Algorithms

FOURTH EDITION

ROBERT SEDGEWICK | KEVIN WAYNE

http://algs4.cs.princeton.edu

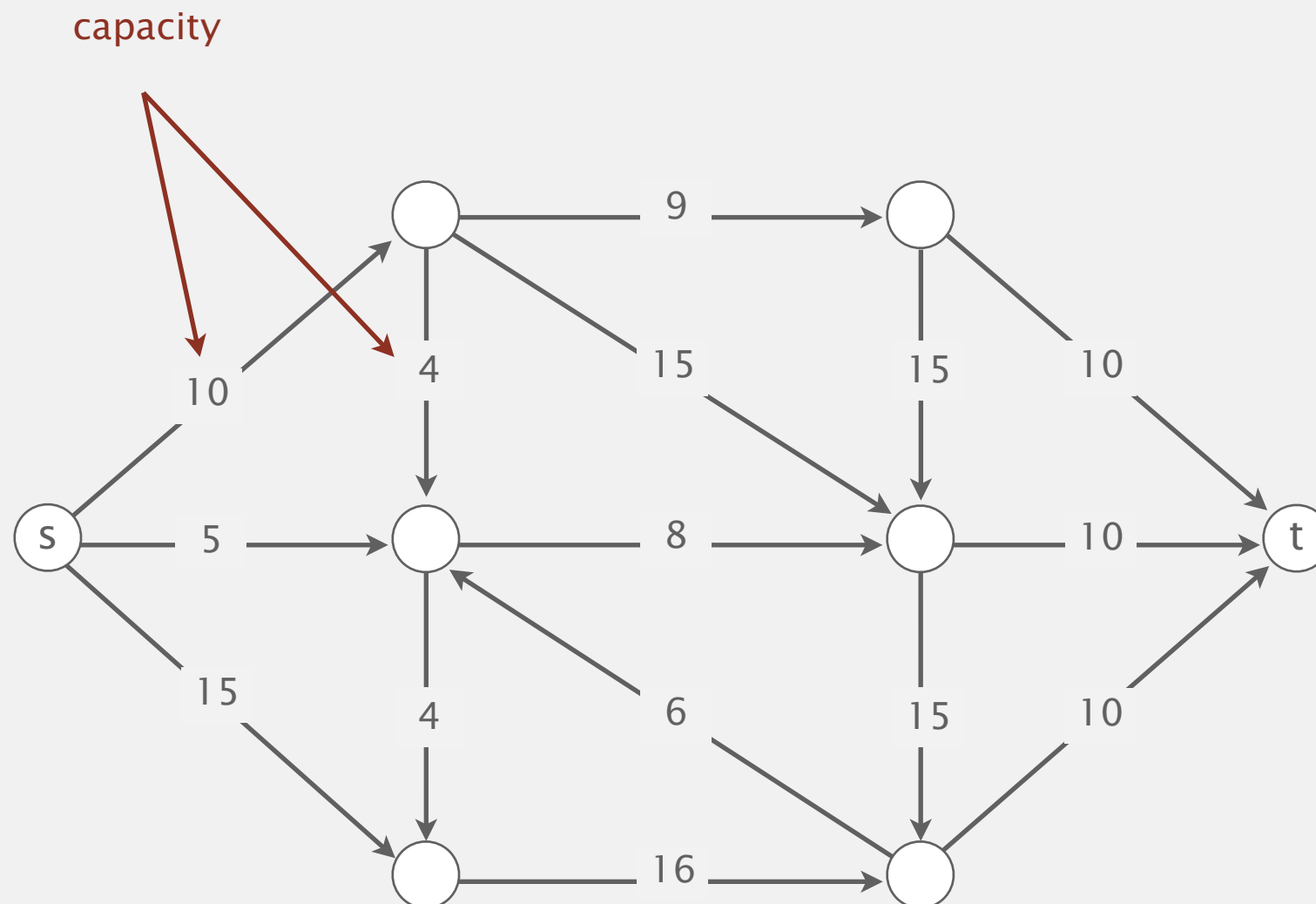# 6.4 MAXIMUM FLOW

▸ introduction

▸ Ford–Fulkerson algorithm

▸ maxflow–mincut theorem

▸ analysis of running time

▸ Java implementation

▸ applications

6.4 MAXIMUM FLOW

Algorithms

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Mincut problem

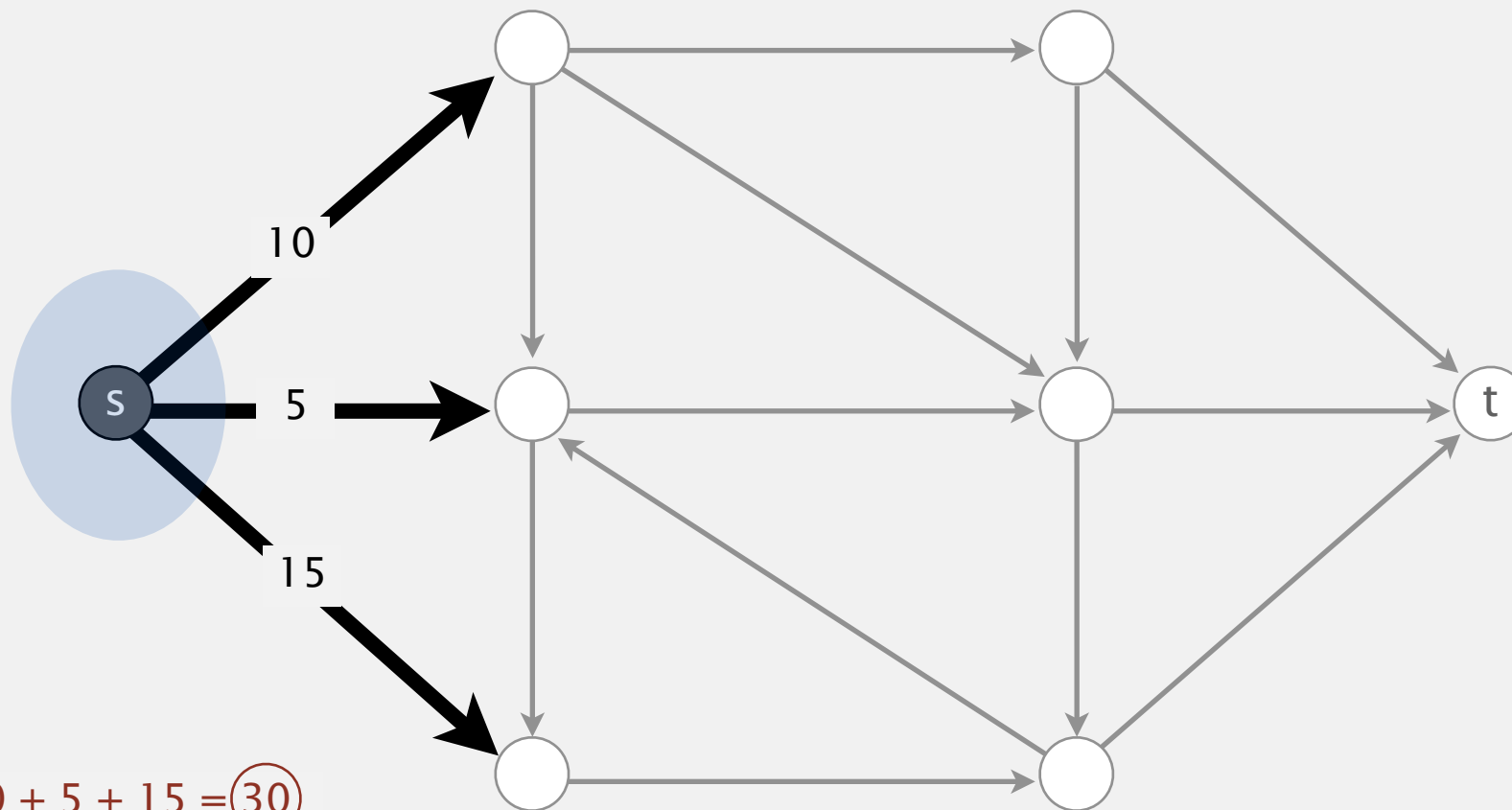**Input.** An edge-weighted digraph, source vertex s, and target vertex t.

each edge has a
positive capacity

capacity

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

capacity = 10 + 5 + 15 = 30

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

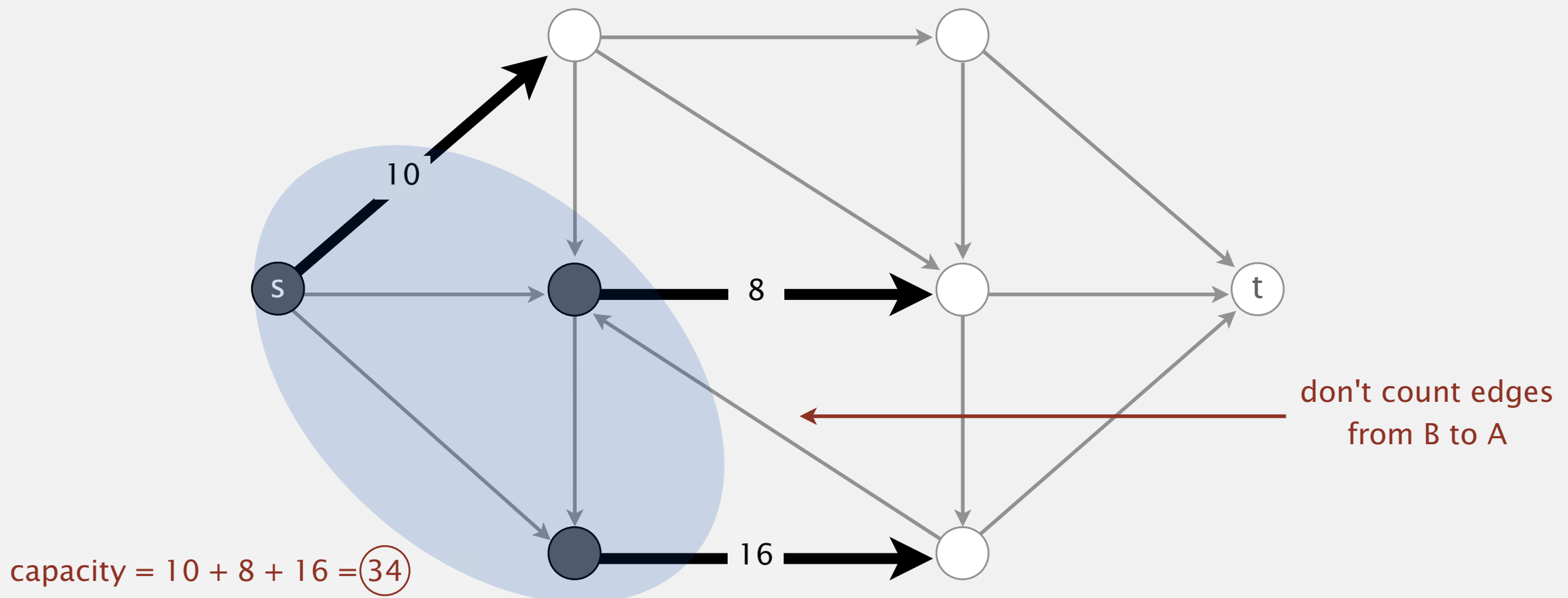Def. Its capacity is the sum of the capacities of the edges from A to B.

10

8

16

t

s

don't count edges
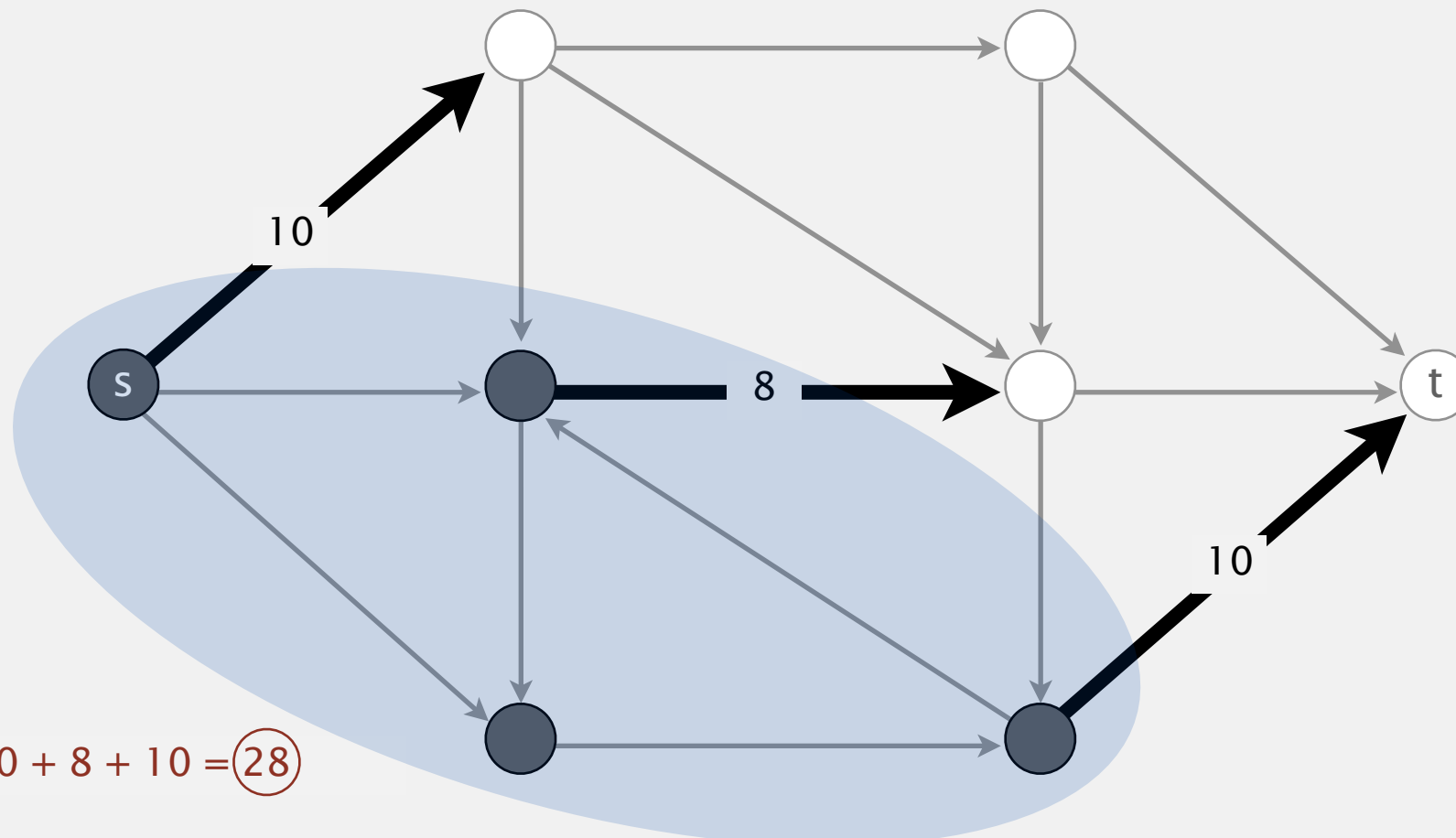from B to A

capacity = 10 + 8 + 16 = 34

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

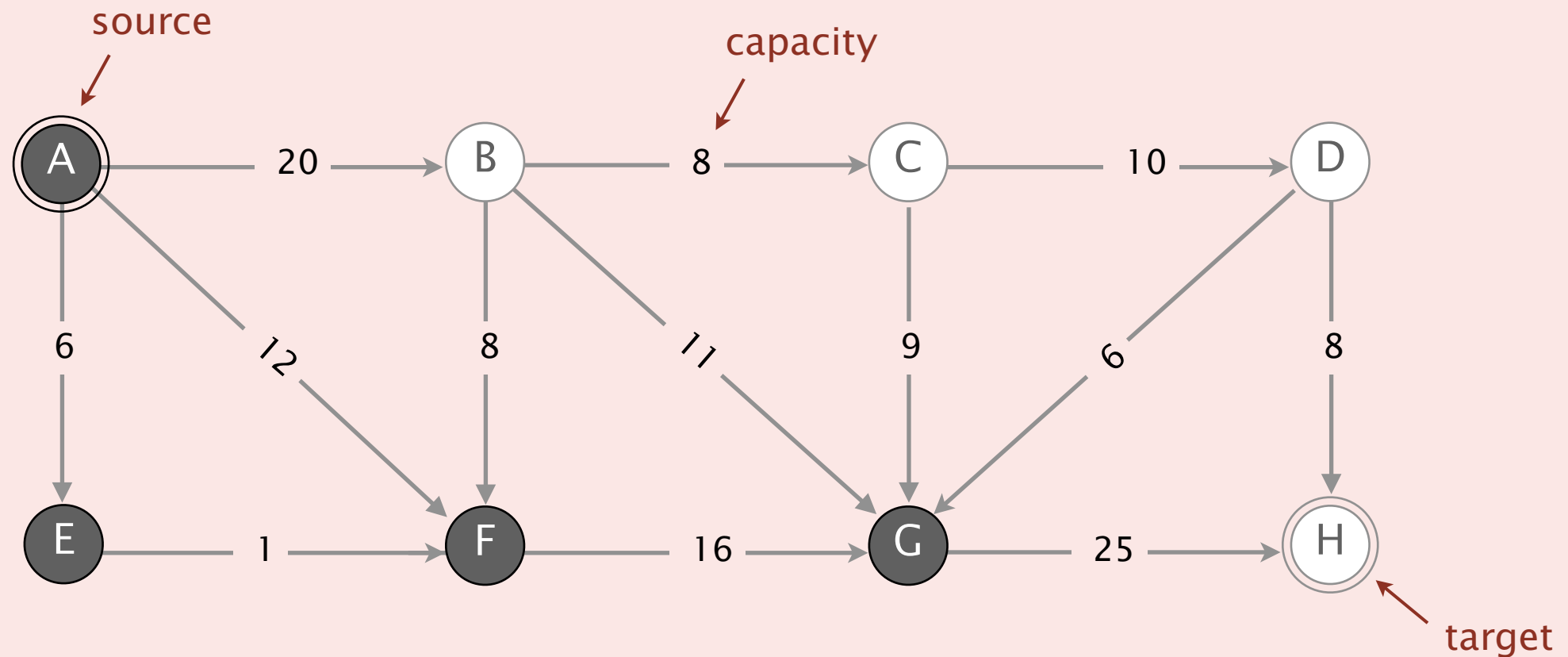Minimum st-cut (mincut) problem. Find a cut of minimum capacity.

