Class Meeting #16

COS 226 — Spring 2018

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(based in part on slides by Kevin Wayne)

Greedy algorithms

A class of algorithms, typically for solving an optimization problem:

- Attain goal with fewest resources
- Get the most out of given fixed resources

Examples:

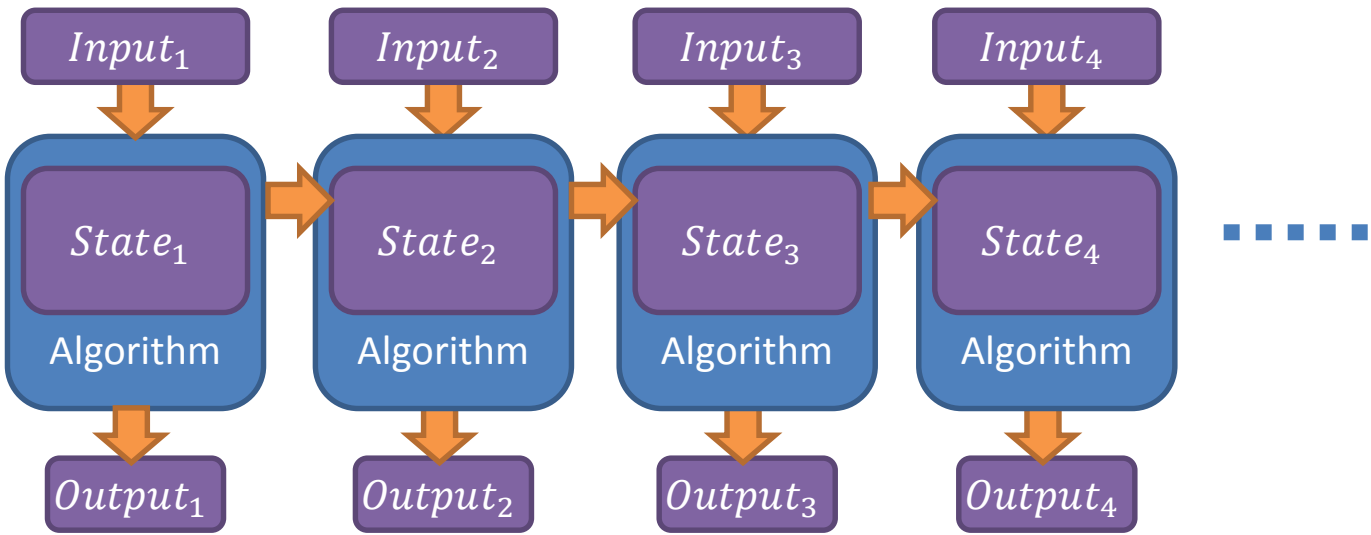
- Produce a given amount of change with fewest coins
- Schedule classes in fewest possible rooms
- Perform the largest possible number of jobs on a machine

Online decision making

- In classical algorithms, we are given an input and need to produce an output.

Online decision making

- In online decision making, need to make irrevocable decisions based on what we've seen so far.



Online decision making

- In online decision making, need to make irrevocable decisions based on what we've seen so far.
- Examples:
  - Games
  - Control
  - Prediction tasks in Machine Learning

Online vs. offline

- Which one will perform better?
  - Offline clearly better.
- Why bother with online?
  - Often no choice…
- When we have a choice?
  - Conceptually simpler.
  - Sometimes lead to optimal performance (often with some preprocessing).
  - When we do have an optimal algorithm that operates by making local irrevocable decisions, such an algorithm is called a *greedy algorithm*.

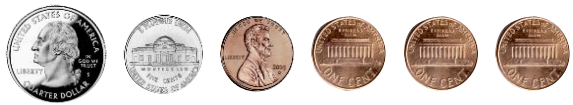
Greedy algorithms

- Make local, irrevocable, decisions
- Work when we can answer the following question easily:
  - Given the current state, can I commit to part of the output right now?

Coin changing

Goal. Given U. S. currency denominations { 1, 5, 10, 25, 100 }, devise a method to pay amount to customer using fewest coins.

Ex. 34¢.



Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex. $2.89.

Cashier's algorithm

At each iteration, add coin of the largest value that does not take us past the amount to be paid.

CASHIERS-ALGORITHM $(x, c_1, c_2, \ldots, c_n)$

SORT n coin denominations so that $0 < c_1 < c_2 < \ldots < c_n$.

$S \leftarrow \varnothing$. ← multiset of coins selected

WHILE $(x > 0)$

  $k \leftarrow$ *largest coin denomination* c_k *such that* $c_k \leq x$.

  IF (no such k)

    RETURN "no solution."

  ELSE

    $x \leftarrow x - c_k$.

    $S \leftarrow S \cup \{ k \}$.

RETURN S.

