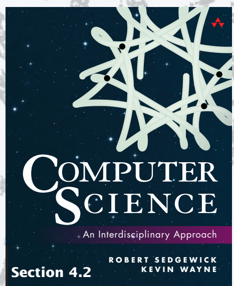
COMPUTER SCIENCE
SEDGEWICK / WAYNE
PART II: ALGORITHMS, THEORY, AND MACHINES



http://introcs.cs.princeton.edu

11. Sorting and Searching

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PART II: ALGORITHMS, THEORY, AND MACHINES

11. Searching and Sorting

- **A typical client**
- Binary search
- Insertion sort
- Mergesort
- Longest repeated substring

CS.11.A.SearchSort.Client

A typical client: Whitelist filter

A blacklist is a list of entities to be *rejected* for service. ← Examples: Overdrawn account, Spammers

A whitelist is a list of entities to be *accepted* for service. ← Examples: Account in good standing, Friends and relatives

Whitelist filter
- Read a list of strings from a *whitelist* file.
- Read strings from StdIn and write to StdOut only those in the whitelist.

Example. Email spam filter
(message contents omitted)

whitelist

```
alice@home
bob@office
carl@beach
dave@boat
```

StdIn

```
bob@office    ✓
carl@beach    ✓
marvin@spam
bob@office    ✓
bob@office    ✓
mallory@spam
dave@boat     ✓
eve@airport
alice@home    ✓
...
```

StdOut

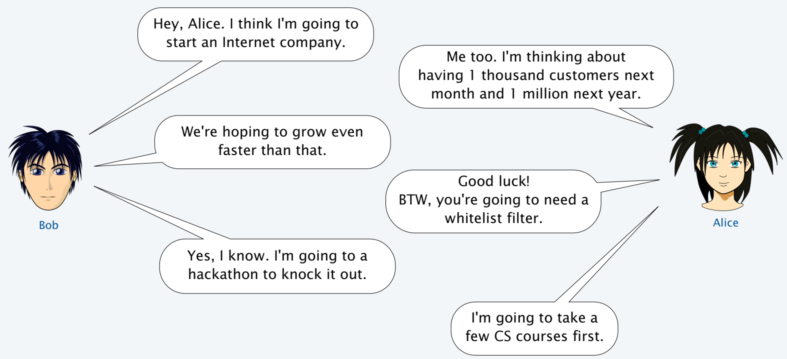
```
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
...
```

Search client: Whitelist filter

```
public class WhiteFilter
{
   public static int search(String key, String[] a)
   // Search method (stay tuned).

   public static void main(String[] args)
   {
      In in = new In(args[0]);
      String[] words = in.readAllStrings();

      while (!StdIn.isEmpty())
      {
         String key = StdIn.readString();
         if (search(key, words) != -1)
            StdOut.println(key);
      }
   }
}
```

```
% more white4.txt
alice@home
bob@office
carl@beach
dave@boat

% more test.txt
bob@office
carl@beach
marvin@spam
bob@office
bob@office
mallory@spam
dave@boat
eve@airport
alice@home

% java WhiteFilter white4.txt < test.txt
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
```

Alice and Bob



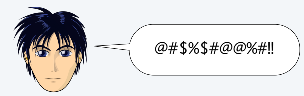
Strawman implementation: Sequential search (first try)

Sequential search

- Check each array entry 0, 1, 2, 3, ... for match with search string.
- If match found, return index of matching string.
- If not, return −1.

```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if (a[i] == key) return i;
   return -1;
}
```

✘ Compares references, not strings!



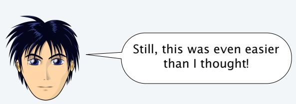
i	a[i]
0	alice ← oscar?
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

Strawman implementation: Sequential search

Sequential search

- Check each array entry 0, 1, 2, 3, ... for match with search string.
- If match found, return index of matching string.
- If not, return −1.

```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if (a[i].compareTo(key) == 0) return i;
   return -1;
}
```



Match found. Return 10

i	a[i]
0	alice ← oscar?
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

Mathematical analysis of whitelist filter using sequential search

Model

- N strings on the whitelist.
- cN transactions for constant c.
- String length not long.