capacity = 10 + 8 + 10 = 28
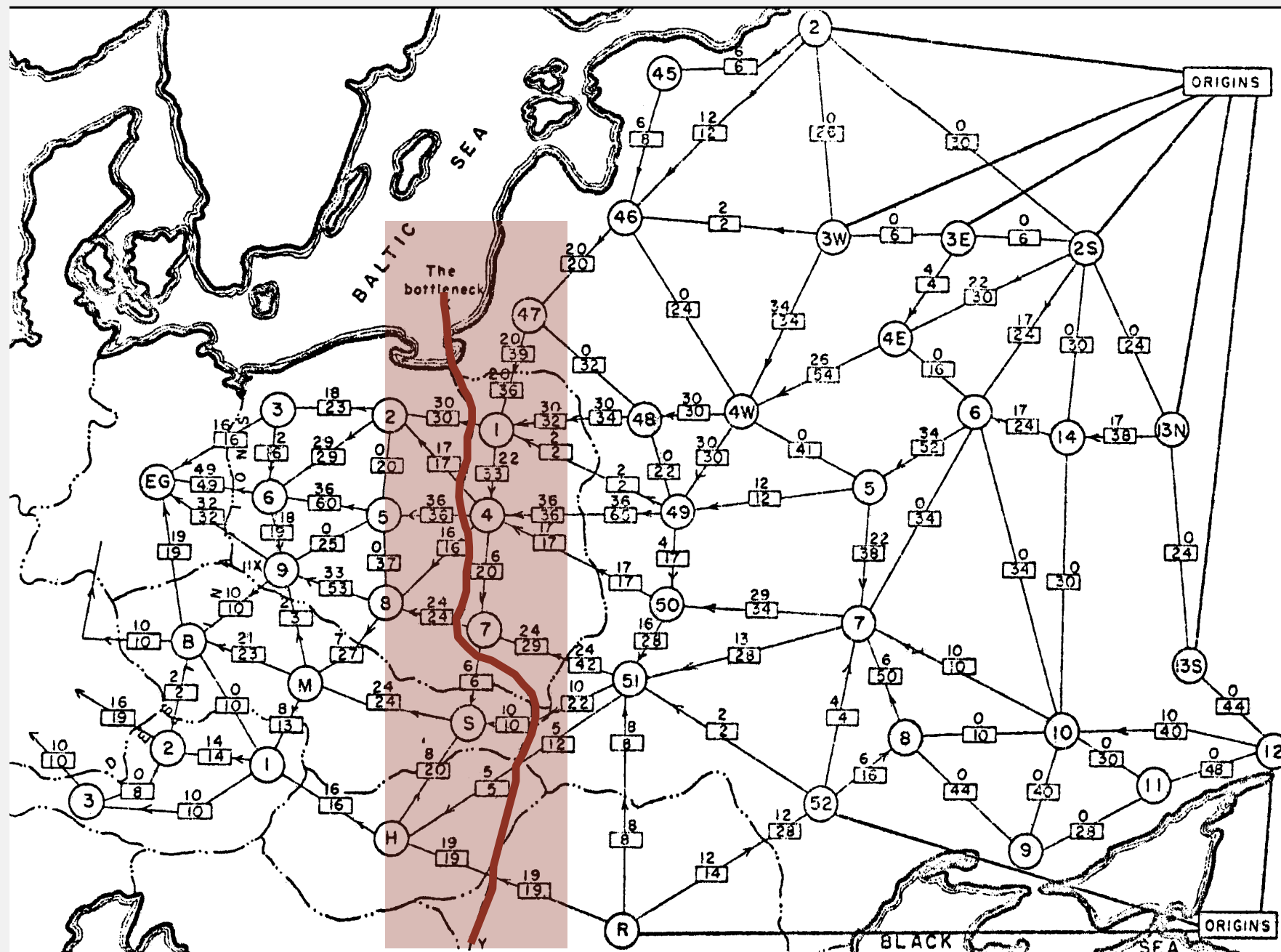
What is the capacity of the st-cut $\{A, E, F, G\}$?

A. 34 $(8 + 11 + 9 + 6)$

B. 45 $(20 + 25)$

C. 78 $(20 + 8 + 11 + 9 + 6 + 24)$

D. *I don't know.*

Mincut application (RAND 1950s)

"Free world" goal. Cut supplies (if cold war turns into real war).

rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

Government-in-power's goal. Cut off communication to set of people.

Maxflow problem

Though maximum flow algorithms have a long history, revolutionary progress is still being made.

BY ANDREW V. GOLDBERG AND ROBERT E. TARJAN

Efficient Maximum Flow Algorithms

gorithms in more detail. We restrict ourselves to basic maximum flow algorithms and do not cover interesting special cases (such as undirected graphs, planar graphs, and bipartite matchings) or generalizations (such as minimum-cost and multi-commodity flow problems).

Before formally defining the maximum flow and the minimum cut problems, we give a simple example of each problem: For the maximum flow example, suppose we have a graph that represents an oil pipeline network from an oil well to an oil depot. Each arc has a capacity, or maximum number of liters per second that can flow through the corresponding pipe. The goal is to find the maximum number of liters per second (maximum flow) that can be shipped from well to depot. For the minimum cut problem, we want to find the set of pipes of the smallest total capacity such that removing the pipes disconnects the oil well from the oil depot (minimum cut).
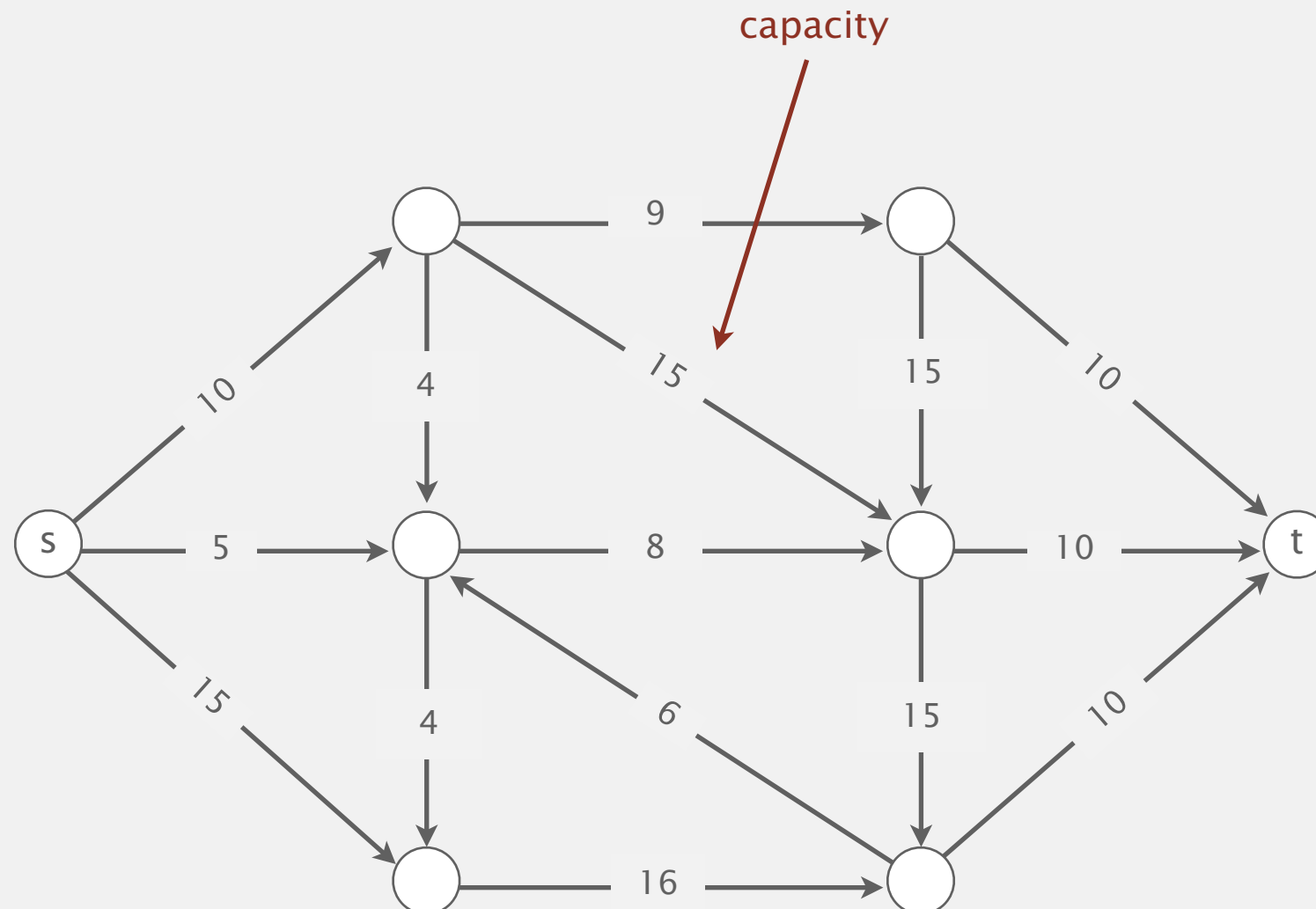
The maximum flow, minimum cut

Efficient Maximum Flow Algorithms by Andrew Goldberg and Bob Tarjan

http://vimeo.com/100774435

Maxflow problem

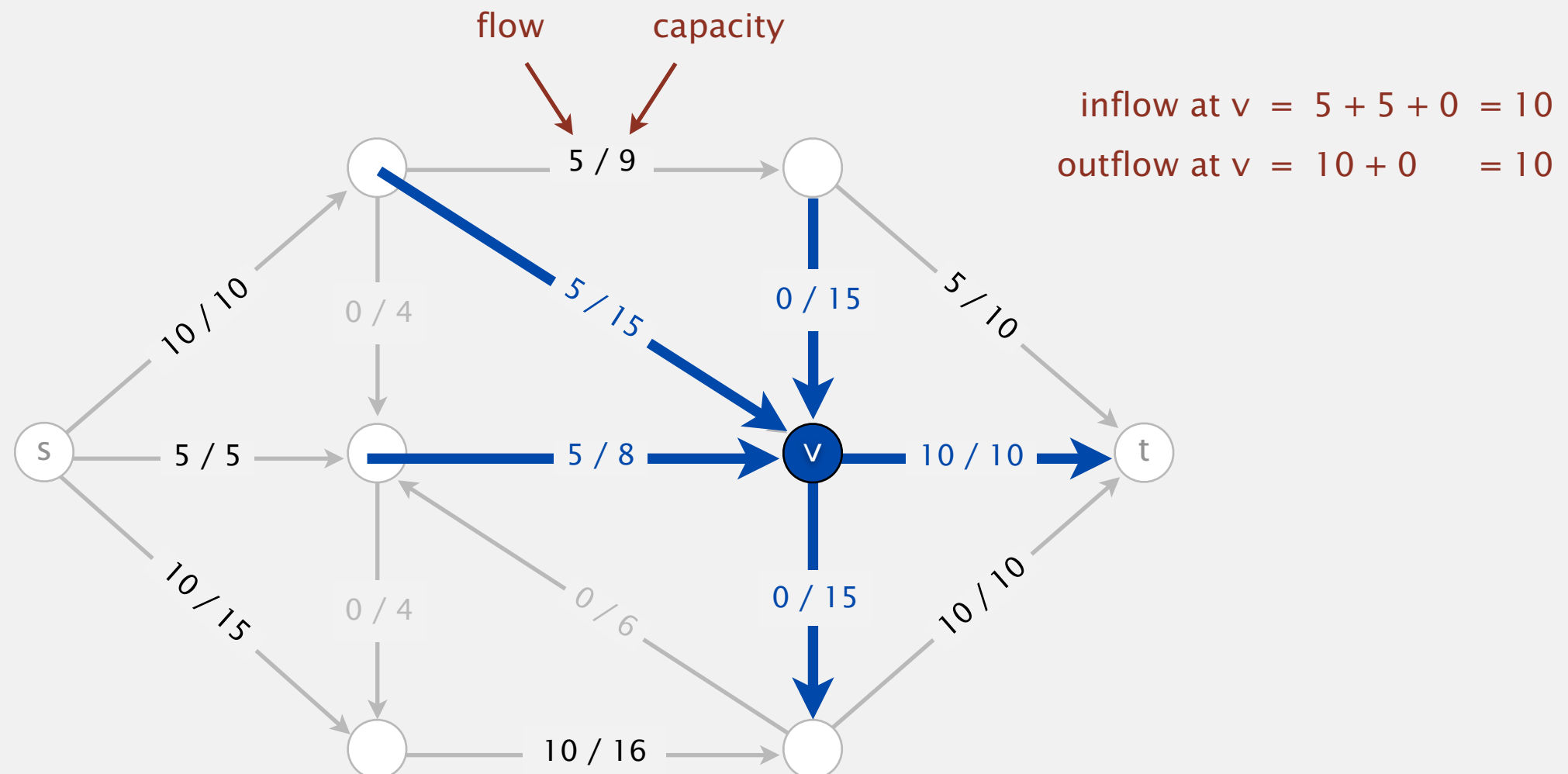Input. An edge-weighted digraph, source vertex s, and target vertex t.

each edge has a
positive capacity

capacity

Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:
- Capacity constraint: $0 \leq$ edge's flow $\leq$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except *s* and *t*).

flow   capacity

inflow at v = 5 + 5 + 0 = 10
outflow at v = 10 + 0   = 10

5 / 9

10 / 10
0 / 4
5 / 15
0 / 15
5 / 10

s   5 / 5   5 / 8   v   10 / 10   t
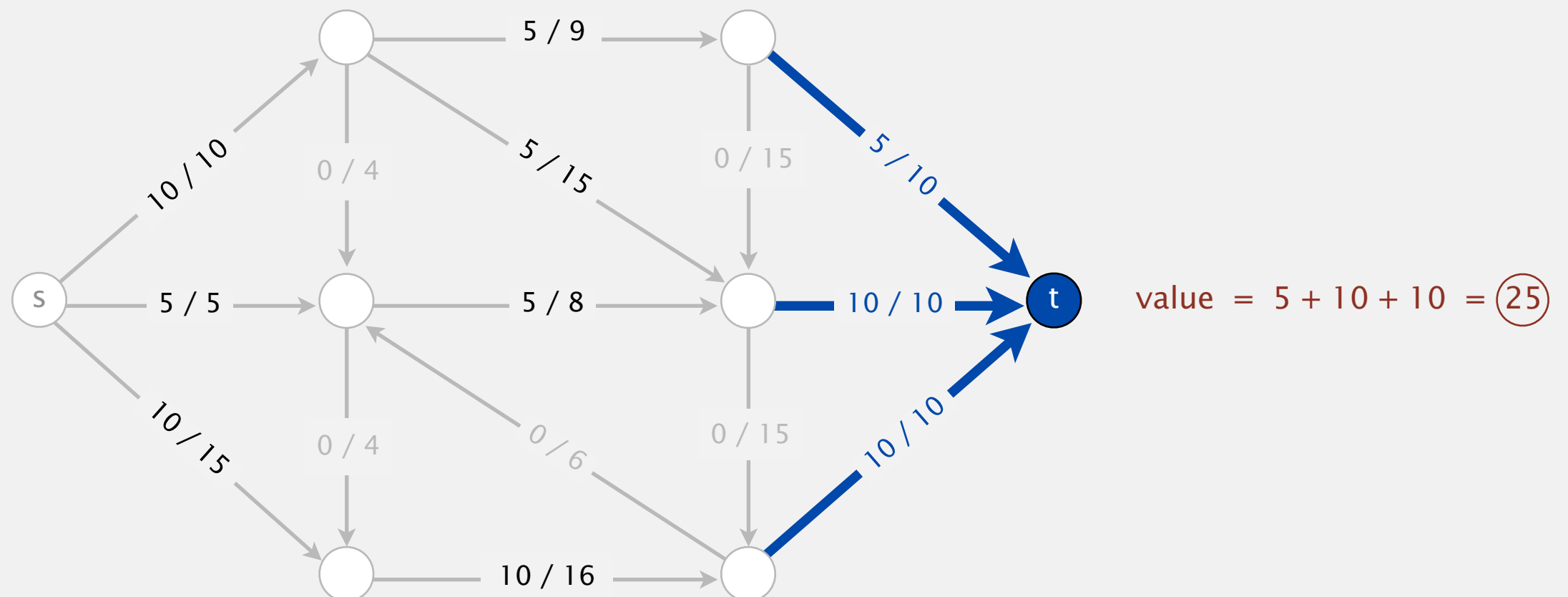
10 / 15
0 / 4
0 / 6
0 / 15
10 / 10

10 / 16

Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:
- Capacity constraint: $0 \le$ edge's flow $\le$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except *s* and *t*).

