COS 302 Precept 8

Spring 2020

Princeton University

Outline

Outline

Transformations

Distribution Function Technique

Change of Variables

Univariate Gaussian Distributions

Review of Distributions

We have seen many named distributions:

- Bernoulli - a coin flip
- Binomial - a series of coin flips
- Gaussian/Normal - height
- Poisson - amount of (e)mail(s) you receive daily

Review of Distributions cont.

For every distribution there are several things to keep in mind:

- Discrete or Continuous
- Parameters
- Probability mass/density function
- Support - nonzero parts
- Expectation or Mean
- Variance

Motivating Example

For named distribution we have a lot of information.

$$X \sim \mathcal{N}(0, 1)$$

But what about X^2? or $\log(X)$?

More generally if I have a a function $U(X)$, what can our information about X tell us about $U(X)$?

Approaches

- Discrete
  - Direct Change
- Continuous
  - Distribution Function Technique
  - Change of Variables

Discrete Case

Suppose X is distributed according to any discrete distribution, and we have an invertible function $U(X) = Y$, then

$$P(Y = y) = P(U(X) = y) = P(X = U^{-1}(y)).$$

Implies we can use X's pmf on the event $U^{-1}(y)$.

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Continuous Case

For a continuous random variable X, a function $Y = U(X)$:

1. Find the cdf:

$$F_Y(y) = P(Y \leq y)$$

2. Differentiate the cdf $F_Y(y)$ to get the pdf $f_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

Example 1 - simple pdf

Let X be a continuous random variable defined on the interval [0,1] with pdf

$$f_X(x) = 3x^2.$$

What is the pdf of the random variable $Y = X^2$?

Example 1 cont.

Step 1: Find the cdf.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(X \leq y^{1/2}) \\
&= F_X(y^{1/2}) \\
&= \int_0^{y^{\frac{1}{2}}} 3t^2 dt \\
&= [t^3]_{t=0}^{t=y^{1/2}} \\
F_Y(y) &= y^{3/2}, \quad y \in [0,1]
\end{aligned}$$

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Change of Variables Steps

X is a univariate random variable (r.v.) with states $x \in [a, b]$ and pdf $f(x)$. Another r.v. $Y = U(X)$, where U is an invertible function. What is pdf $f(y)$?

Steps:

1. Transform cdf of Y into cdf of X.
2. Differentiate cdf to get pdf.

Change of Variables Steps Cont.

1. Transform cdf of Y into cdf of X.

By definition of cdf:

$$F_Y(y) = P(Y \leq y) = P(U(X) \leq y)$$

Assume U is strictly increasing, then U^{-1} is also strictly increasing.

$$\begin{aligned} P(U(X) \leq y) &= P(U^{-1}(U(X)) \leq U^{-1}(y)) \\ &= P(X \leq U^{-1}(y)) \end{aligned}$$

Change of Variables Steps Cont.

2. Differentiate cdf to get pdf.

Based on definition of the cdf of X,

$$F_Y(y) = P(X \leq U^{-1}(y)) = \int_a^{U^{-1}(y)} f(x)dx$$

Differentiate with respect to y,

$$f(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_a^{U^{-1}(y)} f(x)dx$$

Change of Variables Steps Cont.

$$\int f(x)dx = \int f(U^{-1}(y))U^{-1'}(y)dy$$

$$\begin{aligned} f(y) &= \frac{d}{dy}\int_a^{U^{-1}(y)} f_x(U^{-1}(y))U^{-1'}(y)dy \\ &= f_x(U^{-1}(y)) \cdot (\frac{d}{dy}U^{-1}(y)). \end{aligned}$$

For both increasing and decreasing U,

$$f(y) = f_x(U^{-1}(y)) \cdot \mid \frac{d}{dy}U^{-1}(y) \mid .$$

Example 2: Univariate Normal

Theorem

Suppose $X \sim N(\mu, \sigma^2)$ and $Z = U(X) = \frac{X-\mu}{\sigma}$.
Then $Z \sim N(0, 1)$.

Analysis:

$$f(z) = f_x(U^{-1}(z)) \cdot \mid \frac{d}{dz} U^{-1}(z) \mid$$

Example 2: Univariate Normal Cont.