Analysis

- A random search *hit* checks *about half* of the N strings on the whitelist, on average.
- A random search *miss* checks *all* of the N strings on the whitelist, on average.
- Expected order of growth of running time: N^2.

whitelist
dobqi
xwnzb
dqwak
lnuqv
czpwx
bshla
idhld
utfyw
hafah
tsirv

transactions
xwnzb
lnuqv
lnuqv
czpwx
czpwx
dqwak
idhld
dobqi
dobqi
tsirv
dqwak
dobqi
idhld
dqwak
dobqi
lnuqv
xwnzb
idhld
bshla
xwnzb

Random representative inputs for searching and sorting

Generate N random strings of length L from a given alphabet

```
public class Generator
{
   public static String randomString(int L, String alpha)
   {
      char[] a = new char[L];
      for (int i = 0; i < L; i++)
      {
         int t = StdRandom.uniform(alpha.length());
         a[i] = alpha.charAt(t);
      }
      return new String(a);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      int L = Integer.parseInt(args[1]);
      String alpha = args[2];
      for (int i = 0; i < N; i++)
         StdOut.println(randomString(L, alpha));
   }
}
```

```
% java Generator 10 3 abc
bab
bab
bbb
cac
aba
abb
bab
ccb
cbc
bab
```

good chance of duplicates

```
% java Generator 15 8 0123456789
62855405
83179069
79061047
27258805
54441080
76592141
95956542
19442316
75032539
10528640
42496398
34226197
10320073
80072566
87979201
```

not much chance of duplicates

```
% java Generator 1 60 actg
tctatagggtcgtttgcgaagcctacacaaaagtagttgttggacaacgattgacaaaca
```


Test client for sequential search

Print time required for 10N searches in a whitelist of length N

```
public class TestSS
{
   public static int search(String key, String[] a)
   {
      for (int i = 0; i < a.length; i++)
         if ( a[i].compareTo(key) == 0 ) return i;
      return -1;
   }
   public static void main(String[] args)
   {
      String[] words = StdIn.readAllStrings();
      int N = words.length;
      double start = System.currentTimeMillis()/1000.0;
      for (int i = 0; i < 10*N; i++)
      {
         String key = words[StdRandom.uniform(N)];
         if (search(key, words) == -1)
            StdOut.println(key);
      }
      double now = System.currentTimeMillis()/1000.0;
      StdOut.println(Math.round(now-start) + " seconds");
   }
}
```

random successful search (no output)

a-z = abcdefghijklmnopqrstuvwxyz

```
% java Generator 10000 10 a-z | java TestSS
3 seconds
```

generate 10,000 ten-letter words (lowercase)

print time for 100,000 searches

Empirical tests of sequential search

Whitelist filter scenario
- Whitelist of size N.
- $10N$ transactions.

N	T_N (seconds)	$T_N/T_{N/2}$	*transactions per second*
10,000	3		3,333
20,000	9		2,222
40,000	35	3.9	1,143
80,000	149	4.3	536
...			
1.28 million	38,500	4	34

```
% java Generator 10000 ...
3 seconds
% java Generator 20000 ...
9 seconds
% java Generator 40000 ...
35 seconds
% java Generator 80000 ...
149 seconds
```

... = 10 a-z | java TestSS

more than 10.5 hours

1.28 million transactions at a rate of 34 per second and dropping

Doubling method

Hypothesis. The running time of my program is $T_N \sim a N^b$.

Consequence. As N increases, $T_N/T_{N/2}$ approaches 2^b.

Proof: $\frac{a(2N)^b}{aN^b} = 2^b$ ← no need to calculate a (!)

Validates hypothesis that order of growth is N^2. ← Does NOT scale.

Hmmm. That doesn't seem too good.

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Image sources

https://openclipart.org/detail/25617/astrid-graeber-adult-by-anonymous-25617
https://openclipart.org/detail/169320/girl-head-by-jza

CS.13.A.SearchSort.Client

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11. Sorting and Searching

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- Longest repeated substring

CS.11.B.SearchSort.BinarySearch

Binary search

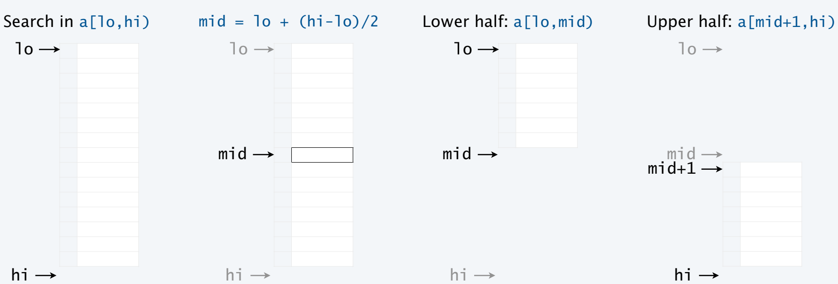
Binary search
- Keep the array in sorted order (stay tuned).
- Examine the middle key.
- If it matches, return its index.
- If it is larger, search the half with lower indices.
- If it is smaller, search the half with upper indices.