Def. The value of a flow is the inflow at *t*.

we assume no edges point to s or from t

value $= 5 + 10 + 10 = 25$
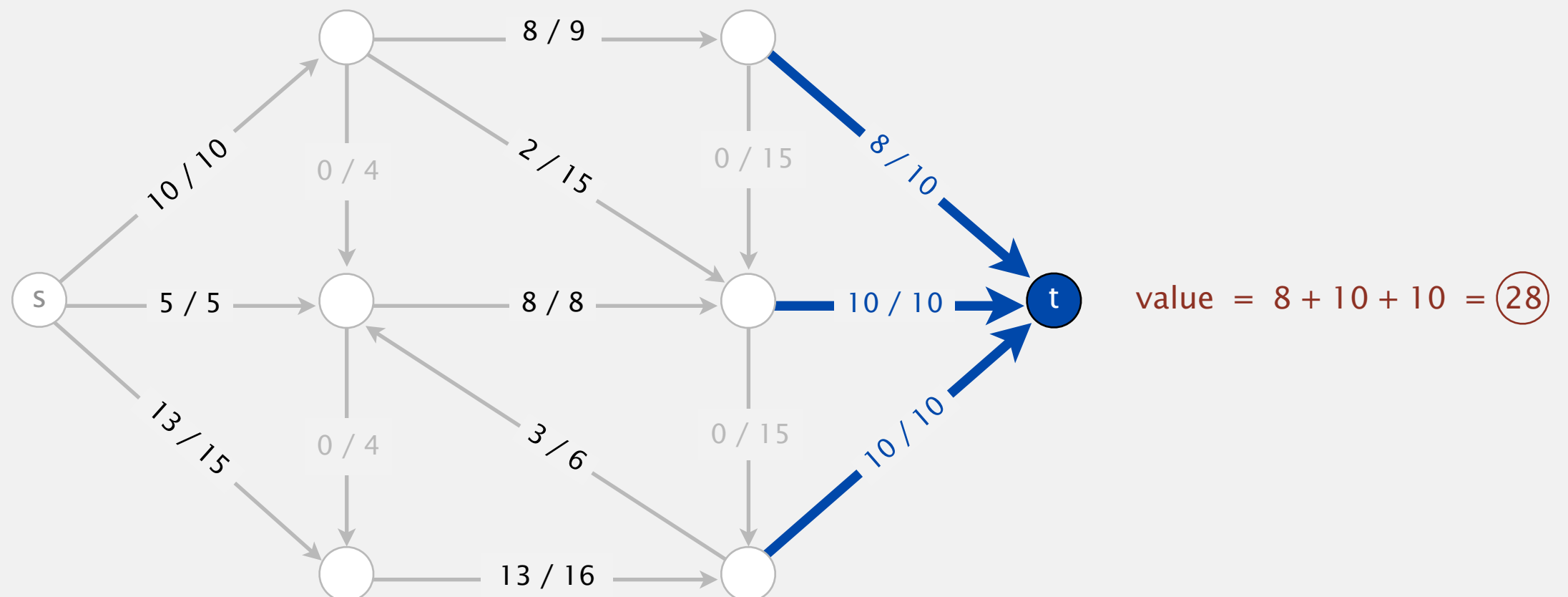
Def. An *st*-flow (flow) is an assignment of values to the edges such that:

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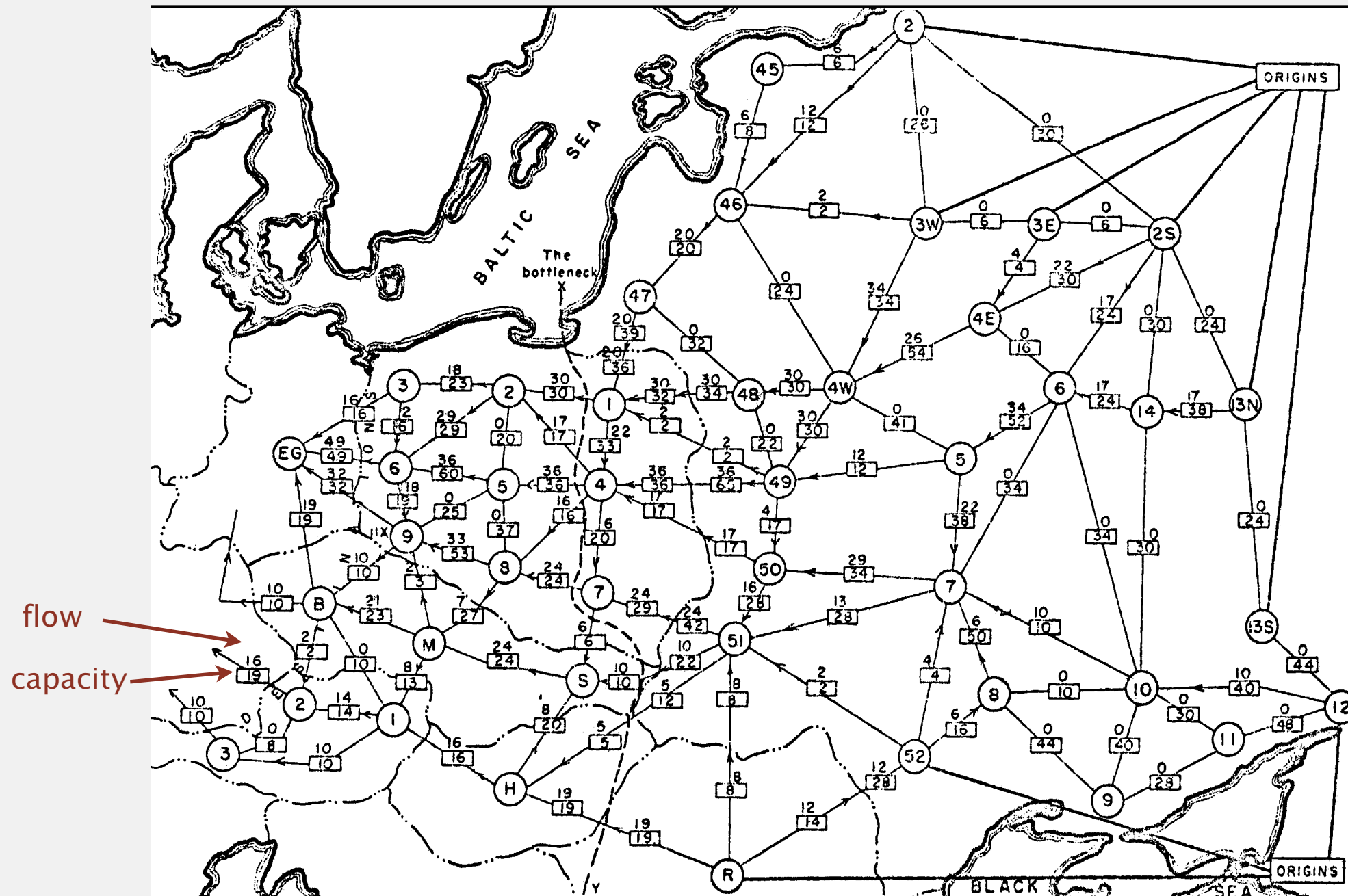
Def. The value of a flow is the inflow at *t*.

Maximum st-flow (maxflow) problem. Find a flow of maximum value.

Maxflow application (Tolstoĭ 1930s)

Soviet Union goal. Maximize flow of supplies to Eastern Europe.

rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

Potential maxflow application (2010s)

"Free world" goal. Maximize flow of information to specified set of people.

facebook graph

Summary

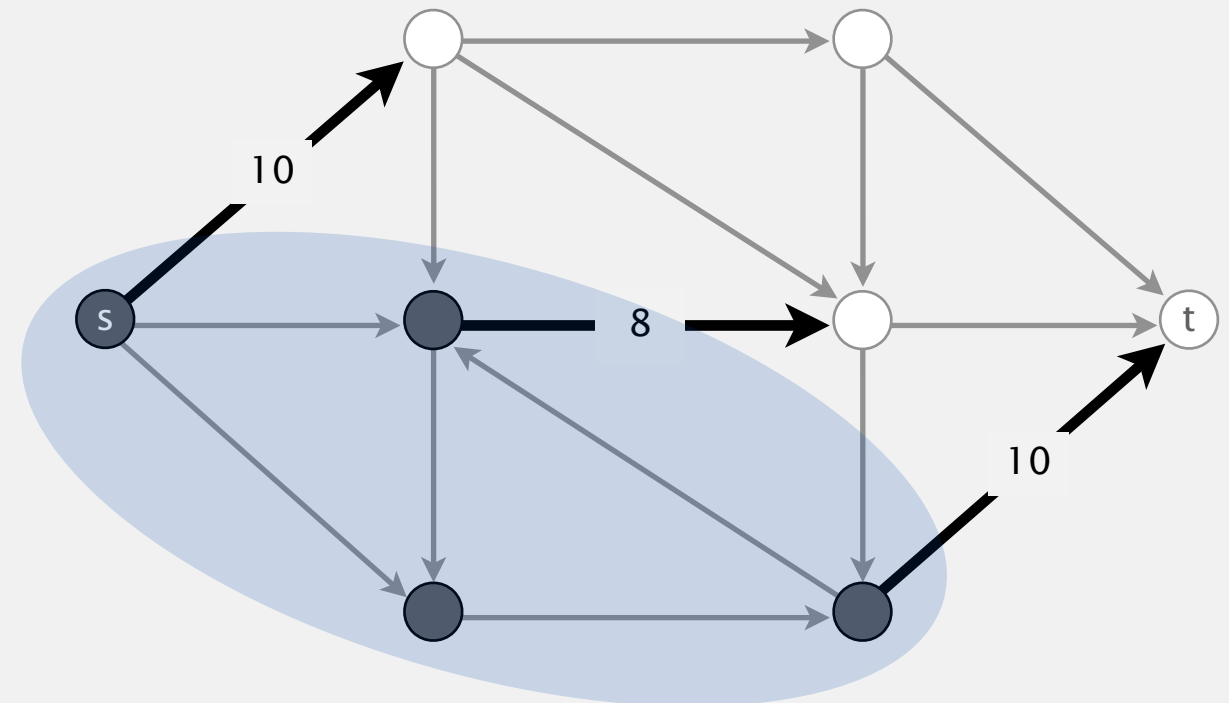Input. A weighted digraph, source vertex s, and target vertex t.

Mincut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.

value of flow = 28

capacity of cut = 28

Remarkable fact. These two problems are dual!

6.4 MAXIMUM FLOW

Algorithms

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Ford–Fulkerson algorithm

Initialization. Start with 0 flow.

Idea: increase flow along augmenting paths
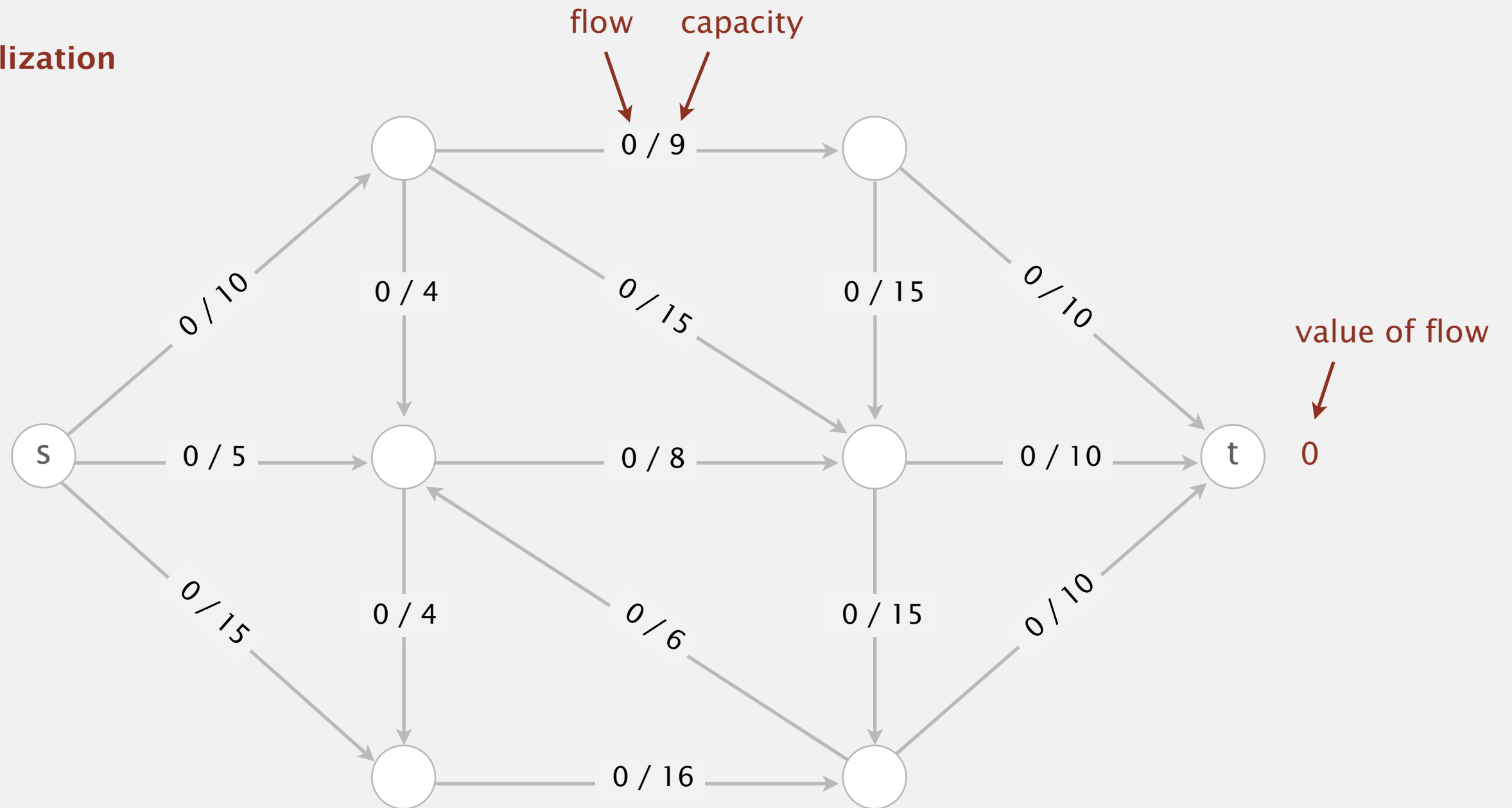
Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

1st augmenting path

bottleneck capacity = 10

$0 + 10 = 10$

Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

2nd augmenting path

Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

3rd augmenting path

backward edge
(not empty)

