y1);   // upper right
   }

   public static void main(String[] args) {
      StdDraw.setPenRadius(0.005);
      int n = Integer.parseInt(args[0]);
      draw(n, 0.5, 0.5, 0.5);
   }
}
```

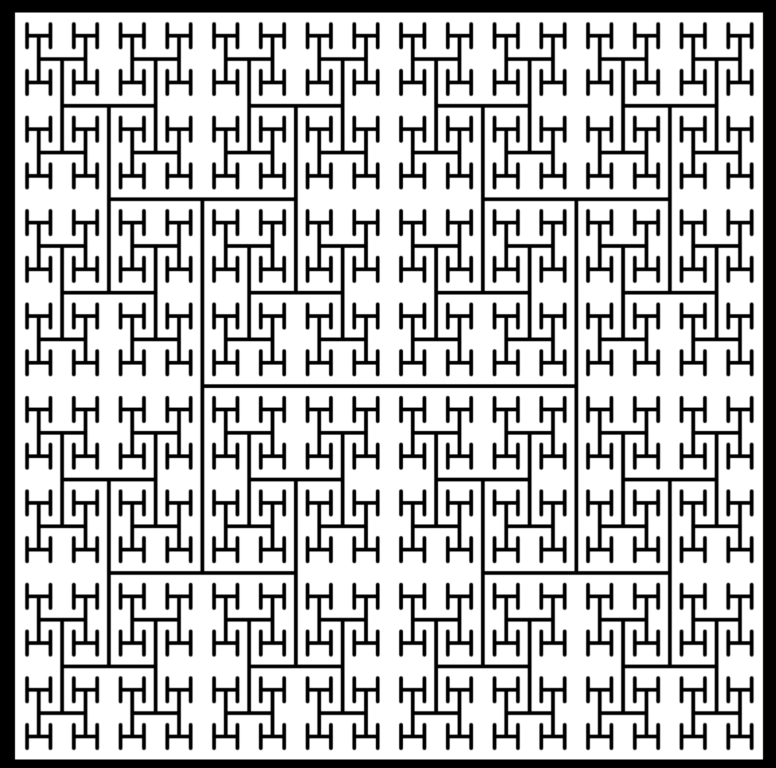
endpoints (the two `double` lines)

draw the H (non-recursive) (the three `StdDraw.line` calls)