Cashier's algorithm (for arbitrary coin denominations)

Q. Is cashier's algorithm optimal for any set of denominations?

A. No. Consider U.S. postage: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

- Cashier's algorithm: 140¢ = 100 + 34 + 1 + 1 + 1 + 1 + 1 + 1.
- Optimal: 140¢ = 70 + 70.

bad, irredeemable decisions

A. No. It may not even lead to a feasible solution if $c_1 > 1$: 7, 8, 9.

- Cashier's algorithm: 15¢ = 9 + ?.
- Optimal: 15¢ = 7 + 8.

Optimality of cashier's algorithm (for U.S. coin denominations)

Theorem. Cashier's algorithm is optimal for U.S. coins { 1, 5, 10, 25, 100 }.

Pf. **Succeed by not failing.**

Enough to show that we never regret adding a coin to the change.

Therefore enough to show that:

(1) Any amount above \$1 has a \$1 coin in optimal solution

(2) Any amount above \$0.25 but below \$1 has a quarter in optimal solution

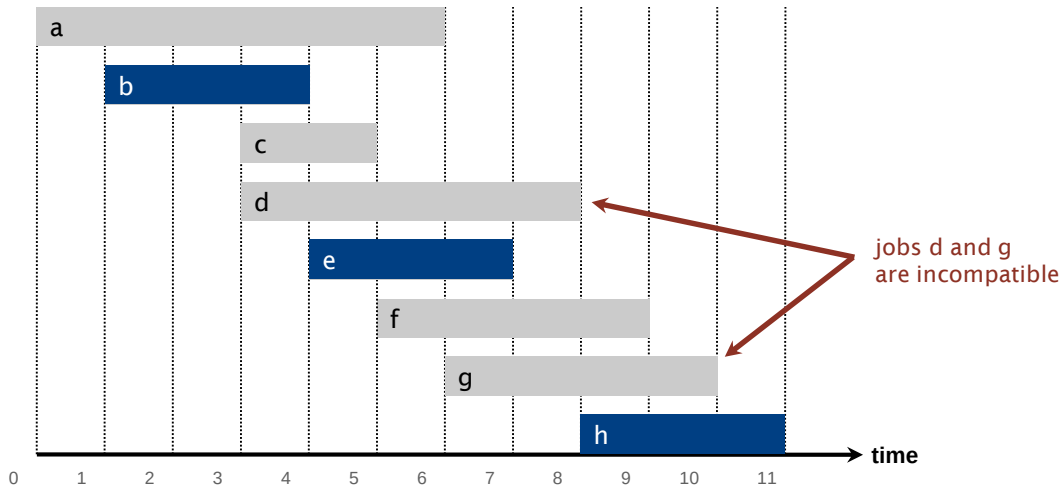
(3) Any amount above \$0.10 but below \$0.25 has a dime in optimal solution

(4) Any amount above \$0.05 but below \$0.10 has a nickel in optimal solution

Prove those by observing that #pennies ≤ 4, #nickels ≤ 1, #dimes≤ 2, #quarters≤ 3.

Interval scheduling

- Job j starts at s_j and finishes at f_j.
- Two jobs are compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval scheduling: greedy algorithms

Greedy template. Consider jobs in some natural order.

Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of s_j.
- [Earliest finish time] Consider jobs in ascending order of f_j.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Interval scheduling: greedy algorithms

Greedy template. Consider jobs in some natural order.

Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

Interval scheduling: earliest-finish-time-first algorithm

EARLIEST-FINISH-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT jobs by finish times and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

$S \leftarrow \varnothing$. ← set of jobs selected

FOR $j = 1$ *TO* n

  IF *(job j is compatible with S)*

    $S \leftarrow S \cup \{ j \}$.

RETURN S.

Proposition. Can implement earliest-finish-time first in $O(n \log n)$ time.

- Keep track of job j* that was added last to S.
- Job j is compatible with S iff $s_j \geq f_{j^*}$.
- Sorting by finish times takes $O(n \log n)$ time.

Note: Can be implemented in an online fashion?

No, because need to know the future to tell whether to accept a job or not.