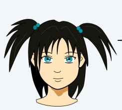
```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if ( a[i].compareTo(key) == 0 ) return i;
   return -1;
}
```

i	a[i]
0	alice
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

Binary search arithmetic

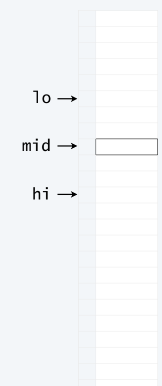
Notation. `a[lo,hi)` means `a[lo], a[lo+1] ... a[hi-1]` (does not include `a[hi]`).

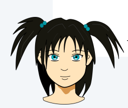




Binary search: Java implementation

```
public static int search(String key, String[] a)
{  return search(key, a, 0, a.length); }

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```





Recursion trace for binary search

```
public static int search(String key, String[] a)
{  return search(key, a, 0, a.length); }

public static int search(String key, String[] a,
                                      int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

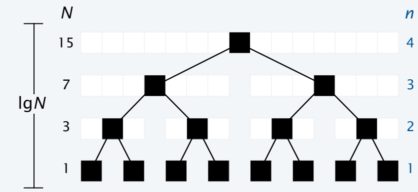
```
search("oscar")
   return            10

   search("oscar", a, 0, 15)
      mid = 7;
      > "eve"
      return         10

   search("oscar", a, 8, 15)
      mid = 11;
      < "peggy"
      return         10

   search("oscar", a, 8, 11)
      mid = 9;
      > "mallory"
      return         10

   search("oscar", a, 10, 11)
      mid = 10;
      == "oscar"
      return 10;
```

10 oscar

Mathematical analysis of binary search

Exact analysis for search miss for $N = 2^n - 1$

- Note that $n = \lg(N+1) \sim \lg N$.
- Subarray size for 1st call is $2^n - 1$.
- Subarray size for 2nd call is $2^{n-1} - 1$.
- Subarray size for 3rd call is $2^{n-2} - 1$.
- ...
- Subarray size for nth call is 1.
- Total # compares (one per call): $n \sim \lg N$.

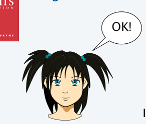


Every search miss is a top-to-bottom path in this tree.

Proposition. Binary search uses ~lg N compares for a search miss.

Proof. An (easy) exercise in discrete math.

Interested in details? Take a course in algorithms.



Proposition. Binary search uses ~lg N compares for a random search hit.

Proof. A slightly more difficult exercise in discrete math.

Empirical tests of binary search

Whitelist filter scenario

- Whitelist of size N.
- $10N$ transactions.

N	T_N (seconds)	$T_N/T_{N/2}$	*transactions per second*
100,000	1		
200,000	3		
400,000	6	2	67,000
800,000	14	2.35	57,000
1,600,000	33	2.33	48,000
10.28 million	264	2	48,000

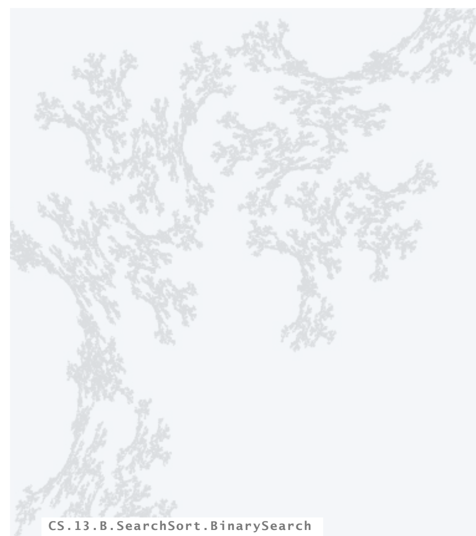
nearly 50,000 transactions per second, and holding

```
% java Generator 100000 ...
1 seconds
% java Generator 200000 ...
3 seconds
% java Generator 400000 ...
6 seconds
% java Generator 800000 ...
14 seconds
% java Generator 1600000 ...
33 seconds
```

```
... = 10 a-z | java TestBS
a-z = abcdefghijklmnopqrstuvwxyz
```

Validates hypothesis that order of growth is $N\log N$.