draw four half-size H-trees (recursively) (the four `draw` calls)



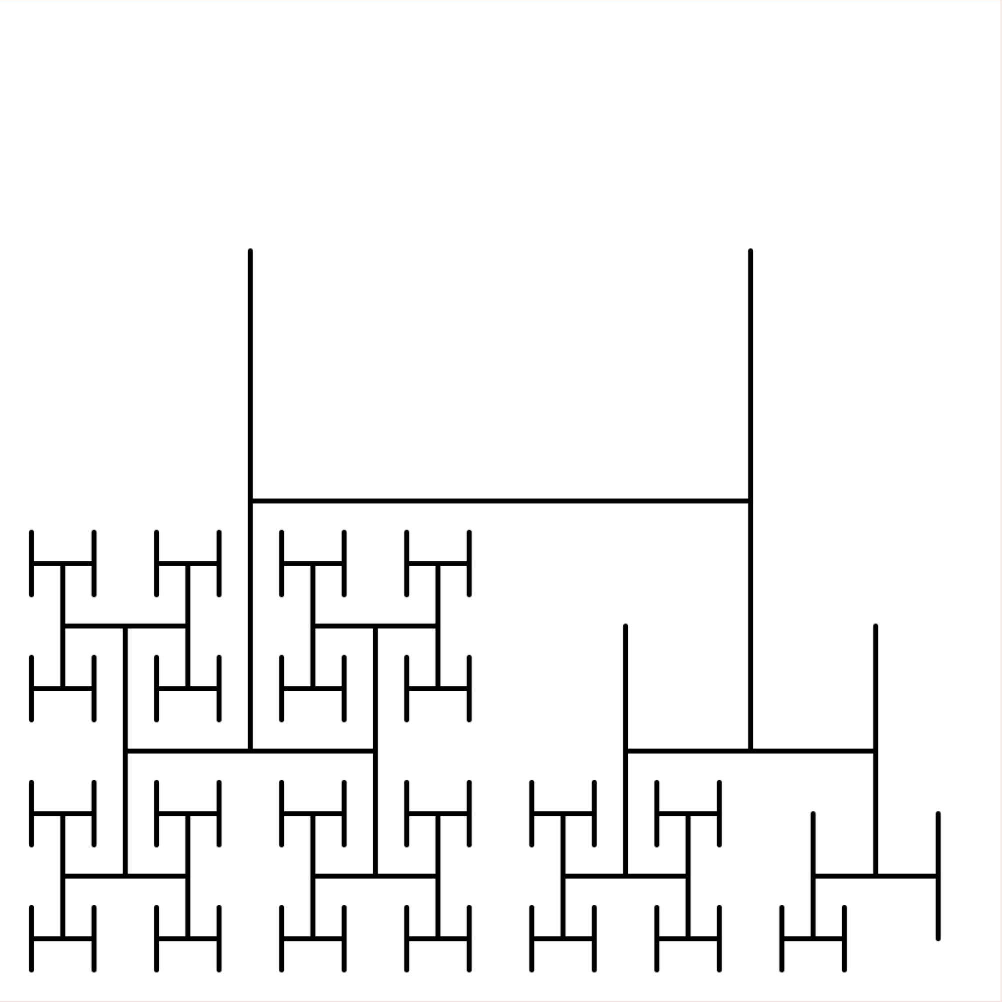
~/cos126/recursion> java-introcs Htree 5



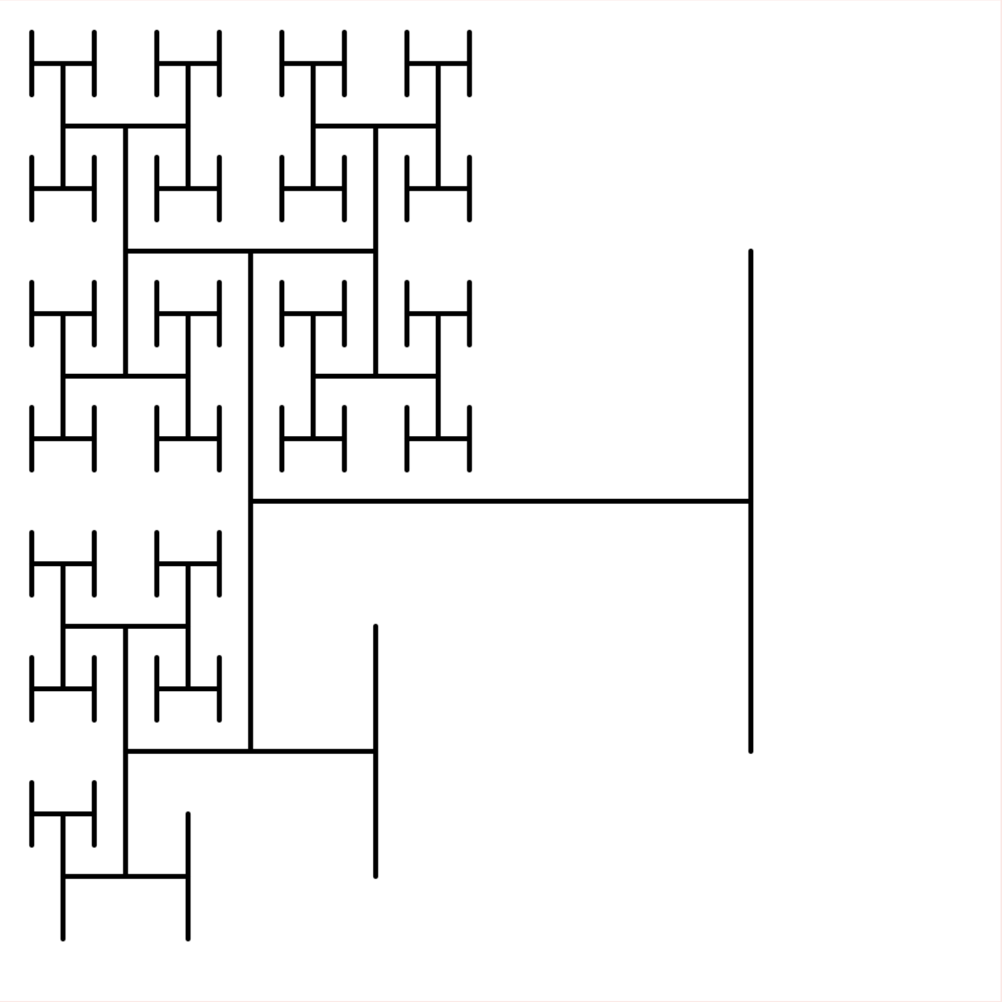
Suppose that `Htree` (with `n = 4`) is stopped after drawing the 30th H.

Which drawing will result?