Proof: $f_x(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$,

$$x = U^{-1}(z) = \sigma z + \mu, \quad \frac{d}{dz} U^{-1}(z) = \sigma.$$

$$\begin{aligned} f(z) = f_x(U^{-1}(z)) \cdot \mid \frac{d}{dz} U^{-1}(z) \mid &= f_x(\sigma z + \mu) \cdot \mid \sigma \mid \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}z^2} \cdot \mid \sigma \mid \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \end{aligned}$$

Multivariate Change of Variables

Theorem

Let X be a multivariate continuous r.v., $f_{\mathbf{x}}(\mathbf{x})$ be the pdf. If the vector-valued function $\mathbf{y} = U(\mathbf{x})$ is differentiable and invertible for all values within the domain of $\mathbf{x}$, then for corresponding values of $\mathbf{y}$, the pdf of $Y = U(X)$ is given by

$$f(\mathbf{y}) = f_{\mathbf{x}}(U^{-1}(\mathbf{y})) \cdot \mid \det(\tfrac{\partial}{\partial \mathbf{y}} U^{-1}(\mathbf{y})) \mid$$

(Univariate: $f(y) = f_x(U^{-1}(y)) \cdot \mid \frac{d}{dy} U^{-1}(y) \mid$)

Example 3: Multivariate Gaussian

Let A be an invertible $p \times p$ matrix , $\boldsymbol{\mu} \in \mathbb{R}^{p\times 1}$, and $Z = (Z_1, \dots, Z_p)' \in \mathbb{R}^{p\times 1}$ be independent standard normal r.v.'s $\{Z_j\} \sim N(0,1)$, with joint pdf $f_{\boldsymbol{z}}(\boldsymbol{z}) = (2\pi)^{-\frac{p}{2}} e^{-\frac{\boldsymbol{z}'\boldsymbol{z}}{2}}$.
Then $X = g(Z) = \boldsymbol{\mu} + AZ \sim N(\boldsymbol{\mu}, C)$, where

$$\begin{aligned} C &= E(X - \boldsymbol{\mu})(X - \boldsymbol{\mu})' \\ &= E(AZ)(AZ)' \\ &= E[AZZ'A'] = AA' \end{aligned}$$

Example 3: Multivariate Gaussian Cont.

Proof: $f(\boldsymbol{x}) = f_{\boldsymbol{z}}(g^{-1}(\boldsymbol{x}))\cdot \mid \det(\frac{\partial}{\partial \boldsymbol{x}} g^{-1}(\boldsymbol{x})) \mid$

$$g^{-1}(\boldsymbol{x}) = A^{-1}(\boldsymbol{x} - \boldsymbol{\mu}), \frac{\partial}{\partial \boldsymbol{x}} g^{-1}(\boldsymbol{x}) = A^{-1}$$

$$\begin{aligned} f(\boldsymbol{x}) &= f_{\boldsymbol{z}}(A^{-1}(x - \boldsymbol{\mu}))\cdot \mid \det A \mid^{-1} \\ &= (2\pi)^{-\frac{p}{2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'(A^{-1})'A^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} / \sqrt{\det AA'} \\ &= \frac{1}{\sqrt{(2\pi)^p \det C}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'C^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \end{aligned}$$

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Example 4: Chi-Square

Let $X \sim \mathcal{N}(0,1)$ and $Y = X^2$. The square function is not one-to-one on the whole real line (i.e. it's inverse only is defined for positive numbers).

However, $X^2 \leq y \implies X \in [-\sqrt{y}, \sqrt{y}]$. Then

$$\begin{aligned} F_Y(y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y}) - 1 \end{aligned}$$

Example 5: Log-Normal

Once again let $X \sim \mathcal{N}(0,1)$, and $Y = e^X$. Since the exponential function is strictly increasing and is one-to-one on the whole real line, then we can just apply the change of variable formula. Recall that $x = \log y$, and $dy/dx = e^x$. We have that

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \varphi(x)\frac{1}{e^x} = \varphi(\log y)\frac{1}{y}, \quad y > 0$$