Will scale.



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CS.13.B.SearchSort.BinarySearch

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PART II: ALGORITHMS, THEORY, AND MACHINES

11. Sorting and Searching

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- Insertion sort
- Mergesort
- Longest repeated substring

CS.11.C.SearchSort.Insertion

Sorting: Rearrange N items to put them in ascending order

Applications
- Binary search
- Statistics
- Databases
- Data compression
- Bioinformatics
- Computer graphics
- Scientific computing
- ...
- [Too numerous to list]

0	wendy
1	alice
2	dave
3	walter
4	carlos
5	carol
6	erin
7	oscar
8	peggy
9	trudy
10	eve
11	trent
12	bob
13	craig
14	frank
15	victor

→

0	alice
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	oscar
10	peggy
11	trent
12	trudy
13	victor
14	walter
15	wendy

Pop quiz 0 on sorting

Q. What's the most efficient way to sort 1 million 32-bit integers?

Insertion sort algorithm

Insertion sort
- Move down through the array.
- Each item *bubbles up* above the larger ones above it.
- Everything above the current item is in order.
- Everything below the current item is untouched.

Like bubble sort, but not bubble sort.
We don't teach bubble sort any more because this is simpler and faster.

0	wendy
1	alice
2	dave
3	walter
4	carlos
5	carol
6	erin
7	oscar
8	peggy
9	trudy
10	eve
11	trent
12	bob
13	craig
14	frank
15	victor

Insertion sort trace

0	wendy	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice
1	alice	wendy	dave	dave	carlos	carlos	carlos	carlos	carlos	carlos	carlos	carlos	bob	bob	bob	bob
2	dave	dave	wendy	walter	dave	carol	carol	carol	carol	carol	carol	carol	carlos	carlos	carlos	carlos
3	walter	walter	walter	wendy	walter	dave	dave	dave	dave	dave	dave	dave	carol	carol	carol	carol
4	carlos	carlos	carlos	carlos	wendy	walter	erin	erin	erin	erin	erin	erin	dave	craig	craig	craig
5	carol	carol	carol	carol	carol	wendy	walter	oscar	oscar	oscar	eve	eve	erin	dave	dave	dave
6	erin	erin	erin	erin	erin	erin	wendy	walter	peggy	peggy	oscar	oscar	eve	erin	erin	erin
7	oscar	oscar	oscar	oscar	oscar	oscar	oscar	wendy	walter	trudy	peggy	peggy	oscar	eve	eve	eve
8	peggy	peggy	peggy	peggy	peggy	peggy	peggy	peggy	wendy	walter	trudy	trent	peggy	oscar	frank	frank
9	trudy	trudy	trudy	trudy	trudy	trudy	trudy	trudy	trudy	wendy	walter	trudy	trent	peggy	oscar	oscar
10	eve	eve	eve	eve	eve	eve	eve	eve	eve	eve	wendy	walter	trudy	trent	peggy	peggy
11	trent	trent	trent	trent	trent	trent	trent	trent	trent	trent	trent	wendy	walter	trudy	trent	trent
12	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	wendy	walter	trudy	trudy
13	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	wendy	walter	victor
14	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	wendy	walter
15	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	wendy

Insertion sort: Java implementation

```
public class Insertion
{
   public static void sort(String[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1].compareTo(a[j]) > 0)
               exch(a, j-1, j);
            else break;
   }

   private static void exch(String[] a, int i, int j)
   {  String t = a[i]; a[i] = a[j]; a[j] = t;  }

   public static void main(String[] args)
   {
      String[] a = StdIn.readAllStrings();
      sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a[i]);
   }
}
```

```
% more names16.txt
wendy
alice
dave
walter
carlos
carol
erin
oscar
peggy
trudy
eve
trent
bob
craig
frank
victor
```

```
% java Insertion < names16.txt
alice
bob
carlos
carol
craig
dave
erin
eve
frank
oscar
peggy
trent
trudy
victor
walter
wendy
```


Empirical tests of insertion sort

Sort random strings
- Array of length *N*.
- 10-character strings.

N	T_N (seconds)	$T_N / T_{N/2}$
20,000	1	
40,000	4	
80,000	35	9
160,000	225	6.4
320,000	1019	4.5
…		
1.28 million	14400	4

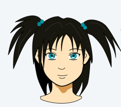
← 4 hours

```
% java Generator 20000 ...
1 seconds
% java Generator 40000 ...
4 seconds
% java Generator 80000 ...
35 seconds
% java Generator 160000 ...
225 seconds
% java Generator 320000 ...
1019 seconds
```

```
... = 10 a-z | java Insertion
a-z = abcdefghijklmnopqrstuvwxyz
```

Confirms hypothesis that order of growth is N^2.

will NOT scale



A rule of thumb

Moore's law. The number of transistors in an integrated circuit doubles about every 2 years.