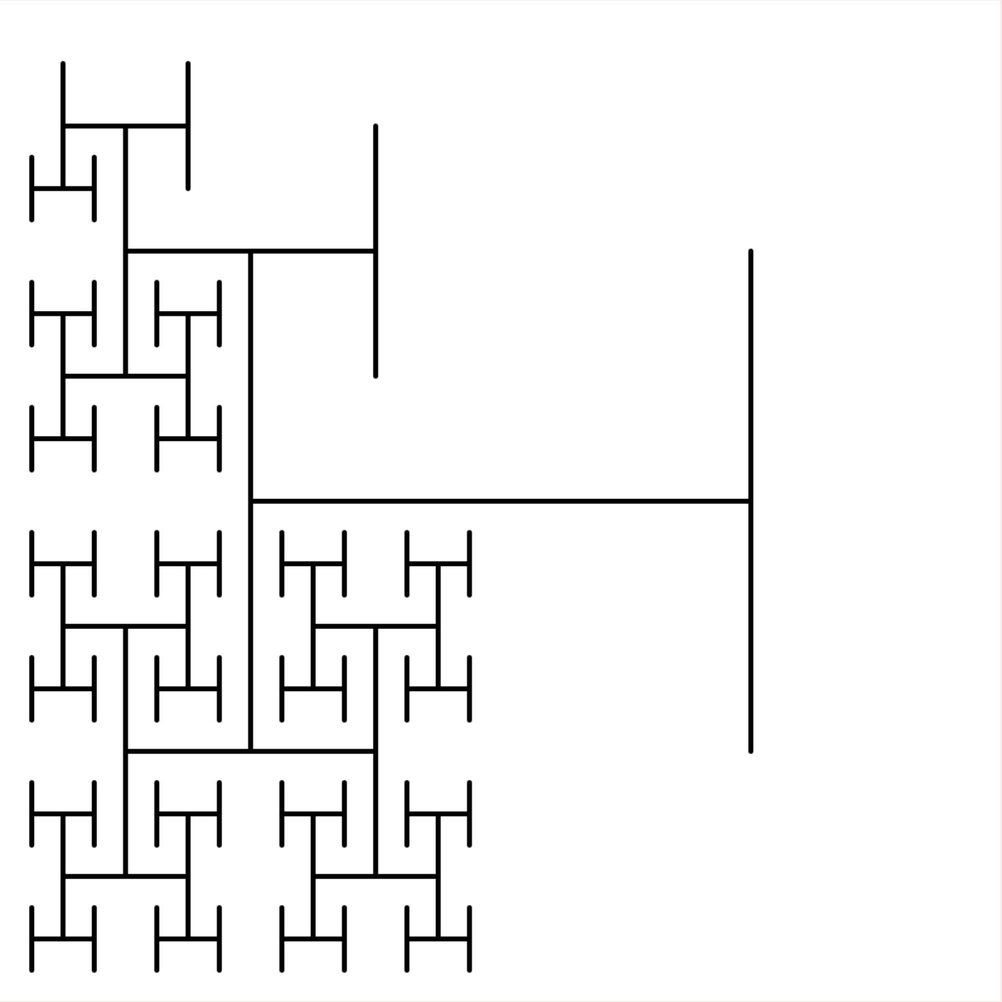
A.



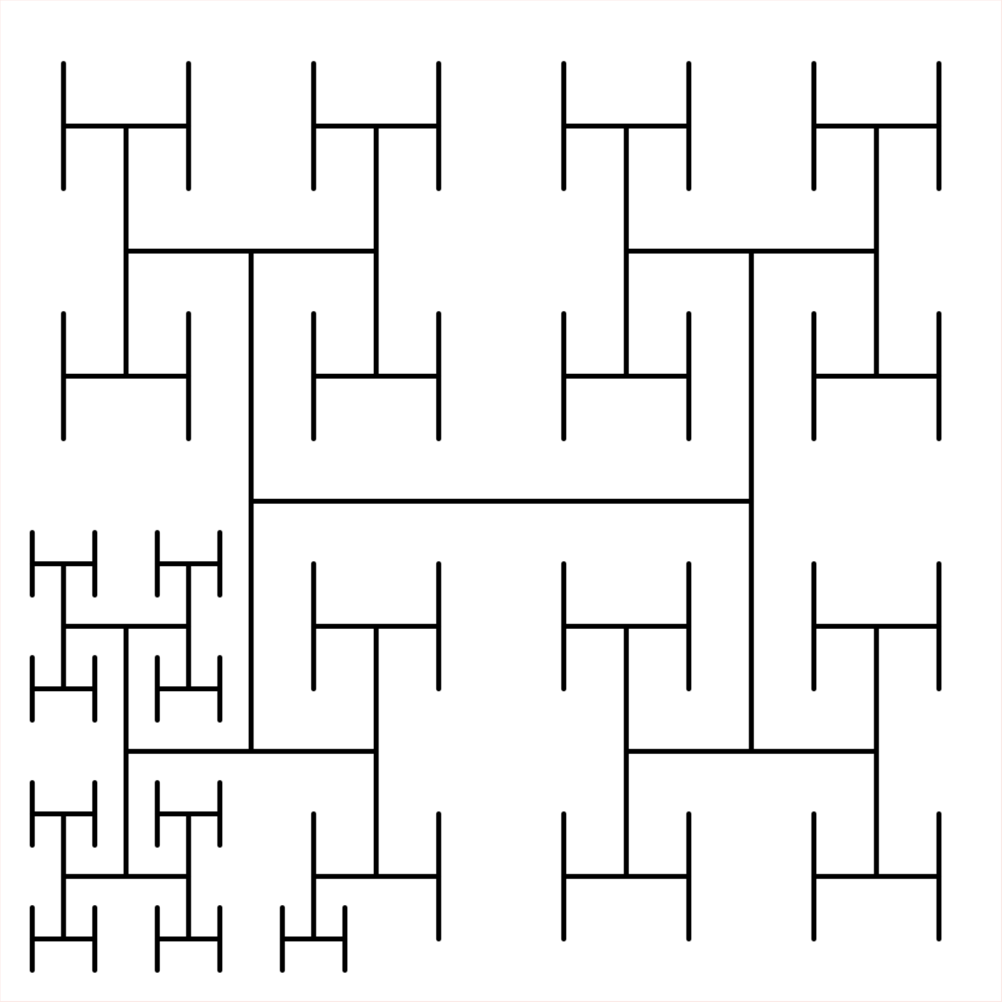
B.