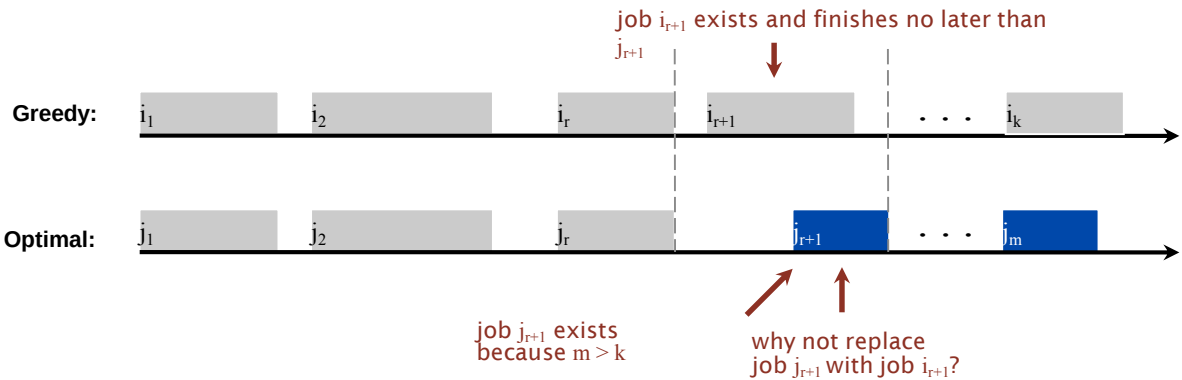
*Yes, if allowed to drop jobs halfway through (and lose the credit)

Interval scheduling: analysis of earliest-finish-time-first algorithm

Theorem. The earliest-finish-time-first algorithm is optimal.

Pf. **Succeed by not failing.**

- Assume greedy made first irredeemable mistake at step r+1.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in an optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$



Interval scheduling: analysis of earliest-finish-time-first algorithm

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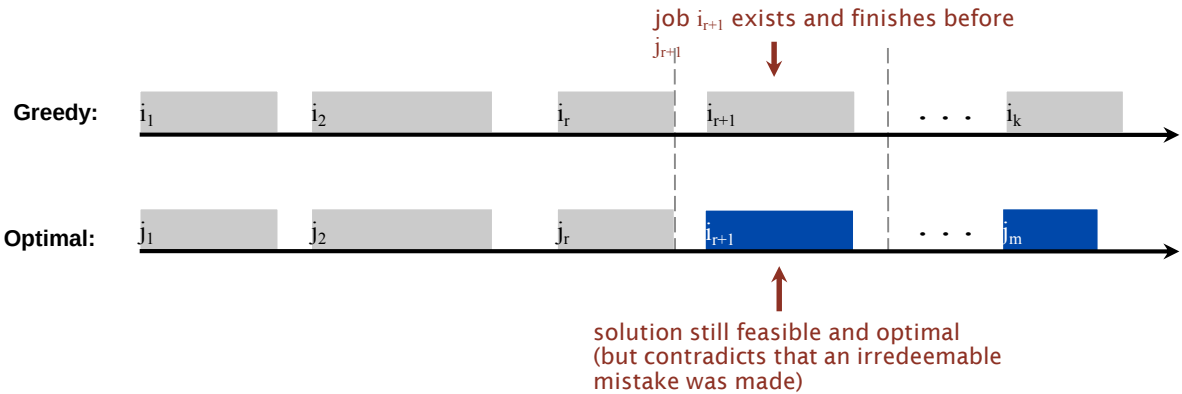


Recap: greedy algorithms

- Greedy algorithms process the input in some order, and make irrevocable decisions.
- They succeed by never making decisions that are irreversibly wrong.
- Design decisions:
  - Order of processing
  - What to be greedy about

Recap: online decision making

- Online decision making algorithms operate on only part of the input, which arrives in order we can't control.
- Typically cannot attain "optimal in hindsight" performance, the goal is to approximate its performance.
- All online decision algorithms are greedy in some sense.
- Algorithm design reduces to two goals:

(1) Objective function design: "what to be greedy about?"
(2) Optimization: "how to find the best next step for the objective?"

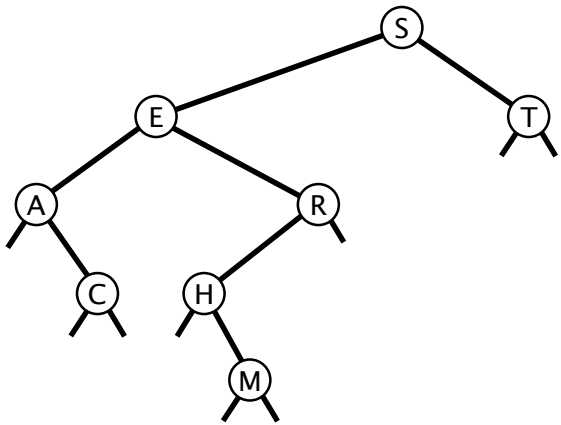
Graph algorithms

BFS vs. level-order-traversal of a binary search tree.

LEVEL-ORDER TRAVERSAL (FROM 5 WEEKS AGO)

Level-order traversal of a binary tree.