Implications
- Memory size doubles every two years.
- Processor speed doubles every two years.

Gordon Moore
Founder of Intel
1929 –

Sedgewick's rule of thumb. It takes *a few seconds* to access every word in a computer.

computer	*instructions per second*	*words of memory*
PDP-9	tens of thousands	tens of thousands
VAX 11-780	millions	millions
CRAY 1	tens of millions	tens of millions
MacBook Air	billions	billions

Scalability

An algorithm *scales* if its running time doubles when the problem size doubles.

2x faster computer with 2x memory using an alg that scales?
- Can solve problems we're solving now in half the time.
- Can solve a 2x-sized problem in the *same* time it took to solve an x-sized problem.
- Progress.

2x faster computer with 2x memory using quadratic alg?
- Can solve problems we're solving now in half the time.
- Takes *twice* as long solve a 2x-sized problem as it took to solve an x-sized problem.
- Frustration.

order of growth	*scales?*
N	✓
$N \log N$	✓
N^2	✗
N^3	✗

Bottom line. Need algorithms that scale to keep pace with Moore's law.

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Image sources

https://www.youtube.com/watch?v=k4RRi_ntQc8

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- **Mergesort**
- Longest repeated substring

Mergesort algorithm

Mergesort
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

Divide	Sort halves	Merge
wendy	alice	
alice	carlos	
dave	carol	
walter	dave	
carlos	erin	
carol	oscar	
erin	walter	
oscar	wendy	
peggy	bob	
trudy	craig	
eve	eve	
trent	frank	
bob	peggy	
craig	trent	
frank	trudy	
victor	victor	

John von Neumann
1903–1957

John von Neumann
- Pioneered computing (stay tuned).
- Early focus on numerical calculations.
- Invented mergesort as a test to see how his machine would measure up on other tasks.

Merge: Java implementation

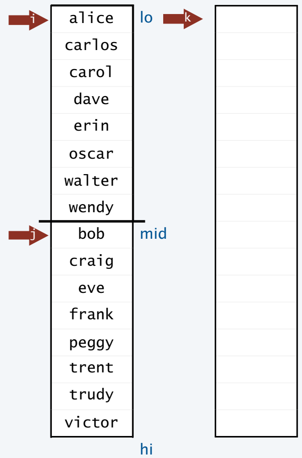
Abstract inplace merge

- Merge a[lo, mid) with a[mid, hi).
- Use auxiliary array for result.
- Copy back when merge is complete.

t

[0,

```
   int i = lo, j = mid, N = hi - lo;
   for (int k = 0; k < N; k++)
   {
      if      (i == mid) aux[k] = a[j++];
      else if (j == hi)  aux[k] = a[i++];
      else if (a[j].compareTo(a[i]) < 0) aux[k] =
a[j++];
      else                                aux[k] =
a[i++];
   }
   // Copy back into a[lo, hi)
   for (int k = 0; k < N; k++)
       a[lo + k] = aux[k];
}
```



Mergesort: Java implementation

Mergesort

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

```
public class Merge
{
   private static String[] aux;
   public static void merge(String[] a, int lo, int mid, int hi)
   { // See previous slide.  }
   public static void sort(String[] a)
   {
      aux = new String[a.length];  // Allocate just once!
      sort(a, 0, a.length);
   }
   public static void sort(String[] a, int lo, int hi)
   {  // Sort a[lo, hi).
      int N = hi - lo;
      if (N <= 1) return;
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);
      merge(a, lo, mid, hi);
   }
   ...
```