C.



D.



Super H-tree implementation

Q. What will happen if we add the following statements to `draw()`, just before the recursive calls?

```
double freq = Synth.midiToFrequency(n + 45);
double duration = 0.25 * n;
double[] a = Synth.supersaw(freq, duration);
StdAudio.play(a);
```

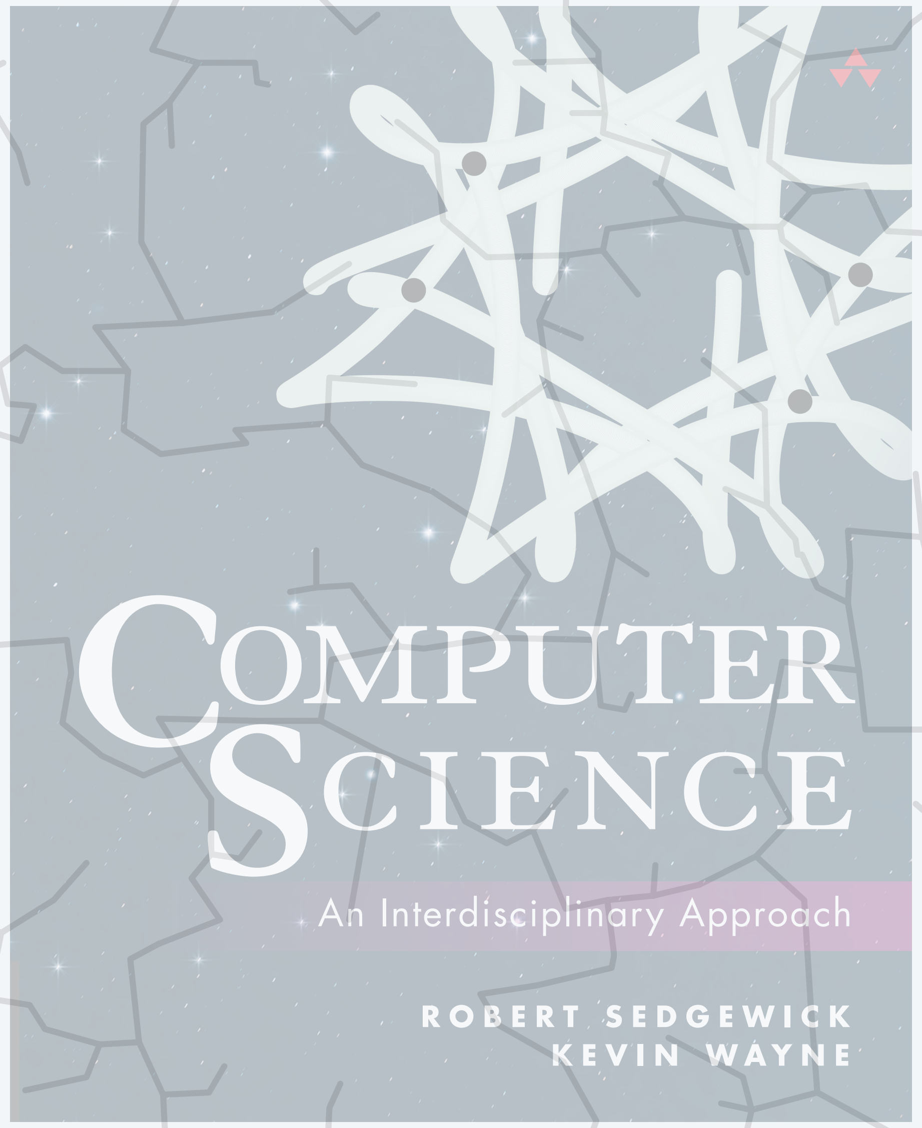
Recursive art (fall 2023)

Every semester, Princeton University's COS 126 invites students to use their newly acquired programming skills to create some amazing pieces of *recursive art*!

