# COS 302 Precept 4

Spring 2020

Princeton University

# Outline

Eigendecomposition

Geometric interpretation

Diagonalization for computation

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# Diagonal Matrix

## Definition

Consider a square matrix $\boldsymbol{D} \in \mathbb{R}^{n \times n}$. $\boldsymbol{D}$ is a diagonal matrix if all $\boldsymbol{D_{ij}} = 0$ if $i \neq j$, i.e., it is of the form:

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

# Properties of Diagonal Matrix

- Diagonal matrices allow fast computation of determinants, powers, and inverses
- The determinant of a diagonal matrix is the product of its diagonal entries
- A matrix power $\boldsymbol{D}^k$ is given by each diagonal element raised to the power $k$
- $\boldsymbol{D}^{-1}$ is the reciprocal of its diagonal elements if all of them are nonzero.

# Diagonalizable Matrix

## Definition

A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is diagonalizable if there exists an invertible matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$, where $\boldsymbol{D}$ is a diagonal matrix.

# Eigendecomposition

**Theorem**

A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ can be factored into

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

where $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, if and only if the eigenvectors of $\boldsymbol{A}$ form a basis of $\mathbb{R}^n$. This factorization is known as the eigendecomposition of $\boldsymbol{A}$.

# Proof of Eigendecomposition Theorem

## Proof

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be a set of scalars, and $\boldsymbol{p_1}, \boldsymbol{p_2}, \cdots, \boldsymbol{p_n}$ be a set of vectors in $\mathbb{R}^n$. We define $\boldsymbol{P} := [\boldsymbol{p_1}, \boldsymbol{p_2}, \cdots, \boldsymbol{p_n}]$. That is, $\boldsymbol{p_i}$'s are the columns of $\boldsymbol{P}$.Let $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ be a diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \cdots, \lambda_n$. That is,

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

# Proof of Eigendecomposition Theorem

## Proof Continued

$$\boldsymbol{AP} = [\boldsymbol{Ap_1}, \boldsymbol{Ap_2}, \cdots, \boldsymbol{Ap_n}] \tag{1}$$

$$\boldsymbol{PD} = [\lambda_1 \boldsymbol{p_1}, \lambda_2 \boldsymbol{p_2}, \cdots, \lambda_n \boldsymbol{p_n}] \tag{2}$$

The above two equations imply that: $\boldsymbol{Ap_i} = \lambda_1 \boldsymbol{p_i}$ $\forall 1 \leq i \leq n$. That is, the $\boldsymbol{p_i}$'s are the eigenvectors of $\boldsymbol{A}$ and $\lambda_i$'s are the corresponding eigenvalues.

# Proof of Eigendecomposition Theorem

## Proof Continued

In the definition of diagonalizability, we require that $\boldsymbol{P}$ is invertible, which implies that the columns of $\boldsymbol{P}$ are linearly independent of each other. Therefore, the columns of $\boldsymbol{P}$, which are also the eigenvectors of $\boldsymbol{A}$, form a basis of $\mathbb{R}^n$.

# Symmetric Square Matrix

**Theorem**

Symmetric Square Matrices are always diagonalizable.

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# Geometric interpretation of eigen-decomposition

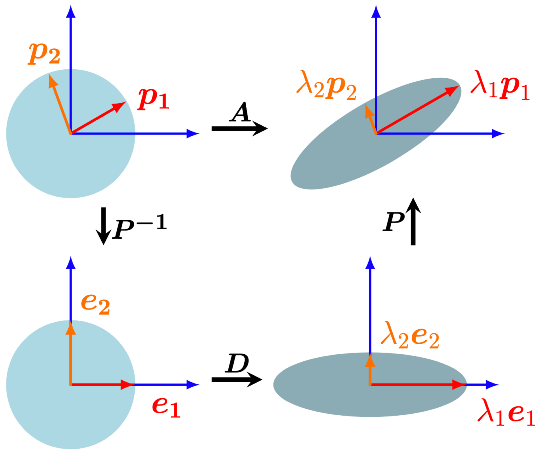


**Figure 1:** Intuition behind the eigendecomposition as sequential transformations.

## Geometric interpretation of eigen-decomposition

1. $P^{-1}$ performs a basis change (here drawn in $\mathbb{R}^2$ and depicted as a rotation-like operation), mapping the eigenvectors into the standard basis.
    - $P\tilde{x}_1 = x \iff \tilde{x}_1 = P^{-1}x$
2. $D$ performs a scaling along the remapped orthogonal eigenvectors, depicted here by a circle being stretched to an ellipse
    - $\tilde{x}_2 = D\tilde{x}_1$

# Geometric interpretation of eigen-decomposition

3. $P$ undoes the basis change (depicted as a reverse rotation) and restores the original coordinate frame.
    - $\tilde{x} = P\tilde{x}_2$

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# Diagonalization for efficient computation

Assume a matrix has eigen-decomposition:
$A = PDP^{-1}$.

- Calculating matrix power:
  $A^k = \left(PDP^{-1}\right)^k = PD^kP^{-1}$
- Calculating determinant:

$$\begin{aligned}\det(A) &= \det(PDP^{-1}) \\ &= \det(P)\det(D)\det(P^{-1}) \\ &= \det(D)\det(P)\det(P^{-1}) \\ &= \det(D)\det(PP^{-1}) \\ &= \det(D)\end{aligned}$$