5
~0~ / 9

5
~10~ / 15

5
~0~ / 10

0 / 4

0 / 15

10 / 10

5
~0~ / 5

5
~0~ / 8

10 / 10

10 / 15

0 / 4

0 / 6

0 / 15
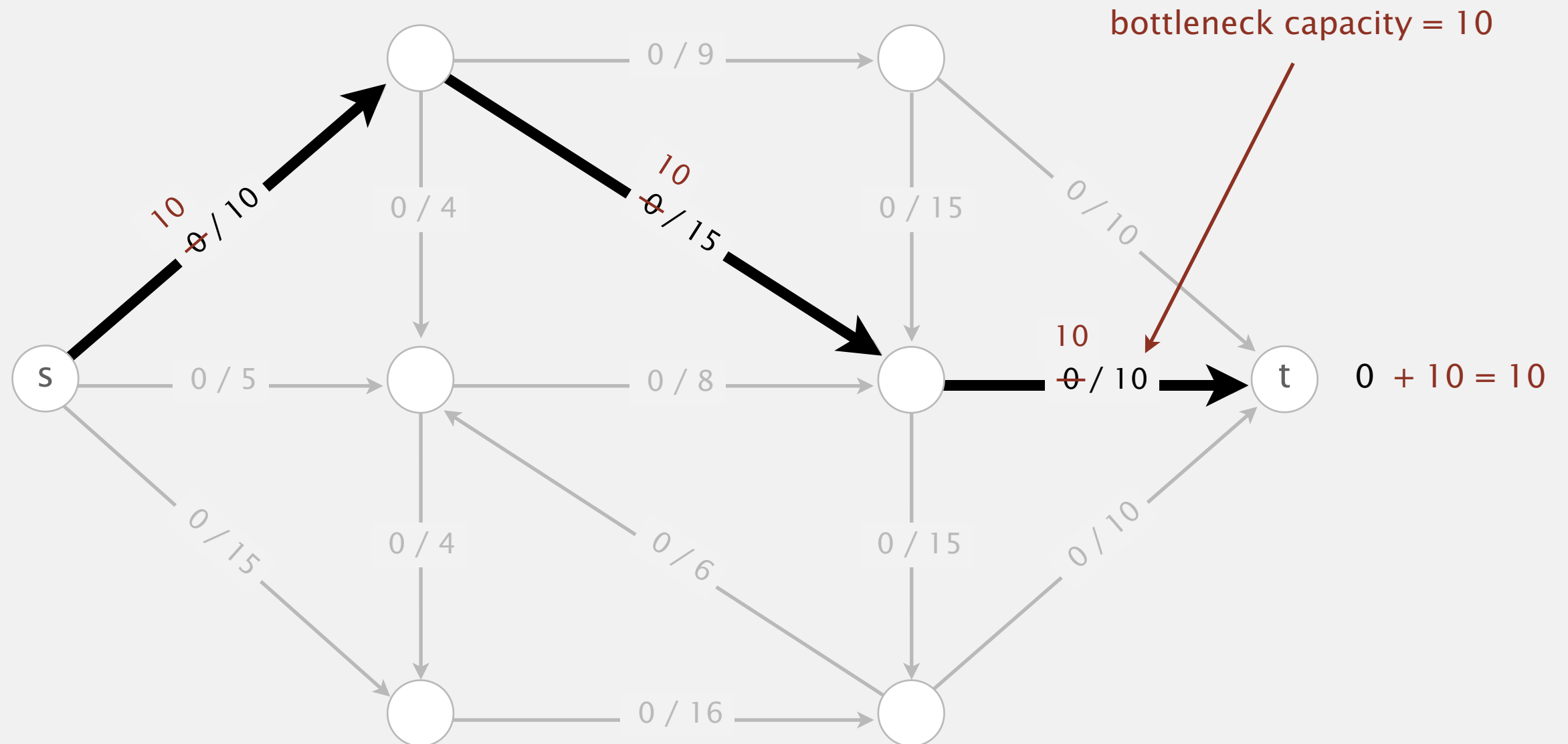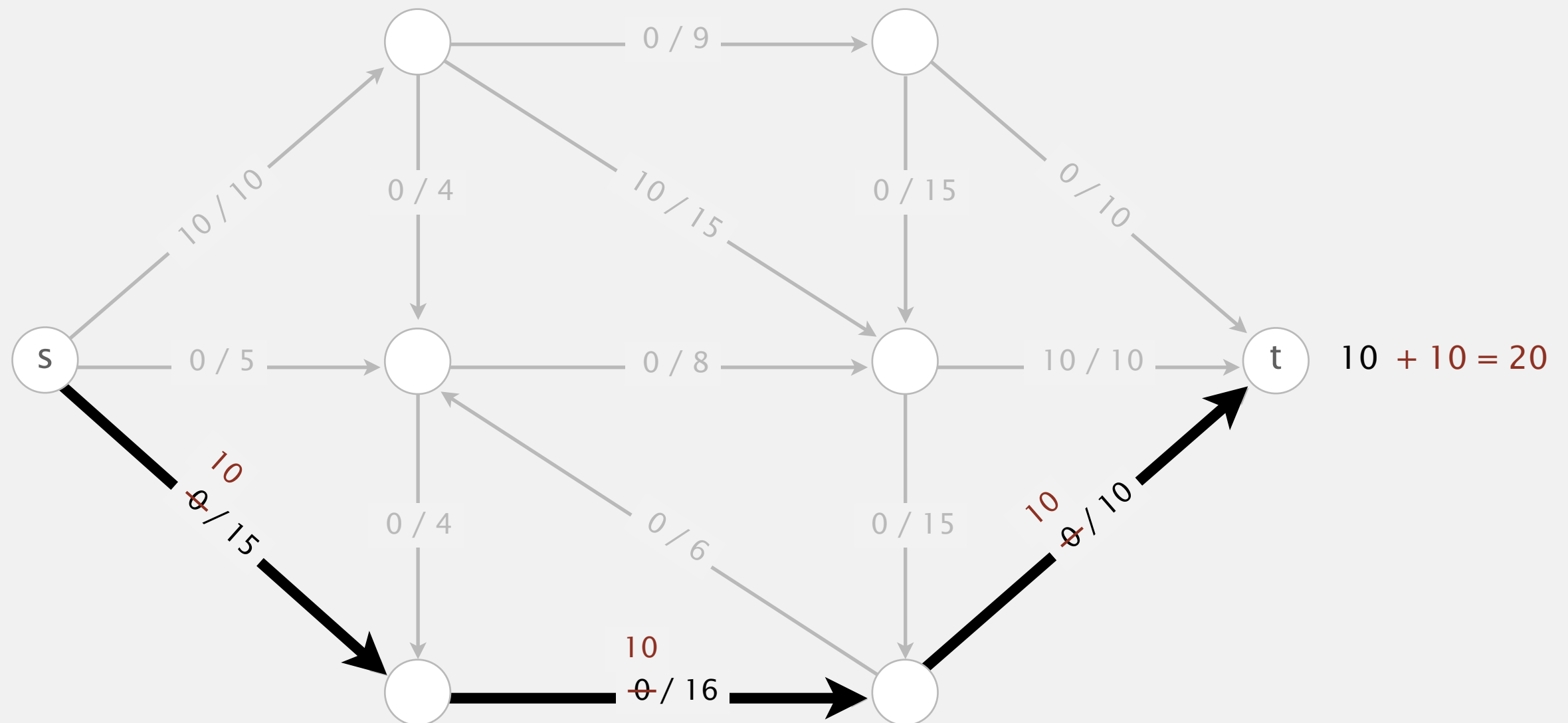
10 / 10

10 / 16

s

t

20 + 5 = 25

Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:
- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).
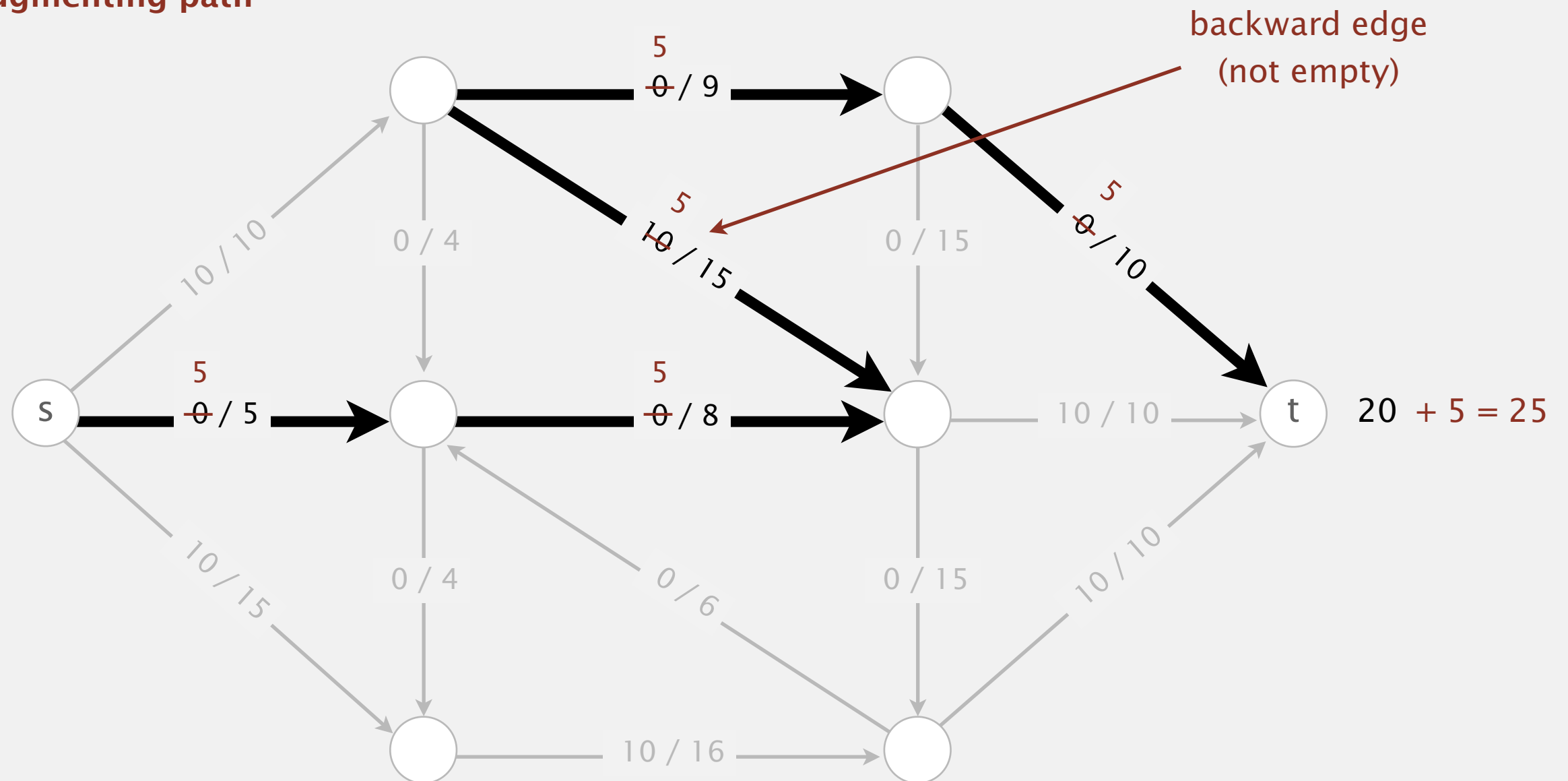
4ᵗʰ augmenting path

backward edge
(not empty)

8
~~5~~ / 9

2
~~5~~ / 15

8
~~5~~ / 10

10 / 10

0 / 4

0 / 15

8
~~5~~ / 8

5 / 5

10 / 10

13
~~10~~ / 15

0 / 4

3
~~0~~ / 6

0 / 15

10 / 10

13
~~10~~ / 16

s

t   25 + 3 = 28
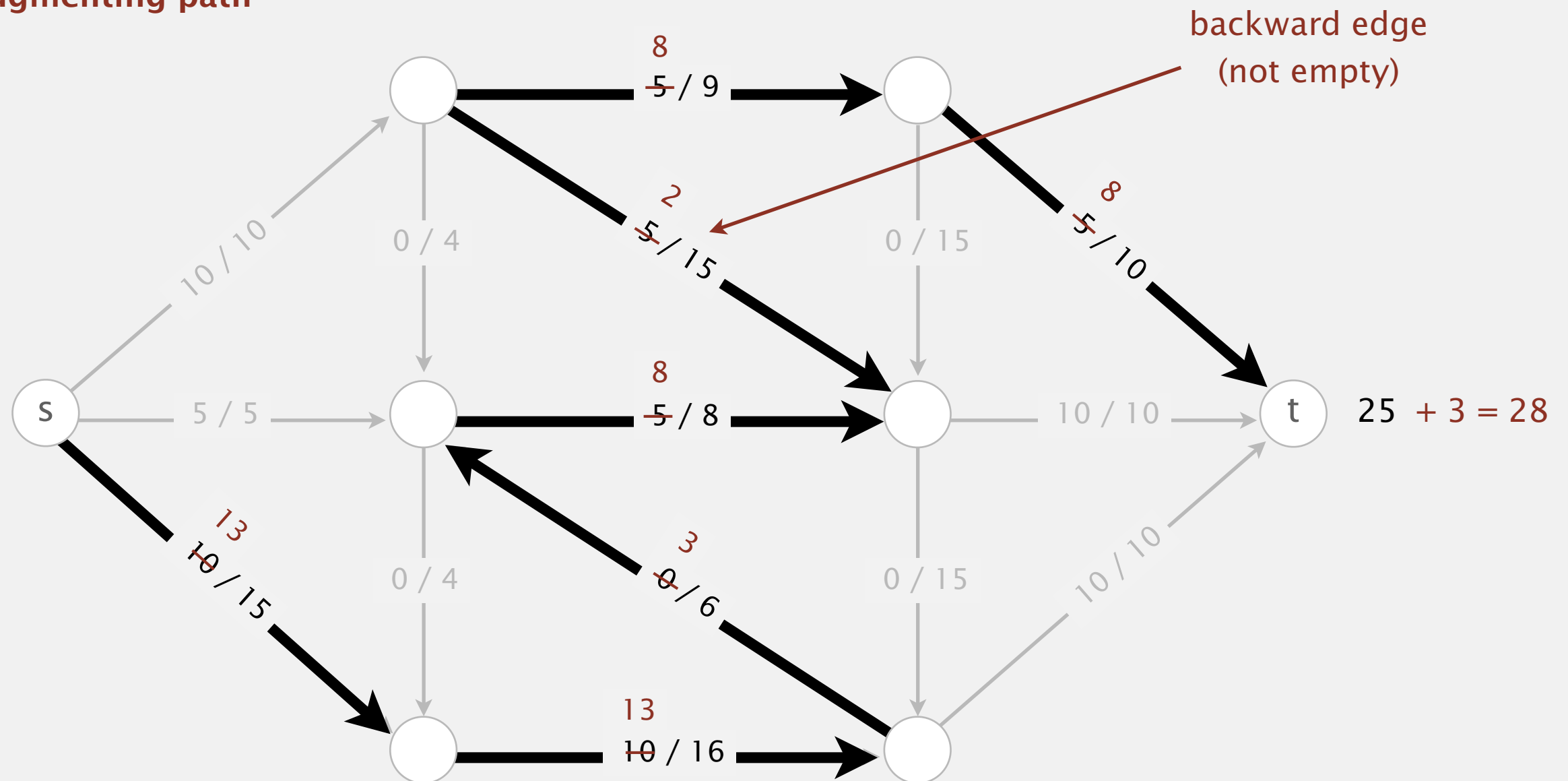
Idea: increase flow along augmenting paths

Termination. All paths from s to t are blocked by either a
- Full forward edge.
- Empty backward edge.

no more augmenting paths

full forward edge

empty backward edge
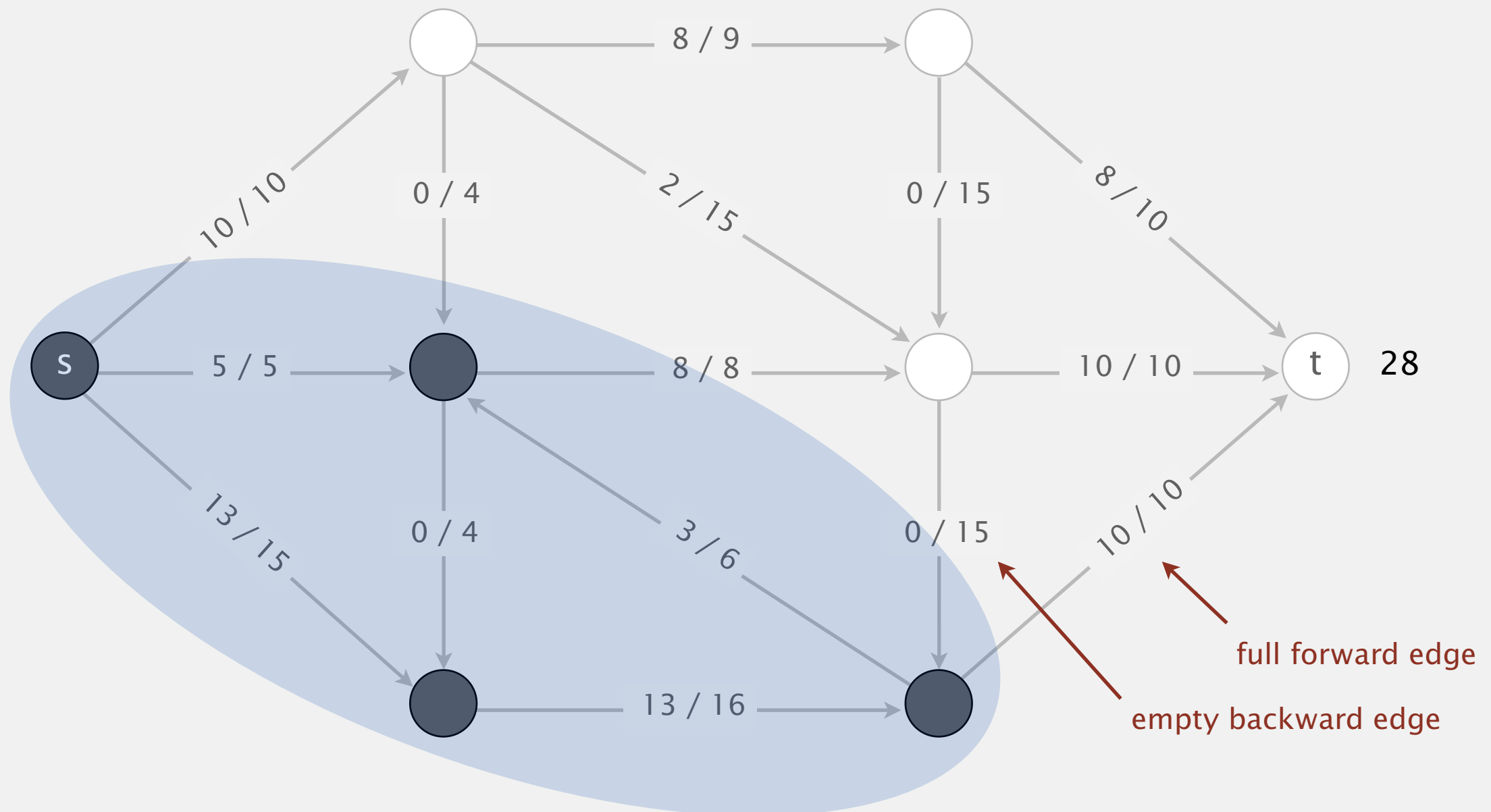
Which is the augmenting path of highest bottleneck capacity?