Here is what the Fall 2023 class has come up with!

PRINCETON UNIVERSITY

flashing images



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2.3 Recursion

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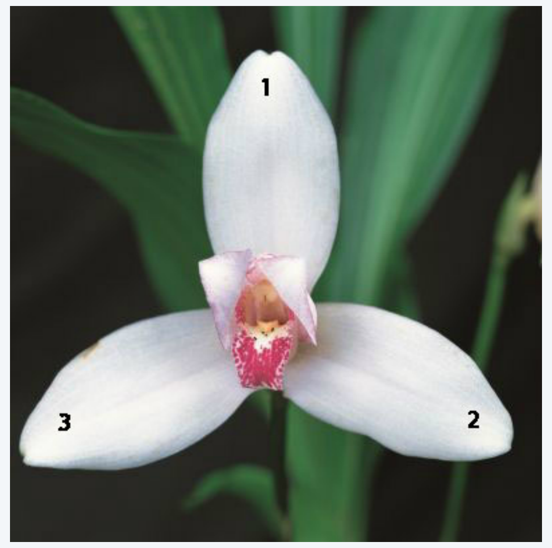
- *exponential waste*

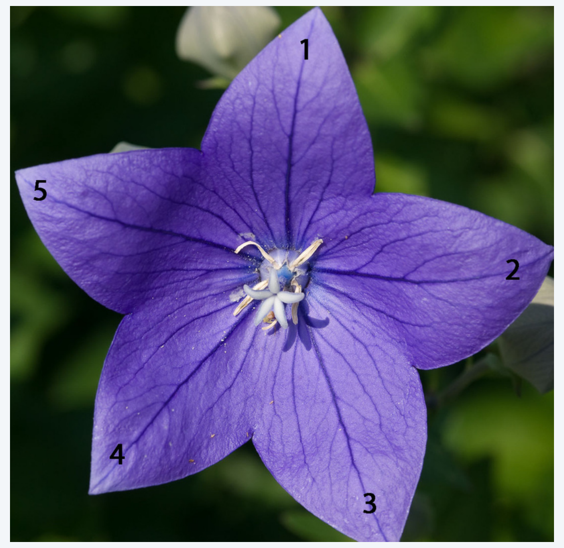
Fibonacci numbers

Fibonacci numbers. $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Leonardo Fibonacci



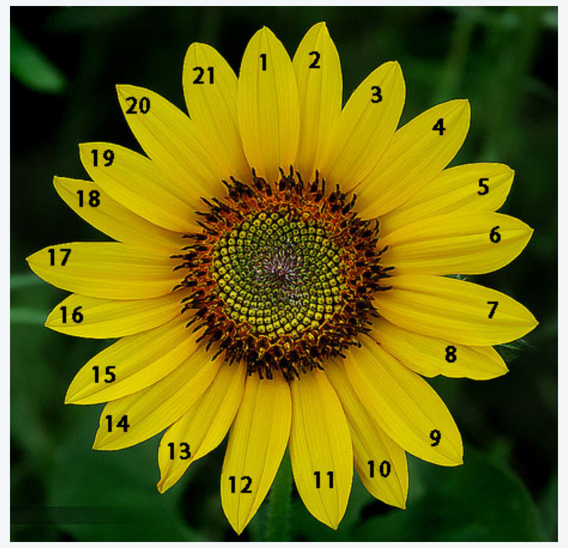
3



5

8

13



21

34

55

89

Fibonacci numbers: recursive approach

Fibonacci numbers. $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

Goal. Given n, compute F_n.

Recursive approach.

- Base cases: $F_0 = 0, \; F_1 = 1$.
- Reduction step: $F_n = F_{n-1} + F_{n-2}$.

```
public static long fib(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return fib(n-1) + fib(n-2);
}
```

How long dose it take to compute `fib(80)` ?

A. Much less than 1 second.

B. About 1 second.

C. About 1 minute.