← same test client as for Insertion

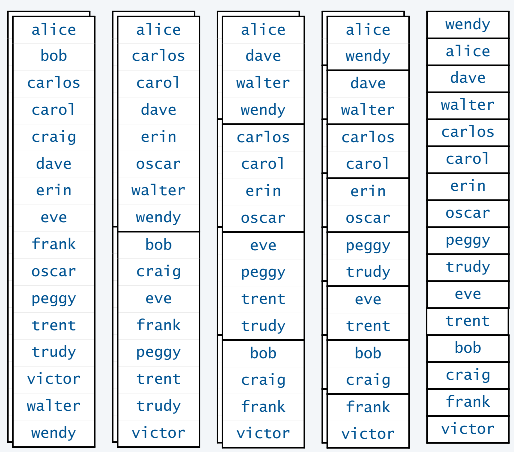
```
% more names16.txt
wendy
alice
dave
walter
carlos
carol
erin
oscar
peggy
trudy
eve
trent
bob
craig
frank
victor
```

```
% java Merge < names16.txt
alice
bob
carlos
carol
craig
dave
erin
eve
frank
oscar
peggy
trent
trudy
victor
walter
wendy
```


Mergesort trace

Mergesort

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



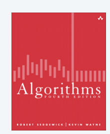
Mergesort analysis

Cost model. Count *data moves*.

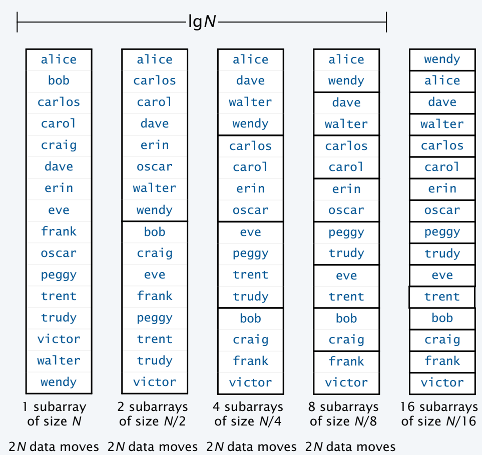
# of times a string moves from one array to another

Exact analysis for $N = 2^n$.

- Note that $n = \lg N$.
- 1 subarray of size 2^n.
- 2 subarrays of size 2^{n-1}.
- 4 subarrays of size 2^{n-2}.
- ...
- 2^n subarrays of size 1.
- Total # data moves: $2N \lg N$.



Interested in details? Take a course in algorithms.



Empirical tests of mergesort

Sort random strings
- Array of length N.
- 10-character strings.

N	T_N (seconds)	$T_N/T_{N/2}$
1 million	1	
2 million	2	
4 million	5	2.5
8 million	10	2
16 million	20	2.5
...		
1.02 billion	1280	2

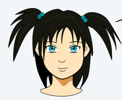
1280 seconds: 20 minutes

```
% java Generator 1000000 ...
1 seconds
% java Generator 2000000 ...
2 seconds
% java Generator 4000000 ...
5 seconds
% java Generator 8000000 ...
10 seconds
% java Generator 16000000 ...
20 seconds
```

```
... = 10 a-z | java Merge
a-z = abcdefghijklmnopqrstuvwxyz
```

Confirms hypothesis that order of growth is $N \log N$

WILL scale



COMPUTER SCIENCE
SEDGEWICK / WAYNE
PART I: PROGRAMMING IN JAVA

CS.13.D.SearchSort.Mergesort

COMPUTER SCIENCE
SEDGEWICK / WAYNE
PART II: ALGORITHMS, THEORY, AND MACHINES

11. Sorting and Searching

- A typical client
- Binary search
- Insertion sort
- Mergesort
- **Longest repeated substring**

CS.11.E.SearchSort.LRS

Detecting repeats in a string

Longest repeated substring
- Given: A string s.
- Task: Find the longest substring in s that appears at least twice.

Example 1. a a c a a g t t t a c a a g c

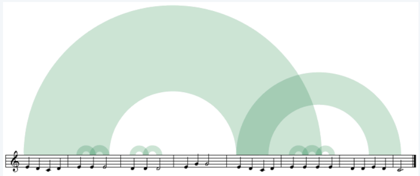
Example 2. a a c a a g t t t a c a a g t t t a c a a g c t a g c

Example 3 (first 100 digits of π).

3 . 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4
6 2 6 4 3 3 8 3 2 7 9 5 0 2 8 8 4 1 9 7
1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4
4 5 9 2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9
9 8 6 2 8 0 3 4 8 2 5 3 4 2 1 1 7 0 6 9

LRS example: repetitive structure in music

Mary had a little lamb



Für Elise

LRS applications

Analysts seek repeated sequences in real-world data because they are causal.