A. $A \to F \to G \to H$

B. $A \to F \to B \to G \to H$

C. $A \to F \to B \to G \to D \to H$

D. *I don't know.*

flow   capacity

A — 20 / 20 → B — 8 / 8 → C — 4 / 10 → D

source

1 / 6   5 / 12   8 / 8   4 / 11   4 / 9   0 / 6   4 / 8

E — 1 / 1 → F — 14 / 16 → G — 22 / 25 → H

target

Ford–Fulkerson algorithm

Ford–Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**
- **compute bottleneck capacity**
- **increase flow on that path by bottleneck capacity**

Fundamental questions.

- How to compute a mincut?
- How to find an augmenting path?
- If FF terminates, does it always compute a maxflow?
- Does FF always terminate? If so, after how many augmentations?

6.4 Maximum Flow

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

net flow across cut $= 5 + 10 + 10 = 25$

value of flow $= 25$

Relationship between flows and cuts

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

net flow across cut = 10 + 5 + 10 = 25

value of flow = 25

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from B to A.

net flow across cut = (10 + 10 + 5 + 10 + 0 + 0) − (5 + 5 + 0 + 0) = 25

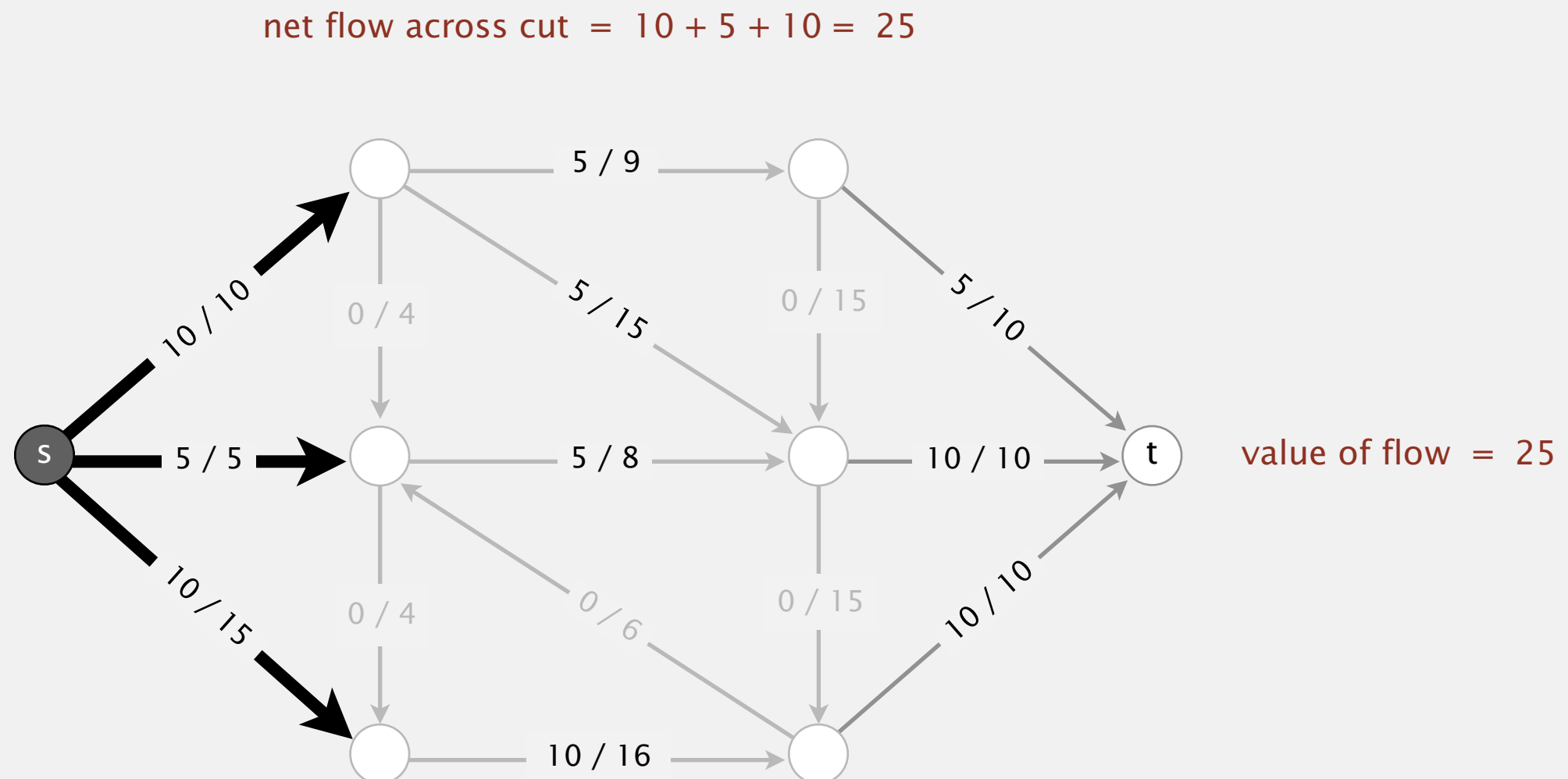

edges from B to A

value of flow = 25
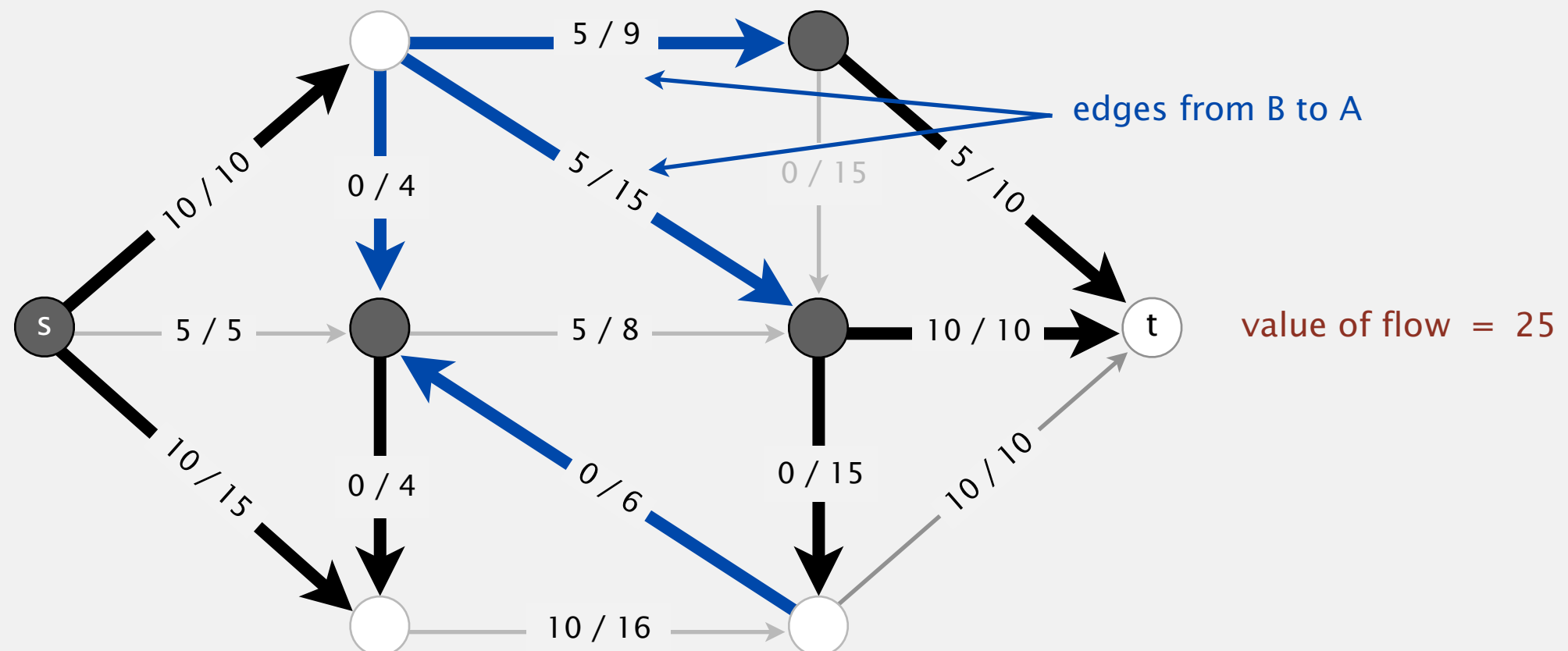
Which is the net flow across the st-cut { A, E, F, G }?

A.   11 $(20 + 25 - 8 - 11 - 9 - 6)$

B.   26 $(20 + 22 - 8 - 4 - 4)$

C.   42 $(20 + 22)$

D.   45 $(20 + 25)$

E.   *I don't know.*

flow   capacity

source

A — 20 / 20 → B — 8 / 8 → C — 4 / 10 → D

1 / 6   5 / 12   8 / 8   4 / 11   4 / 9   0 / 6   4 / 8

E — 1 / 1 → F — 14 / 16 → G — 22 / 25 → H

target
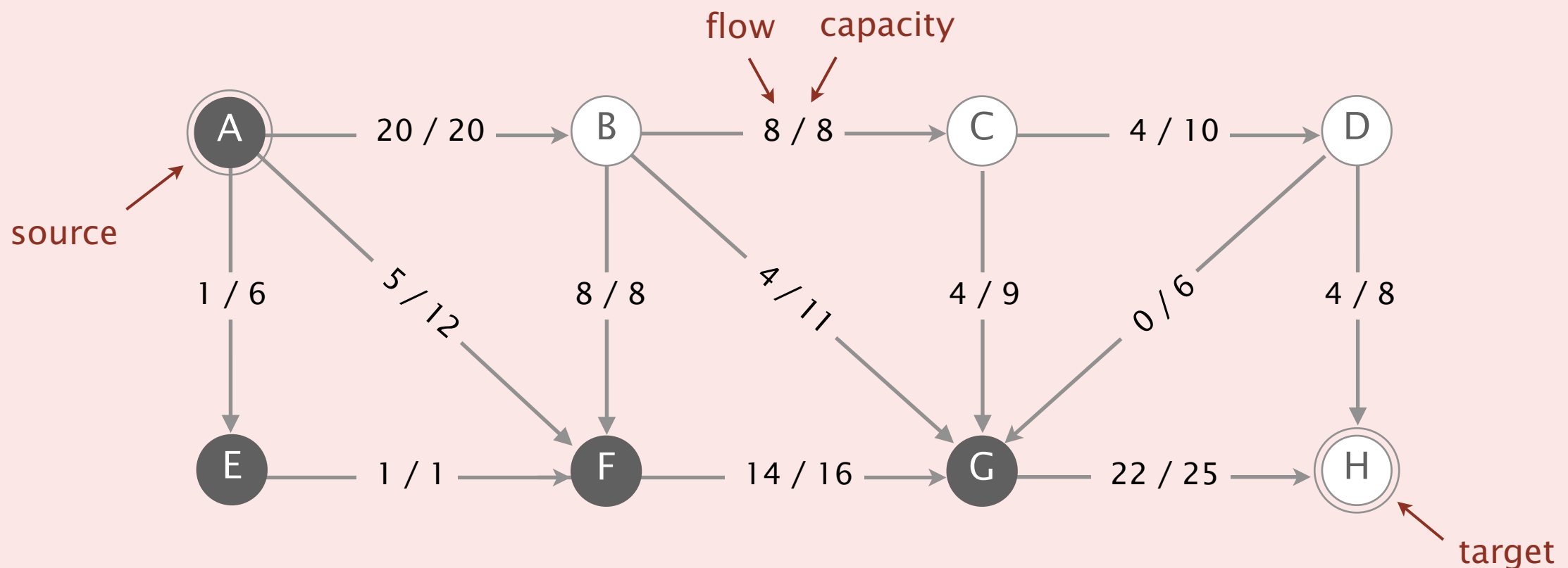
Relationship between flows and cuts

Flow-value lemma. Let f be any flow and let (A, B) be any cut. Then, the net flow across (A, B) equals the value of f.

Intuition. Conservation of flow.

Pf. By induction on the size of B.
- Base case: $B = \{\, t \,\}$.
- Induction step: remains true by local equilibrium when moving any vertex from A to B.

Corollary. Outflow from s = inflow to t = value of flow.

Weak duality. Let f be any flow and let (A, B) be any cut.

Then, the value of the flow $\leq$ the capacity of the cut.

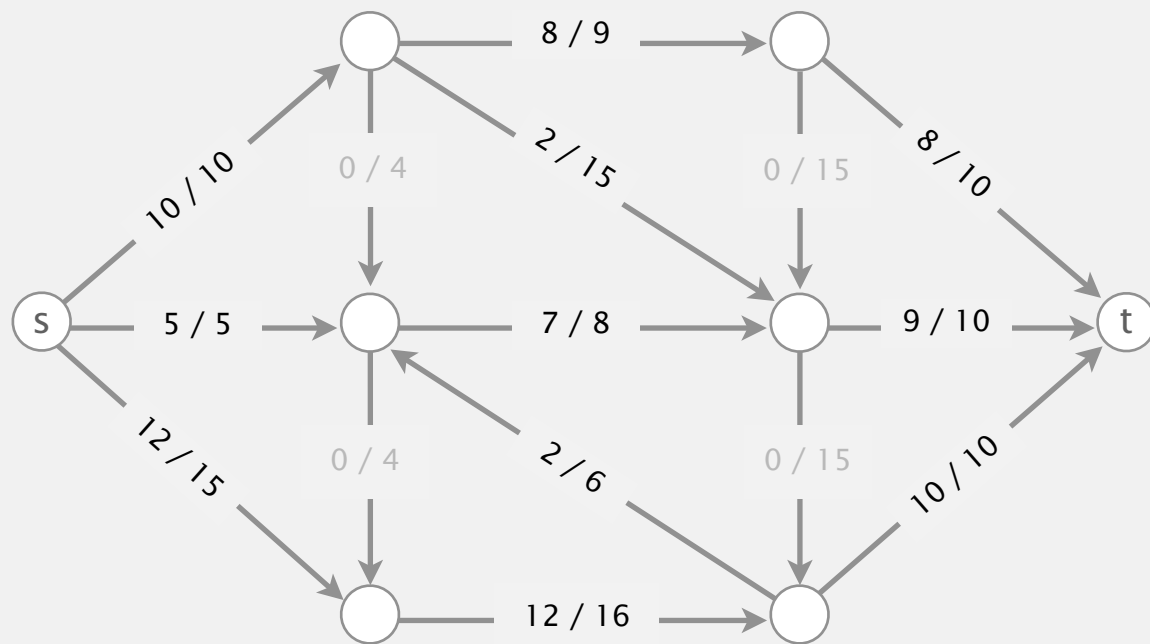Pf. Value of flow f = net flow across cut (A, B) $\leq$ capacity of cut (A, B).