D. About 1 hour.

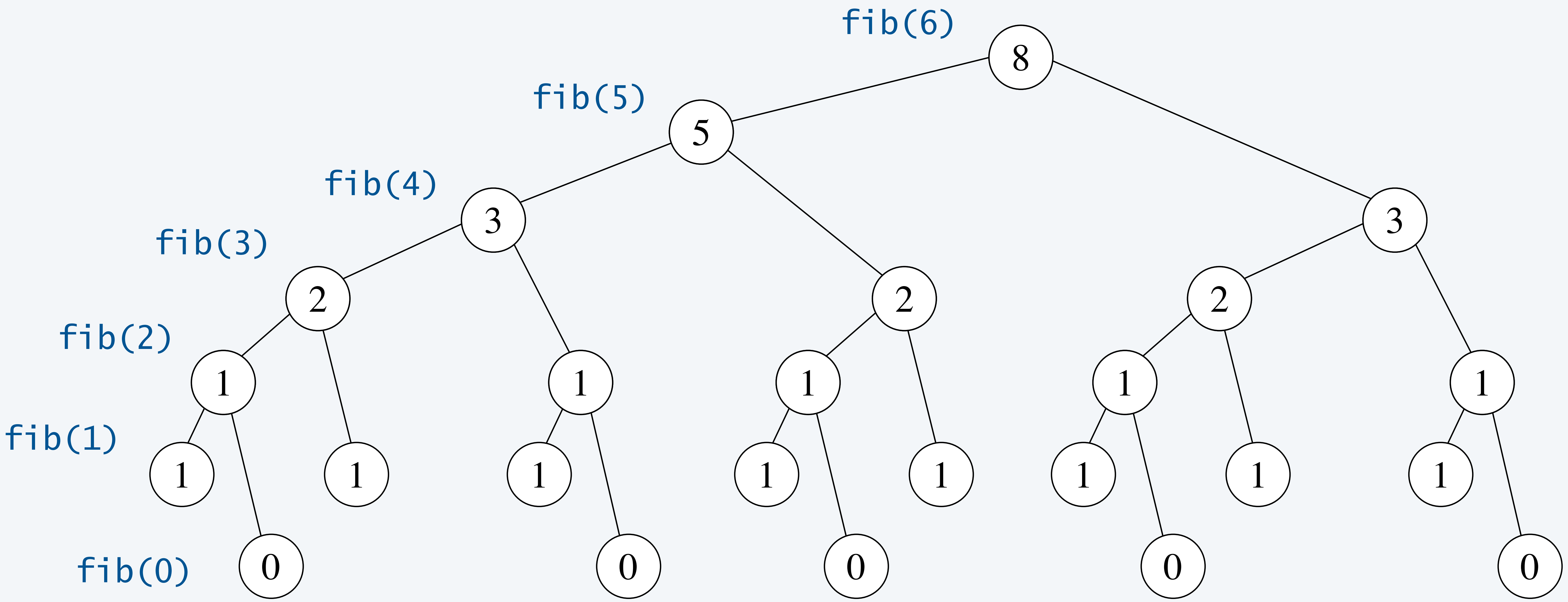
E. More than 1 hour.

```
public static long fib(int n) {
   if (n == 0) return 0;
   if (n == 1) return 1;
   return fib(n-1) + fib(n-2);
}
```

Recursion tree for Fibonacci numbers

Recursion tree.

- One node for each recursive call.
- Label node with return value after children are labelled.

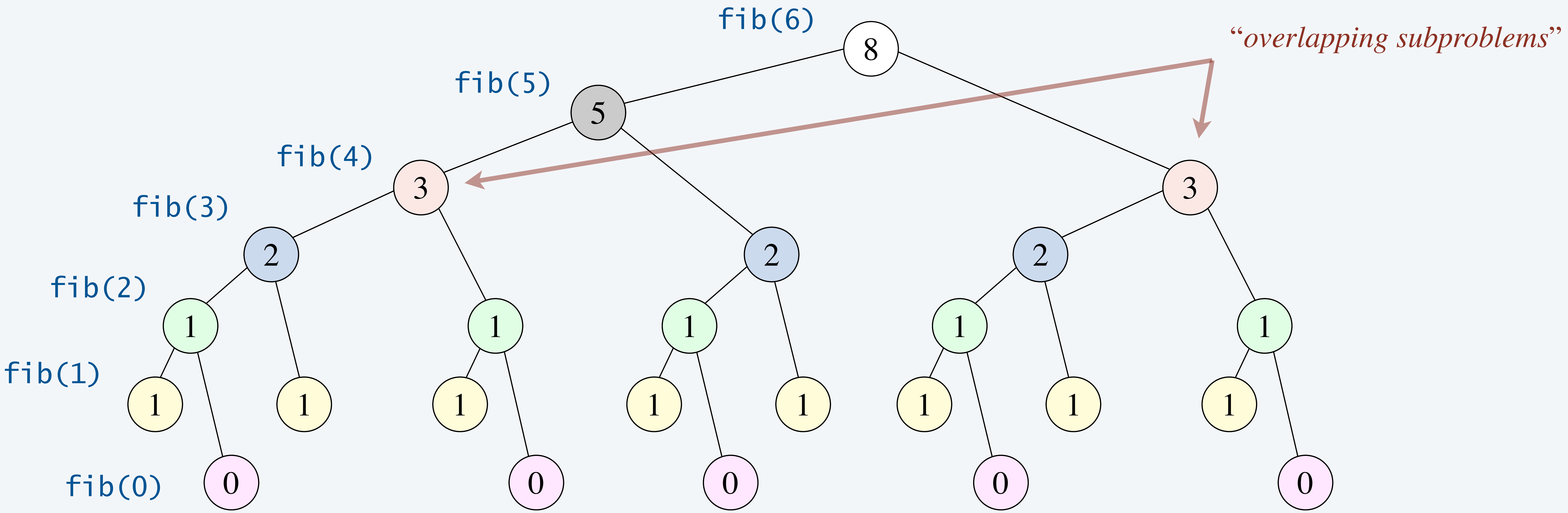


Exponential waste

Exponential waste. Same overlapping subproblems are solved repeatedly.