Example 1: Digits of π

- Q. Are they "random" ?
- A. No, but we can't tell the difference.
- Ex. Length of LRS in first 10 million digits is 14.

```
3.141592653589793238462643383279502884
19716939937510582097494459230781640628
```

```
3.141592653589793238462643383279502884
19716939937510582097494459230781640628
62089986280348253421170679821480865132
82306647093844609550582231725359408128
48111745028410270193852110555964462294
89549303819644288109756659334461284756
48233786783165271201909145648566923460
34861045432664821339360726024914127372
45870066063155881748815209209628292540
```

Example 2: Cryptography

- Find LRS.
- Check for "known" message header information.
- Break code.

```
11001001001111011011100101101011100110
00101111110100100001001101001011110011
00100111111101110000010101100010000111
01010011010000111100100110011101111111
01010000010000100010100101010001100000
10111100010010011010110111100011010011
01110011110101111001000100111010101110
10000010100100010001101010101110000000
10110000010011100010111011010010101100
```

Example 3: DNA

- Find LRS
- Look somewhere else for causal mechanisms
- Ex. Chromosome 11 has 7.1 million nucleotides

```
tgactaatccagtatccagggcaaattaggttacccac
gtgattacgagaggttccgccgctaatcgggtgcgtcc
gaaacgtatgccctcttctgctcgatgtgattggccgg
cctgtgtcatgccggcacttaaacgatcaaatagtgaa
aatcaaaatcgccggtctgtgagcctagcggatgcaag
atgggcgtacatgcccagcccaccttcggaccgagctg
cgcgtagggccgtagtgctaaagtctgagaatacccca
gtcgttcgttgaggcgcacgtctatgcataatttatgg
aggtcagtgctcttcagaggttgcagtttactctattc
```


Warmup: Longest common prefix

Longest common prefix

- Given: Two strings string s and t.
- Task: Find the longest substring that appears at the beginning of both

Example.

```
[a a c a a g t t t a c a a g] c
[a a c a a g t t t a c a a g] t t t a c a a g c t a g c
```

Implementation (easy)

```
private static String lcp(String s, String t)
{
   int N = Math.min(s.length(), t.length());
   for (int i = 0; i < N; i++)
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   return s.substring(0, N);
}
```


LRS: Brute-force implementation

```
public class LRS
{
   public static String lcp(String s)
   { // See previous slide. }

   public static String lrs(String s)
   {
      int N = s.length();
      String lrs = "";
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
         {
            String x = lcp(s.substring(i, N), s.substring(j, N));
            if (x.length() > lrs.length()) lrs = x;
         }
      return lrs;
   }
   public static void main(String[] args)
   {
      String s= StdIn.readAll();
      StdOut.println(lrs(s));
   }
}
```

```
% more tiny.txt
aacaagtttacaagc

% java LRSbrute
acaag
```

Analysis

- $\sim N^2/2$ calls on `lcp()`.
- Obviously does not scale.

LRS: An efficient solution that uses sorting

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
a a c a a g t t t a c a a g c
```

1. Form suffix strings

```
 0  a a c a a g t t t a c a a g c
 1  a c a a g t t t a c a a g c
 2  c a a g t t t a c a a g c
 3  a a g t t t a c a a g c
 4  a g t t t a c a a g c
 5  g t t t a c a a g c
 6  t t t a c a a g c
 7  t t a c a a g c
 8  t a c a a g c
 9  a c a a g c
10  c a a g c
11  a a g c
12  a g c
13  g c
14  c
```

2. Sort suffix strings

```
 0  a a c a a g t t t a c a a g c
11  a a g c
 3  a a g t t t a c a a g c
 9  a c a a g c
 1  a c a a g t t t a c a a g c
12  a g c
 4  a g t t t a c a a g c
14  c
10  c a a g c
 2  c a a g t t t a c a a g c
13  g c
 5  g t t t a c a a g c
 8  t a c a a g c
 7  t t a c a a g c
 6  t t t a c a a g c
```

3. Find longest LCP among adjacent entries.

LRS: Suffix array implementation

Form suffix strings / Sort suffix strings / Find longest LCP among adjacent entries.

```
public static String lrs(String s)
{
   int N = s.length();
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);

   Merge.sort(suffixes);

   String lrs = "";
   for (int i = 0; i < N-1; i++)
   {
      String x = lcp(suffixes[i], suffixes[i+1]);
      if (x.length() > lrs.length()) lrs = x;
   }
   return lrs;
}
```

```
% more tiny.txt
aacaagtttacaagc

% java LRS
acaag
```

Analysis
- *N* calls on `substring()`.
- *N* calls on `lcp()`.
- Potentially scales.

LRS: Empirical analysis (1995-2012)

Model
- Alphabet: `actg`.
- *N*-character random strings.