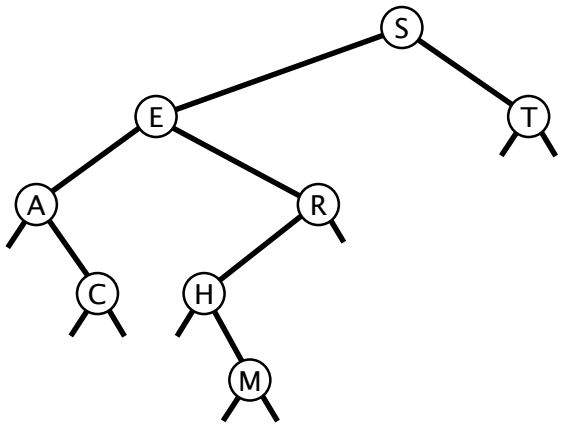
- Process root.
- Process children of root, from left to right.
- Process grandchildren of root, from left to right.
- …



level-order traversal: S E T A R C H M

LEVEL-ORDER TRAVERSAL

Q1. Given binary tree, how to compute level-order traversal?




```
public Iterable<Key> levelOrder() {
    Queue<Key> keys = new Queue<Key>();
    Queue<Node> queue = new Queue<Node>();
    queue.enqueue(root);
    while (!queue.isEmpty()) {
        Node x = queue.dequeue();
        if (x == null) continue;
        keys.enqueue(x.key);
        queue.enqueue(x.left);
        queue.enqueue(x.right);
    }
    return keys;
}
```

level-order traversal: S E T A R C H M

Graph algorithms

BFS vs. level-order-traversal of a binary search tree.

```
// BFS from single source
private void bfs(Digraph G, int s) {
    Queue<Integer> q = new Queue<Integer>();
    marked[s] = true;
    distTo[s] = 0;
    q.enqueue(s);
    while (!q.isEmpty()) {
        int v = q.dequeue();
        for (int w : G.adj(v)) {
            if (!marked[w]) {
                edgeTo[w] = v;
                distTo[w] = distTo[v] + 1;
                marked[w] = true;
                q.enqueue(w);
            }
        }
    }
}
```

```
public Iterable<Key> levelOrder() {
    Queue<Key> keys = new Queue<Key>();
    Queue<Node> queue = new Queue<Node>();
    queue.enqueue(root);
    while (!queue.isEmpty()) {
        Node x = queue.dequeue();
        if (x == null) continue;
        keys.enqueue(x.key);
        queue.enqueue(x.left);
        queue.enqueue(x.right);
    }
    return keys;
}
```

Cycle detection in digraphs

An example of timing analysis (similar to WordNet readme).

```
public DirectedCycle(Digraph G) {
    marked  = new boolean[G.V()];
    onStack = new boolean[G.V()];
    edgeTo  = new int[G.V()];
    for (int v = 0; v < G.V(); v++)
        if (!marked[v] && cycle == null) dfs(G, v);
}
```

Running time?

V (always) + cost of dfs calls.

Cost of dfs(G, v) calls?

```
private void dfs(Digraph G, int v) {
    onStack[v] = true;
    marked[v] = true;
    for (int w : G.adj(v)) {

        // short circuit if directed cycle found
        if (cycle != null) return;

        // found new vertex, so recur
        else if (!marked[w]) {
            edgeTo[w] = v;
            dfs(G, w);
        }

        // trace back directed cycle
        else if (onStack[w]) {
            cycle = new Stack<Integer>();
            for (int x = v; x != w; x = edgeTo[x]) {
                cycle.push(x);
            }
            cycle.push(w);
            cycle.push(v);
            assert check();
        }
    }
    onStack[v] = false;
}
```

Each edge visited at most once throughout the execution of DirectedCycle(..)
Cost: $\leq E$
Exactly once if no cycles.

Run at most once, throughout the execution of DirectedCycle(..)
Cost: $\leq V$

Cycle detection in digraphs

```
public DirectedCycle(Digraph G) {
    marked  = new boolean[G.V()];
    onStack = new boolean[G.V()];
    edgeTo  = new int[G.V()];
    for (int v = 0; v < G.V(); v++)
        if (!marked[v] && cycle == null) dfs(G, v);
}
```

Running time?

V (always) + cost of dfs calls.

Worst case: O(V+E);

Best case: O(V);

Best case if there are no cycles: O(V+E).

Example: exam problem from Spring'12

Given an edge-weighted digraph G the *bottleneck capacity of a path* is the minimum weight of an edge on the path.

(a) Given an edge-weighted digraph G, two distinguished vertices s and t, and a threshold value T, design an algorithm to find any one path from s to t of bottleneck capacity greater than or equal to T or report that no such path exists. The order of growth of the worst case running time of your algorithm should be $E + V$.

(b) Using the subroutine from (a), design an algorithm to find a *maximum bottleneck capacity path* from s to t in an edge-weighted digraph G. The order of growth of the worst case running time of your algorithm should be $(E + V) \log E$.

Note: solution is on the course webpage (Final Spring'12)