- fib(5) is called 1 time.
- fib(4) is called 2 times.
- fib(3) is called 3 times.
- fib(2) is called 5 times.
- fib(1) is called 8 times.

number of recursive calls are Fibonacci numbers (and grow exponentially)



Exponential waste dwarfs progress in technology

Lesson. If you engage in exponential waste, you will not be able to solve a large problem.

n	recursive calls	VAX-11 (1970s)	MacBook Pro (2020s)
30	2,692,536	*minute*	
40	331,160,280	*hours*	
50	40,730,022,146	*weeks*	*minute*
60	5,009,461,563,920	*years*	*hours*
70	616,123,042,340,256	*centuries*	*weeks*
80	75,778,124,746,287,810	*millenia*	*years*
90	9,320,093,220,751,060,616	⋮	*centuries*
100	1,146,295,688,027,634,168,200		*millenia*
⋮			⋮

↑ *exponential growth* (!)

time to compute `fib(n)` using recursive code

VAX-11/780

Macbook Pro (10,000× faster)

Avoiding exponential waste with memoization

Memoization.

- Maintain an array to remember all computed values.
- If value to compute is known, just return it; otherwise, compute it; remember it; and return it.

Impact. Calls `fibR(i)` at most twice for each `i`.

```
~/cos126/recursion> java-introcs FibonacciMemo 6
8

~/cos126/recursion> java-introcs FibonacciMemo 80
23416728348467685
```

instantaneous (!)

```java
public class FibonacciMemo {
   private static long[] memo;