```
% java Generator 1 1000000 actg | java LRS
2 seconds
% java Generator 1 10000000 actg | java LRS
21 seconds
```

Doubling

N	T_N	$T_N/T_{N/2}$
2,000,000	3	
4,000,000	7	2.3
8,000,000	16	2.3
16,000,000	39	2.4

x10

N	T_N	$T_N/T_{N/10}$
1,000,000	2	
10,000,000	21	10

Confirms hypothesis that the order of growth is $N \log N$ (for the sort).

Bottom line. Scales with the size of the input and enables new research and development.

LRS: Empirical analysis (since 2012)

Model
- Alphabet: `actg`.
- *N*-character random strings.

```
% java Generator 1 10000 actg | java LRS
Exception in thread "main" java.lang.OutOfMemoryError: Java heap space
        at java.util.Arrays.copyOfRange(Arrays.java:3664)
        at java.lang.String.<init>(String.java:201)
        at java.lang.String.substring(String.java:1956)
        at LRS.LRS(LRS.java:17)
        at LRS.main(LRS.java:33)
```

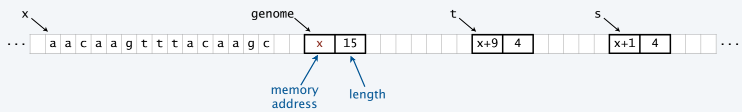
?

Change in the system ***breaks a working program*** (not good).

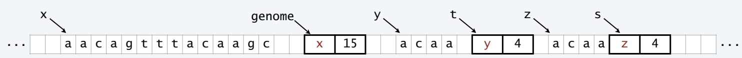
Explanation: Two alternatives for implementing substrings

1. Refer to original string (1995-2102).
   - No need to copy characters.
   - *Constant* time and space.

```
String genome = "aacaagtttacaagc";
String s = genome.substring(1, 5);
String t = genome.substring(9, 13);
```

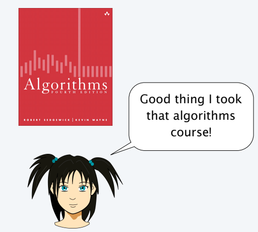


2. Copy the characters to make a new string (since 2012).
   - Allows potential to free up memory when the original string is no longer needed.
   - *Linear* time and space (in the length of the substring).



Fixing the LRS implementation

Implement our own constant-time suffix operation.
- Imitate old `substring()` implementation.
- Need `compareTo()` to enable sort.
- (Details in *Algorithms*)



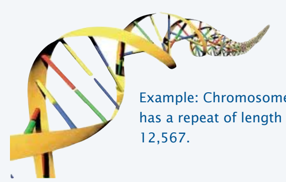

```
% java Generator 1 1000000 actg | java LRSfixed
2 seconds
% java Generator 1 10000000 actg | java LRSfixed
21 seconds
```

Lesson. Trust the *algorithm*, not the system.

Bottom line. New research and development can continue.

Final note on LRS implementation

Long repeats
- More precise analysis reveals that running time is *quadratic* in the length of the longest repeat.
- Model has no long repeats.
- Real data may have long repeats.
- Linear time algorithm (guarantee) is known.



Example: Chromosome 11 has a repeat of length 12,567.

Summary

Binary search. Efficient algorithm to search a sorted array.
Mergesort. Efficient algorithm to sort an array.
Applications. Many, many, many things are enabled by fast sort and search.

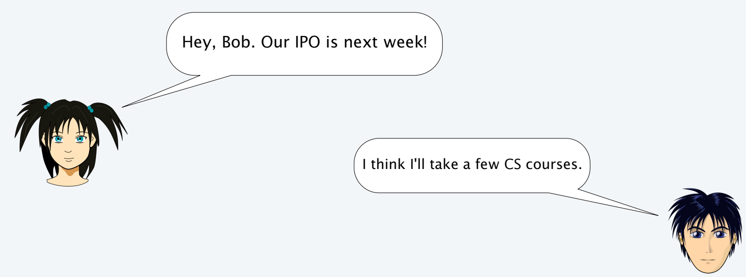
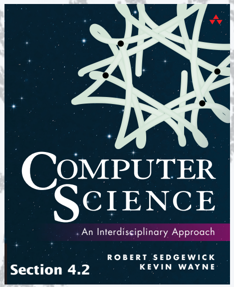


Image sources

https://www.bewitched.com/match



http://introcs.cs.princeton.edu

11. Sorting and Searching