↑

flow-value lemma

↑

flow bounded by capacity

value of flow = 27
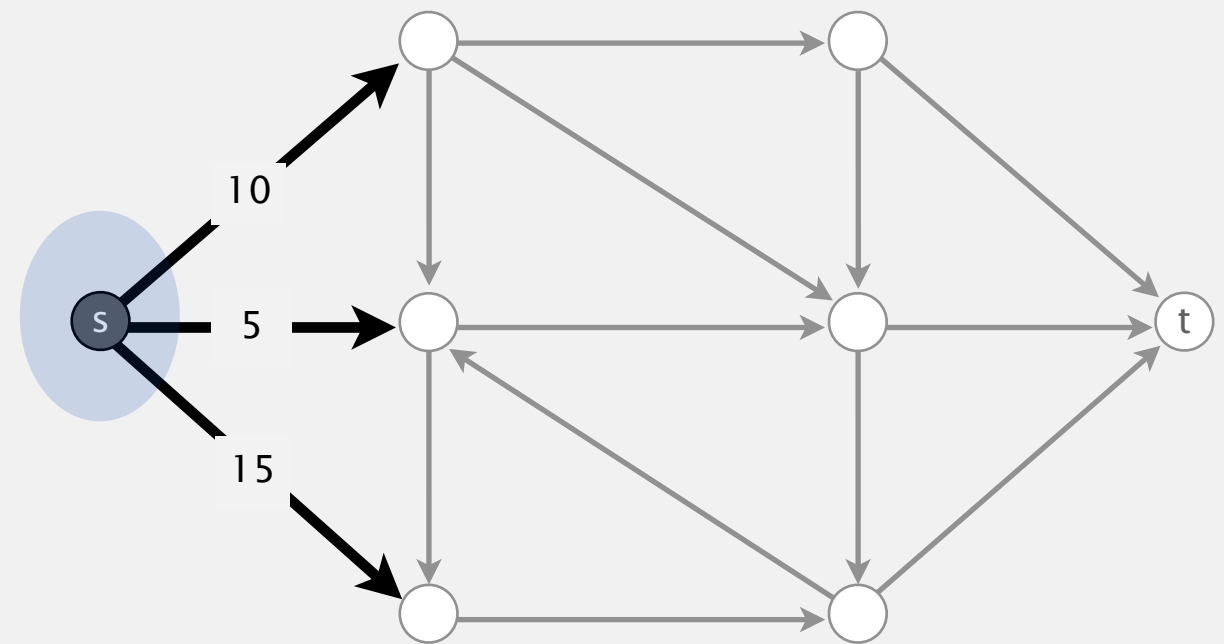
capacity of cut = 30

Maxflow–mincut theorem

Maxflow–mincut theorem. Value of the maxflow = capacity of mincut.

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f:

 i. There exists a cut whose capacity equals the value of the flow f.

 ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[i $\Rightarrow$ ii]

- Suppose that (A, B) is a cut with capacity equal to the value of f.
- Then, the value of any flow $f' \leq$ capacity of $(A, B) =$ value of f.
- Thus, f is a maxflow.

        ↑ weak duality   ↑ by assumption

Maxflow–mincut theorem

Maxflow–mincut theorem. Value of the maxflow = capacity of mincut.

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Pf. The following three conditions are equivalent for any flow f:

  i. There exists a cut whose capacity equals the value of the flow f.

 ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[ii $\Rightarrow$ iii]   We prove contrapositive: ~iii $\Rightarrow$ ~ii.

- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a maxflow.

Maxflow–mincut theorem

[iii ⇒ i]

- Let f be a flow with no augmenting paths.
- Let A be set of vertices connected to s by an undirected path with no full forward or empty backward edges
- By definition of cut A, s is in A.
- By definition of cut A and flow f, t is in B.
- Capacity of cut = net flow across cut
  = value of flow f.

flow-value lemma

G

A

B

t

backward edge from B to A (flow = 0)

forward edge from A to B (flow = capacity)

s

To compute mincut (A, B) from maxflow f :

- By augmenting path theorem, no augmenting paths with respect to f.
- Compute A = set of vertices connected to s by an undirected path with no full forward or empty backward edges.

8 / 9

10 / 10

0 / 4

2 / 15

0 / 15

8 / 10

s

5 / 5

8 / 8

10 / 10

t

A

13 / 15

3 / 4

6 / 6

0 / 15

10 / 10

16 / 16

forward edge
(not full)

backward edge
(not empty)

full forward edge

empty backward edge

6.4 MAXIMUM FLOW

Algorithms

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Ford–Fulkerson algorithm

Ford–Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**

- **compute bottleneck capacity**

- **increase flow on that path by bottleneck capacity**

Fundamental questions.

- How to compute a mincut?   Easy. ✔
- How to find an augmenting path? BFS works well.
- If FF terminates, does it always compute a maxflow? Yes. ✔
- Does FF always terminate? If so, after how many augmentations?

yes, provided edge capacities are integers
(or augmenting paths are chosen carefully)

requires clever analysis

Ford–Fulkerson algorithm with integer capacities

Important special case. Edge capacities are integers between 1 and U.

flow on each edge is an integer

Invariant. The flow is integral throughout Ford–Fulkerson.

Pf. [by induction]

- Bottleneck capacity is an integer.
- Flow on an edge increases/decreases by bottleneck capacity.

Proposition. Number of augmentations ≤ the value of the maxflow.

Pf. Each augmentation increases the value by at least 1.

critical for some applications (stay tuned)

Integrality theorem. There exists an integral maxflow.

Pf. Ford–Fulkerson terminates and maxflow that it finds is integer-valued.

Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

initialize with 0 flow
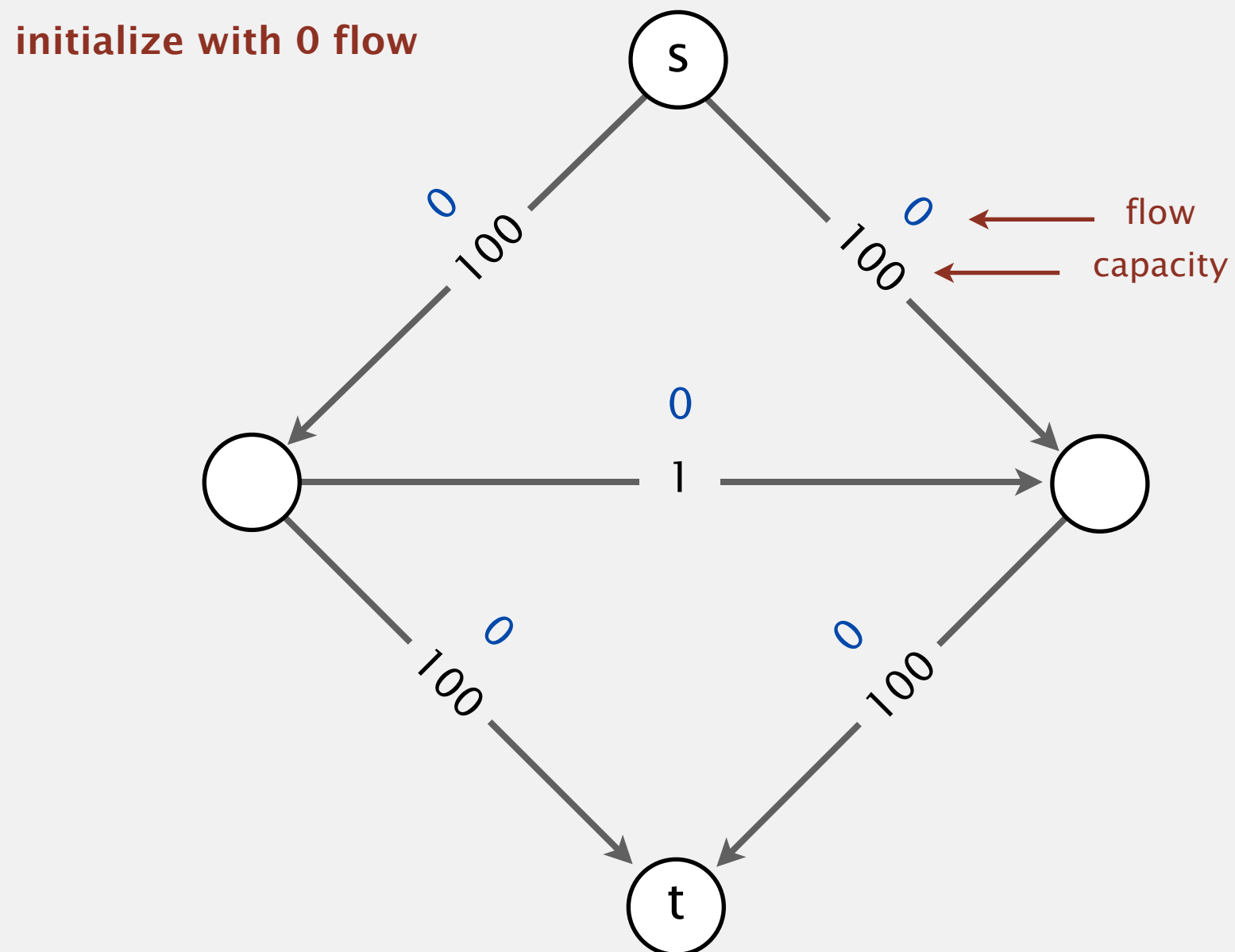
Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

1st iteration

Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

2nd iteration

Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

4th iteration
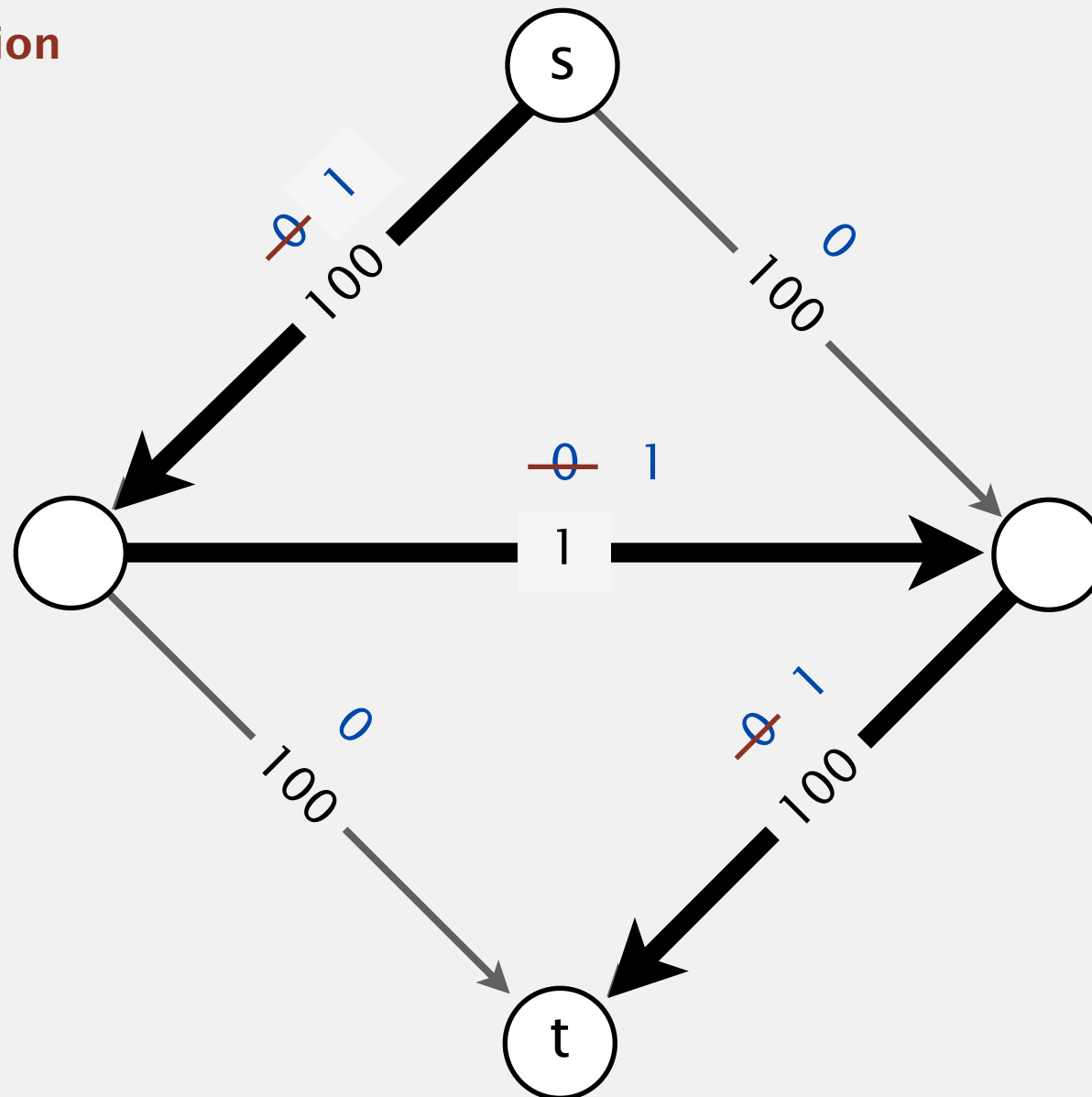
Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.
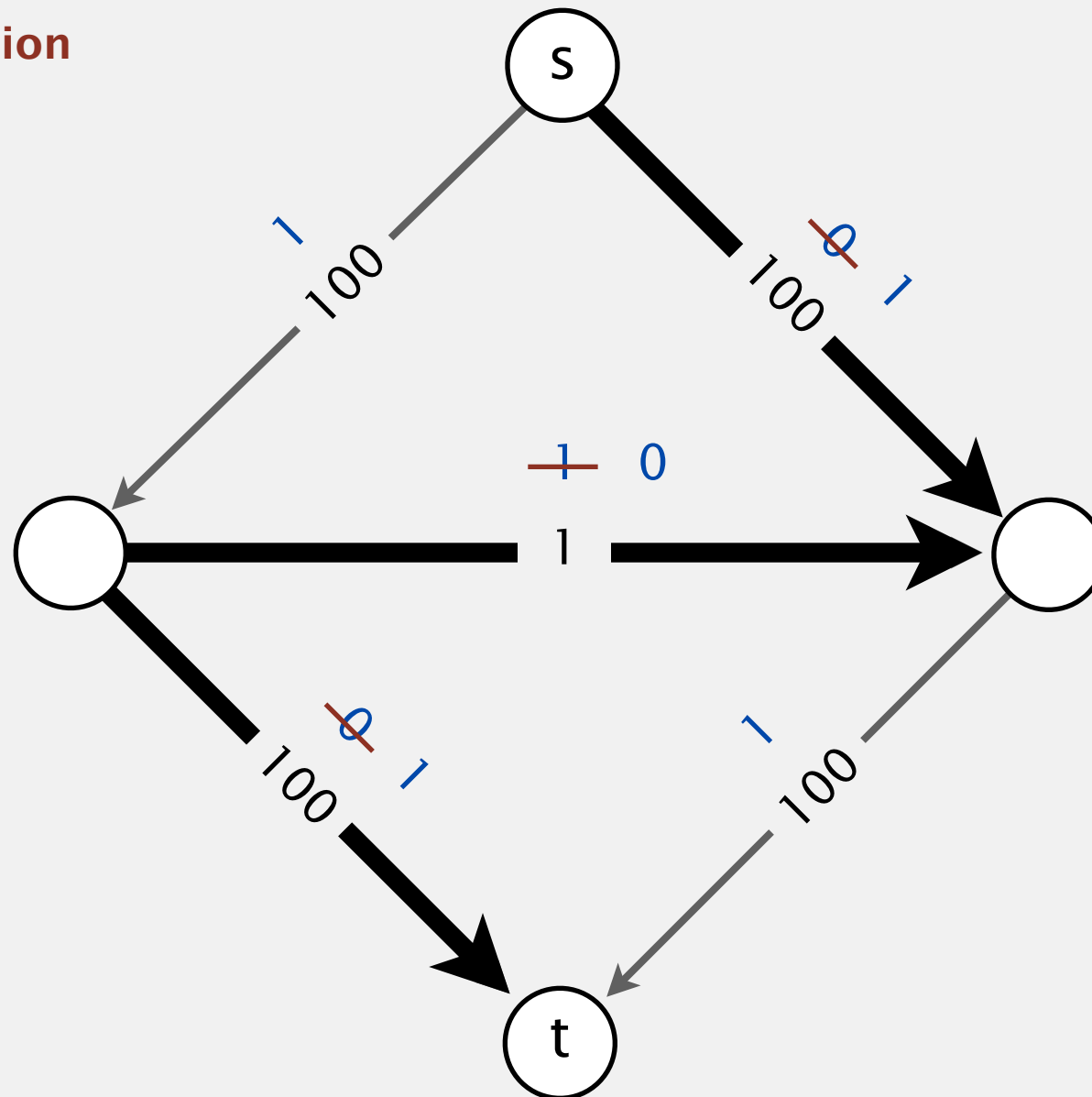
•   •   •

Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

199th iteration

Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.
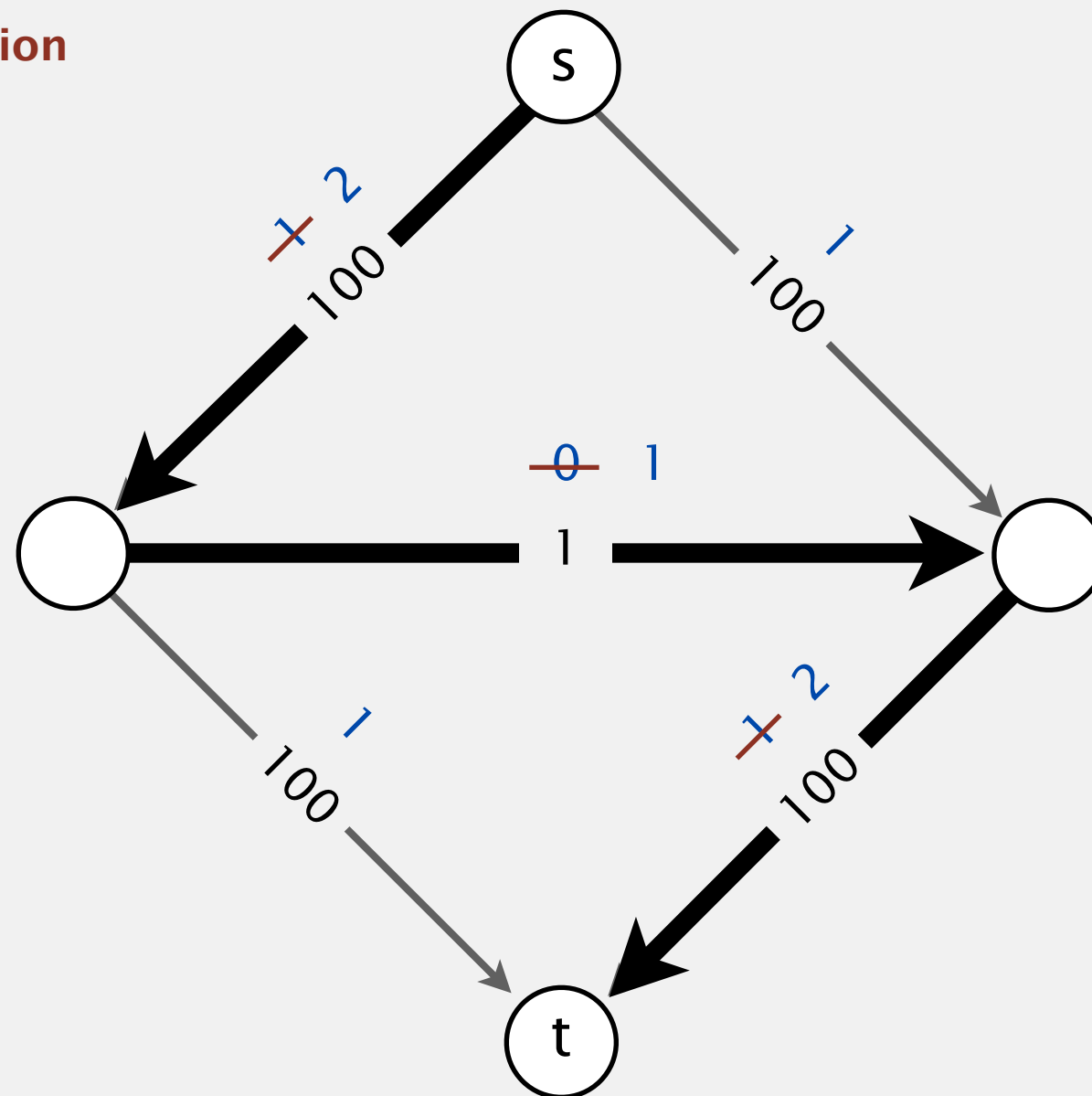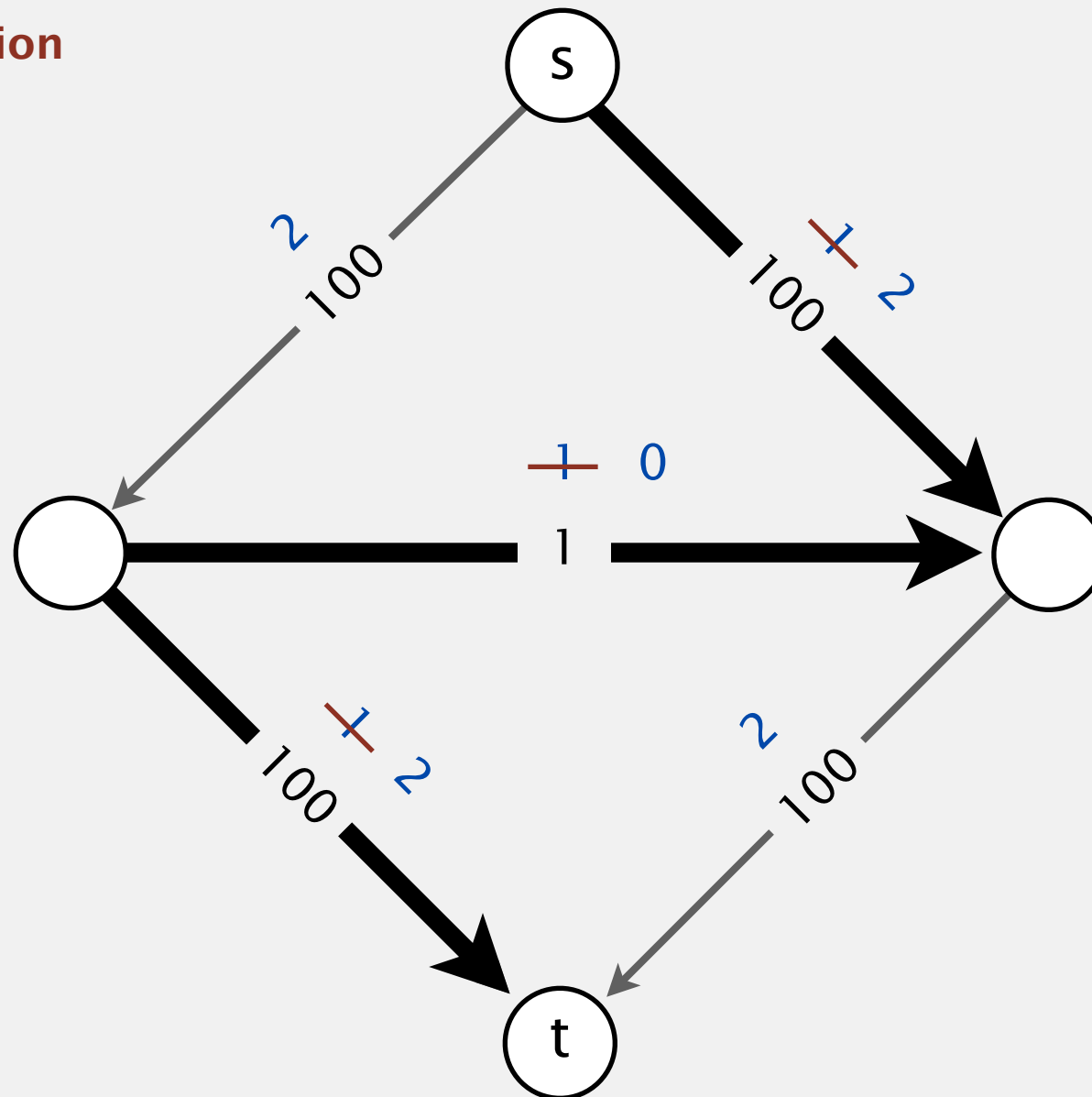
200th iteration

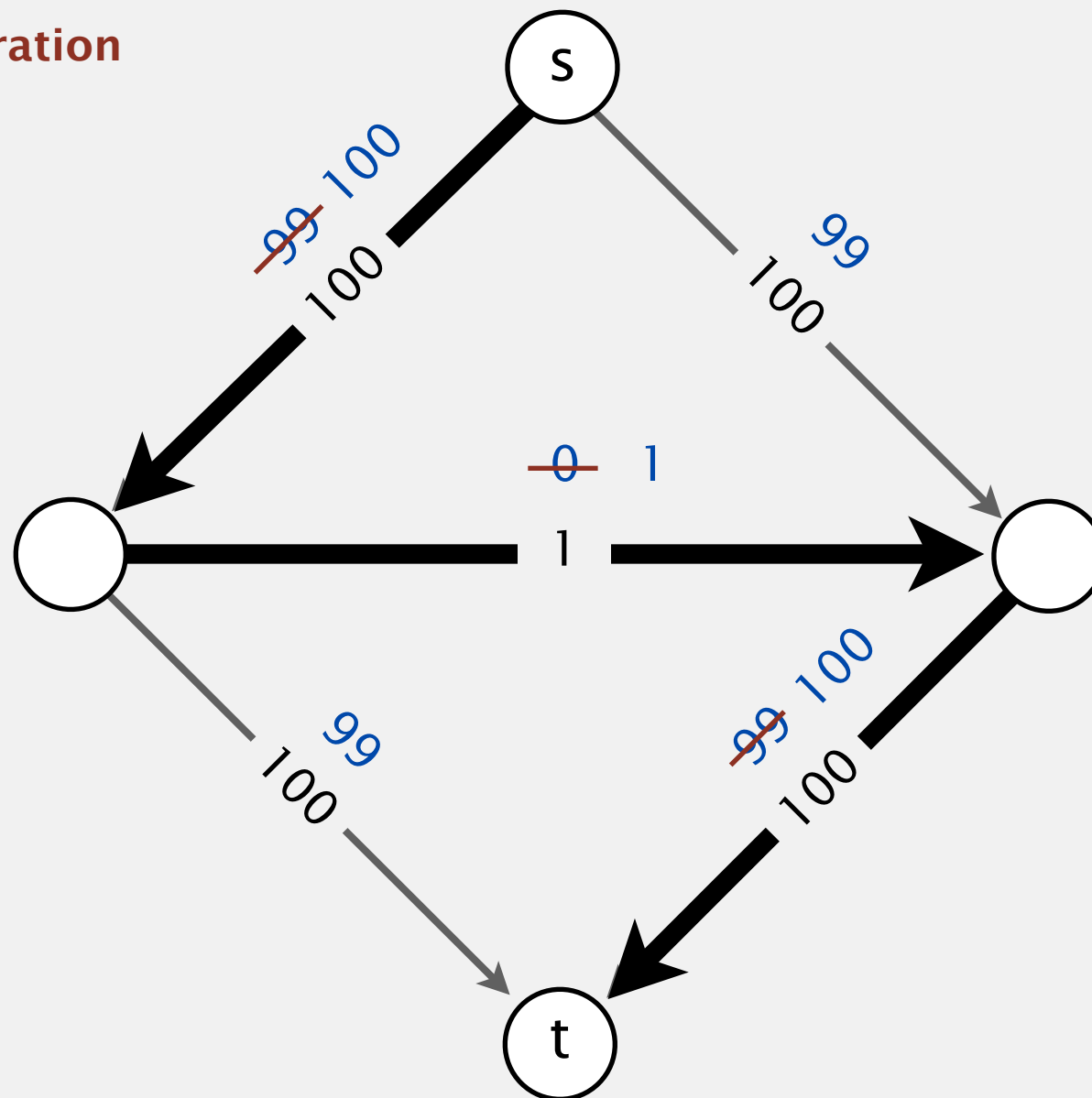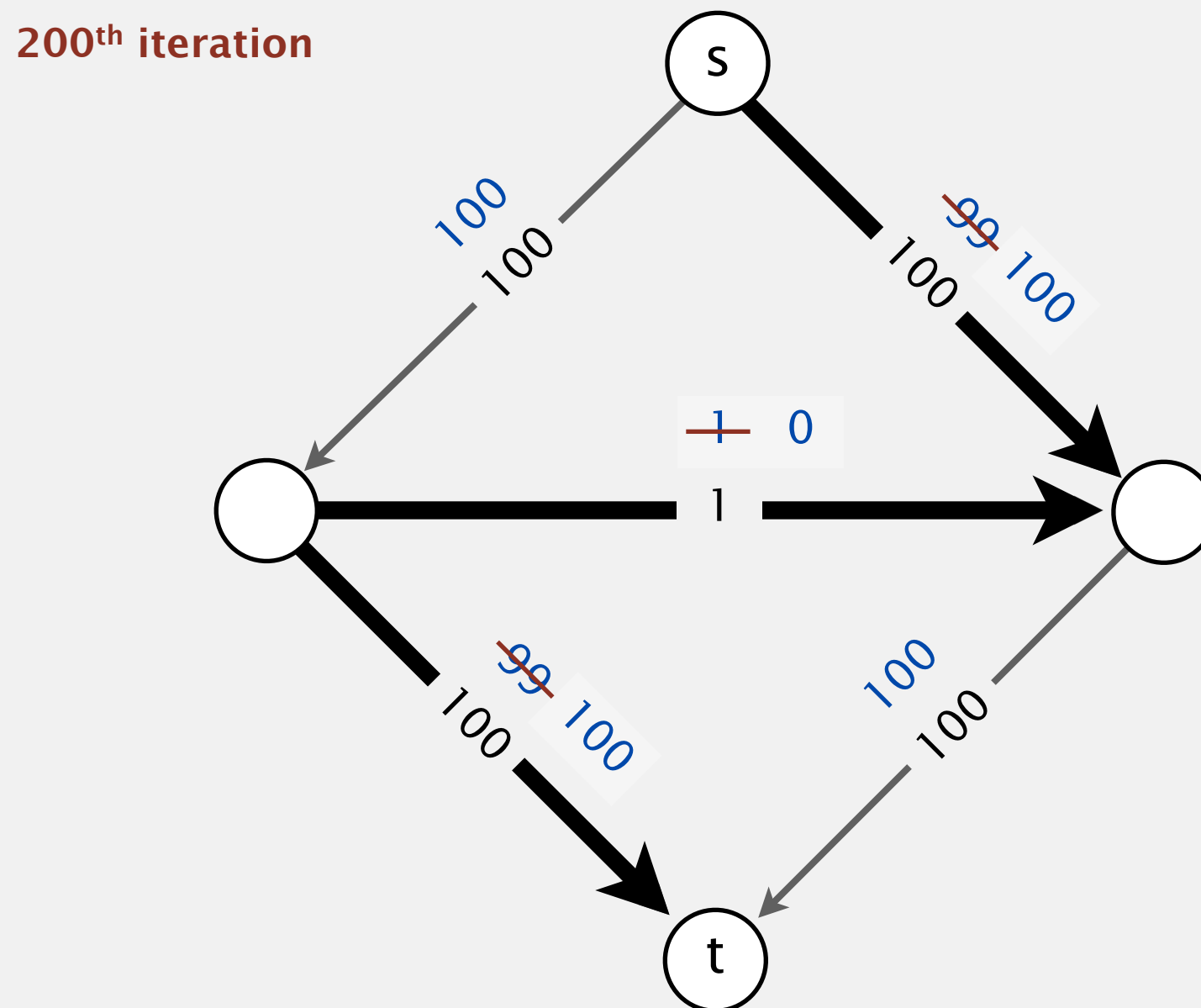# Bad case for Ford–Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.
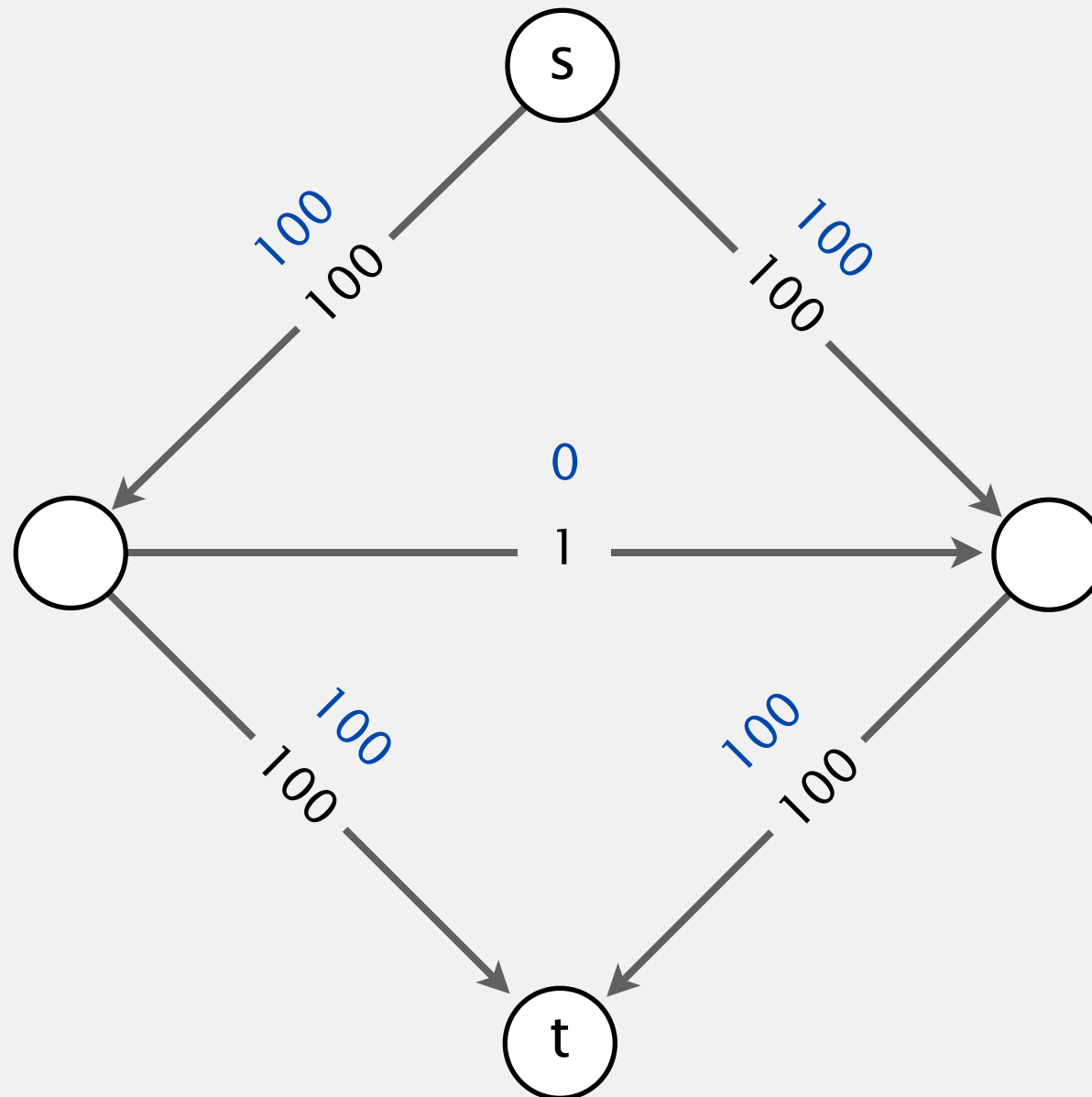
can be exponential in input size

Good news. This case is easily avoided. [use shortest/fattest path]

How to choose augmenting paths?