   public static long fib(int n) {
      memo = new long[n+1];
      return fibR(n);
   }

   private static long fibR(int n) {
      if (memo[n] != 0) return memo[n];
      if       (n == 0) memo[n] = 0;
      else if  (n == 1) memo[n] = 1;
      else memo[n] = fibR(n-1) + fibR(n-2);

      return memo[n];
   }

   ...
}
```

"global" variable (`memo`)

initialize to all 0s (not yet known)

F_n known

compute F_n and store in array

return stored value

Design paradigm. This is a simple example of memoization (top-down dynamic programming).

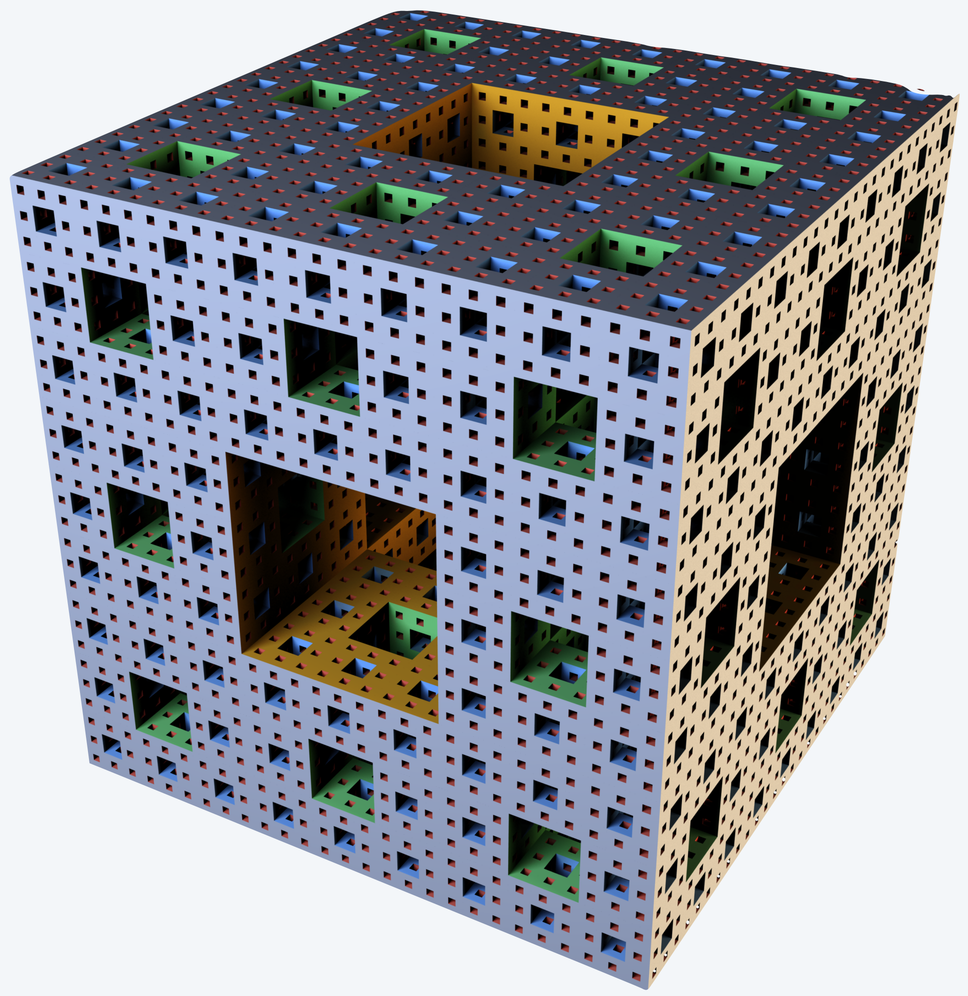
Summary

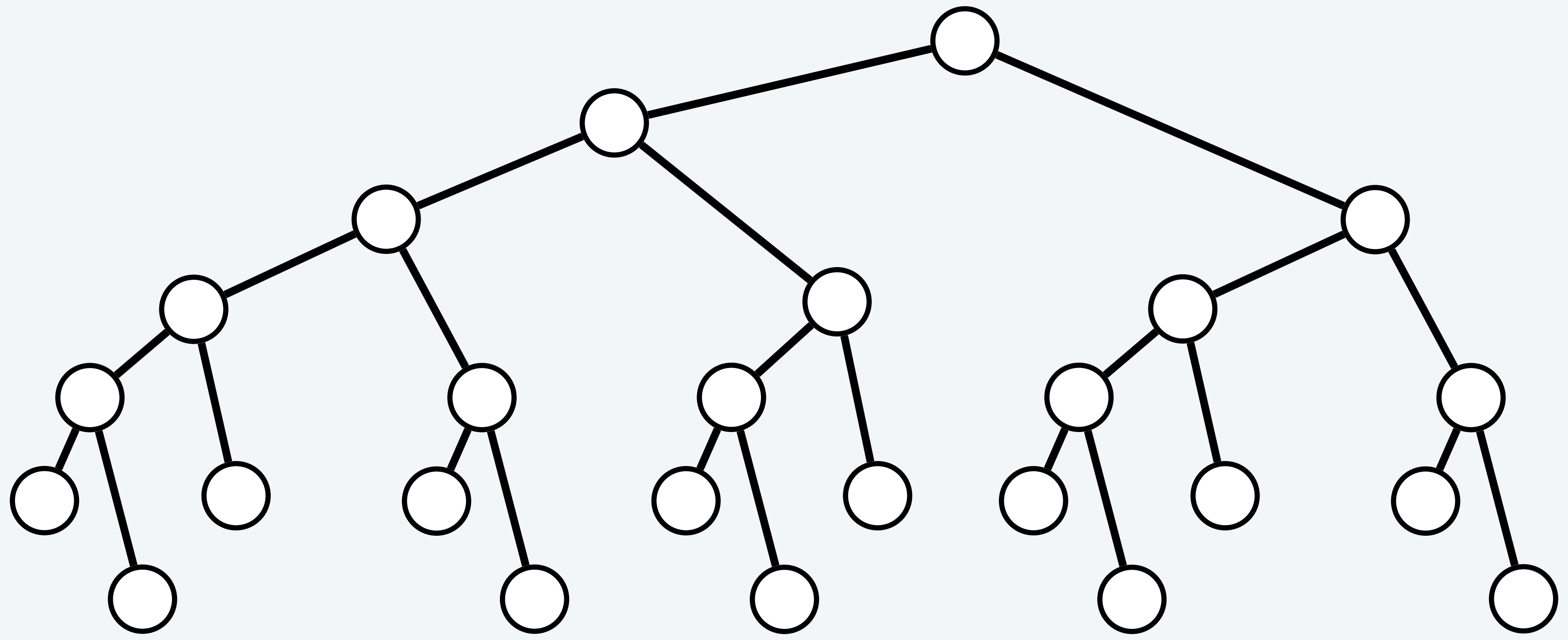
Recursive function. A function that calls itself.

Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Reveals insight into the nature of computation.

Dynamic programming. A powerful technique to avoid exponential waste. ← *see also* COS 226





```
// Ackermann function
public static long ack(long m, long n) {
   if (m == 0) return n+1;
   if (n == 0) return ack(m-1, 1);
   return ack(m-1, ack(m, n-1));
}
```

challenge for bored: compute ack(5, 2)

Credits

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Problems with Recursion	Zach Weinersmith	
You're Eating Recursion	Safely Endangered	
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