Choose augmenting paths with:

- Shortest path: fewest number of edges.
- Fattest path: max bottleneck capacity.

Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems

JACK EDMONDS

University of Waterloo, Waterloo, Ontario, Canada

AND

RICHARD M. KARP

University of California, Berkeley, California

ABSTRACT. This paper presents new algorithms for the maximum flow problem, the Hitchcock transportation problem, and the general minimum-cost flow problem. Upper bounds on the numbers of steps in these algorithms are derived, and are shown to compare favorably with upper bounds on the numbers of steps required by earlier algorithms.

Edmonds–Karp 1972 (USA)

ALGORITHM FOR SOLUTION OF A PROBLEM OF MAXIMUM FLOW IN A NETWORK WITH POWER ESTIMATION

UDC 518.5

E. A. DINIC

Different variants of the formulation of the problem of maximal stationary flow in a network and its many applications are given in [1]. There also is given an algorithm solving the problem in the case where the initial data are integers (or, what is equivalent, commensurable). In the general case this algorithm requires preliminary rounding off of the initial data, i.e. only an approximate solution of the problem is possible. In this connection the rapidity of convergence of the algorithm is inversely proportional to the relative precision.

Dinic 1970 (Soviet Union)

How to choose augmenting paths?

Use care when selecting augmenting paths.
- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

augmenting path	number of paths	implementation
random path	$\leq E\,U$	randomized queue
DFS path	$\leq E\,U$	stack (DFS)
shortest path	$\leq \frac{1}{2}\,E\,V$	queue (BFS)
fattest path	$\leq E\,\ln(E\,U)$	priority queue

flow network with V vertices, E edges, and integer capacities between 1 and U

6.4 MAXIMUM FLOW

Algorithms

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Flow network representation

Flow edge data type. Associate flow f_e and capacity c_e with edge $e = v \twoheadrightarrow w$.

flow f$_e$ capacity c$_e$

v —— 7 / 9 ——→ w
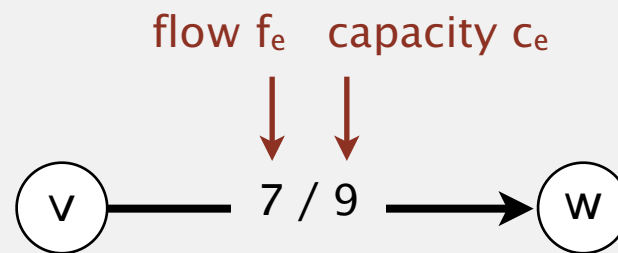
Flow network data type. Must be able to process edge $e = v \rightarrow w$ in either direction: include e in adjacency lists of both v and w.

Residual (spare) capacity.
- Forward edge: residual capacity $= c_e - f_e$.
- Backward edge: residual capacity $= f_e$.

Augment flow.
- Forward edge: add Δ.
- Backward edge: subtract Δ.

residual capacity   forward edge

v 2 w
   7
backward edge
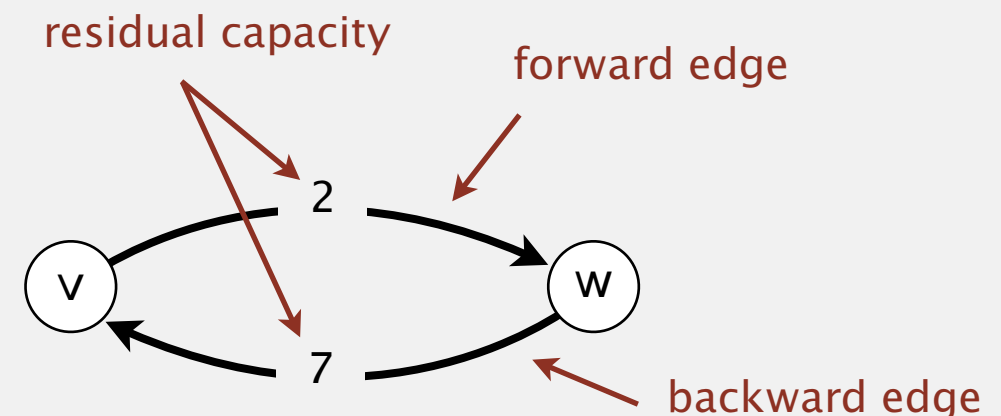
Flow network representation

Residual network. A useful view of a flow network. ← *includes all edges with positive residual capacity*

original network

residual network

backward edge (not empty)

forward edge (not full)
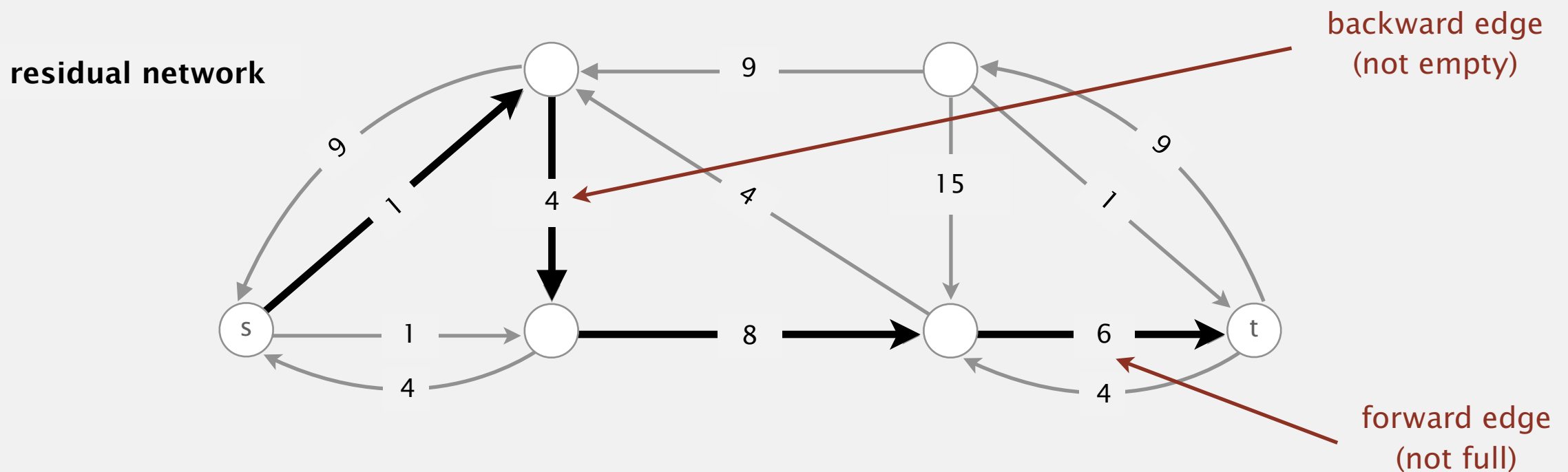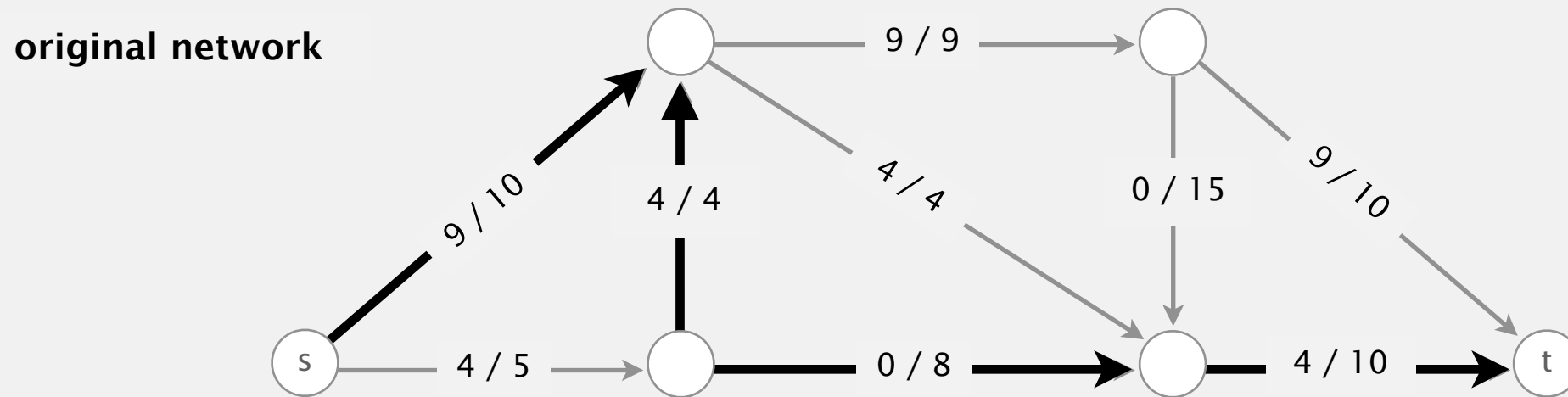
Key point. Augmenting paths in original network are in one-to-one correspondence with directed paths in residual network.

Flow edge API

```
public class FlowEdge
```

	FlowEdge(int v, int w, double capacity)	*create a flow edge v→w*
int	from()	*vertex this edge points from*
int	to()	*vertex this edge points to*
int	other(int v)	*other endpoint*
double	capacity()	*capacity of this edge*
double	flow()	*flow in this edge*
double	residualCapacityTo(int v)	*residual capacity toward v*
void	addResidualFlowTo(int v, double delta)	*add delta flow toward v*

flow f_e   capacity c_e

v —— 7 / 9 ——→ w

residual capacity   forward edge

v   2   w

7

backward edge
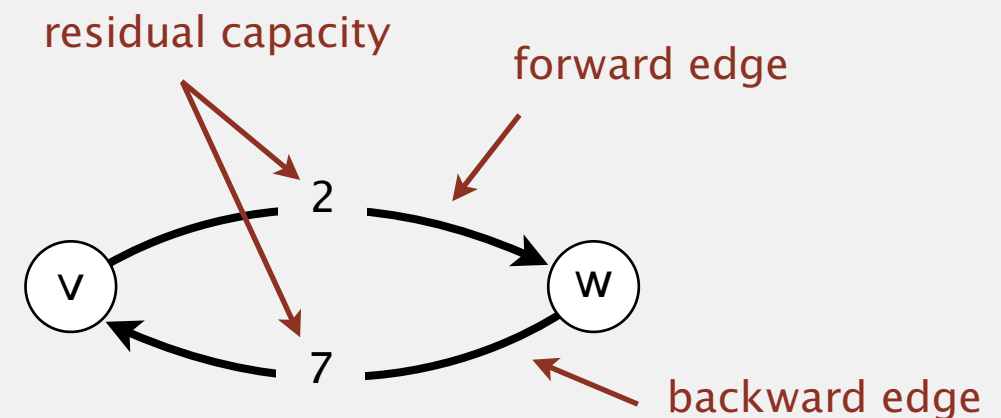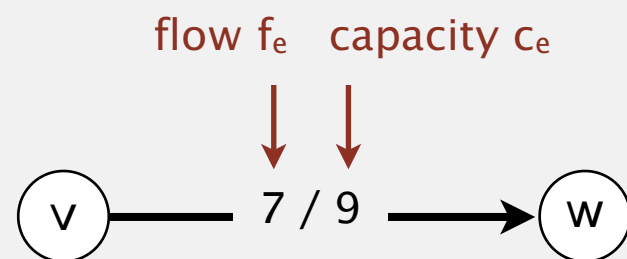
Flow edge: Java implementation

```java
public class FlowEdge
{
    private final int v, w;          // from and to
    private final double capacity;   // capacity
    private double flow;             // flow

    public FlowEdge(int v, int w, double capacity)
    {
        this.v        = v;
        this.w        = w;
        this.capacity = capacity;
    }

    public int from()          { return v;        }
    public int to()            { return w;        }
    public double capacity()   { return capacity; }
    public double flow()       { return flow;     }

    public int other(int vertex)
    {
        if      (vertex == v) return w;
        else if (vertex == w) return v;
        else throw new IllegalArgumentException();
    }

    public double residualCapacityTo(int vertex)              {...}
    public void addResidualFlowTo(int vertex, double delta)  {...}
}
```

flow variable
(mutable)

next slide

Flow edge: Java implementation (continued)

```java
public double residualCapacityTo(int vertex)
{
    if       (vertex == v) return flow;              ← forward edge
    else if (vertex == w) return capacity - flow;    ← backward edge
    else throw new IllegalArgumentException();
}
```

```java
public void addResidualFlowTo(int vertex, double delta)
{
    if       (vertex == v) flow -= delta;            ← forward edge
    else if (vertex == w) flow += delta;             ← backward edge
    else throw new IllegalArgumentException();
}
```

flow f_e capacity c_e

v —— 7 / 9 —→ w

residual capacity   forward edge

v   2   →   w

        7

backward edge
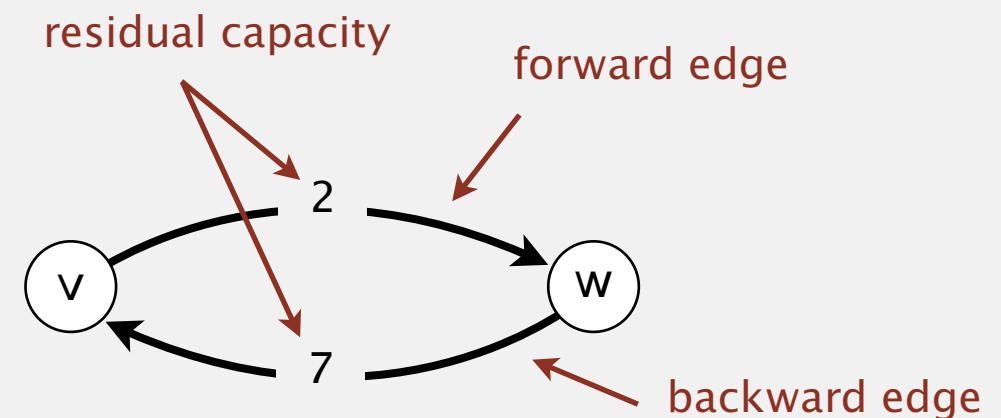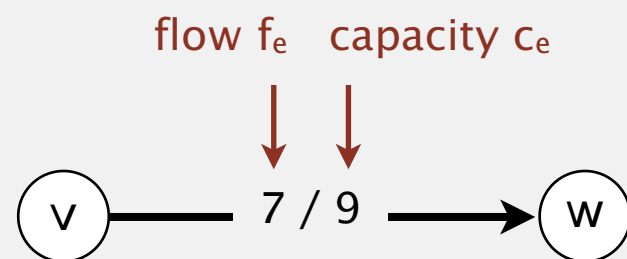
Flow network API

```
    public class FlowNetwork

                   FlowNetwork(int V)        create an empty flow network with V vertices

                   FlowNetwork(In in)        construct flow network input stream

              void addEdge(FlowEdge e)       add flow edge e to this flow network

Iterable<FlowEdge> adj(int v)               forward and backward edges incident to/from v

Iterable<FlowEdge> edges()                  all edges in this flow network

               int V()                      number of vertices

               int E()                      number of edges

            String toString()               string representation
```

Conventions. Allow self-loops and parallel edges.

Flow network: Java implementation

```java
public class FlowNetwork
{
    private final int V;
    private Bag<FlowEdge>[] adj;

    public FlowNetwork(int V)
    {
        this.V = V;
        adj = (Bag<FlowEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<FlowEdge>();
    }

    public void addEdge(FlowEdge e)
    {
        int v = e.from();
        int w = e.to();
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<FlowEdge> adj(int v)
    {   return adj[v]; }
}
```
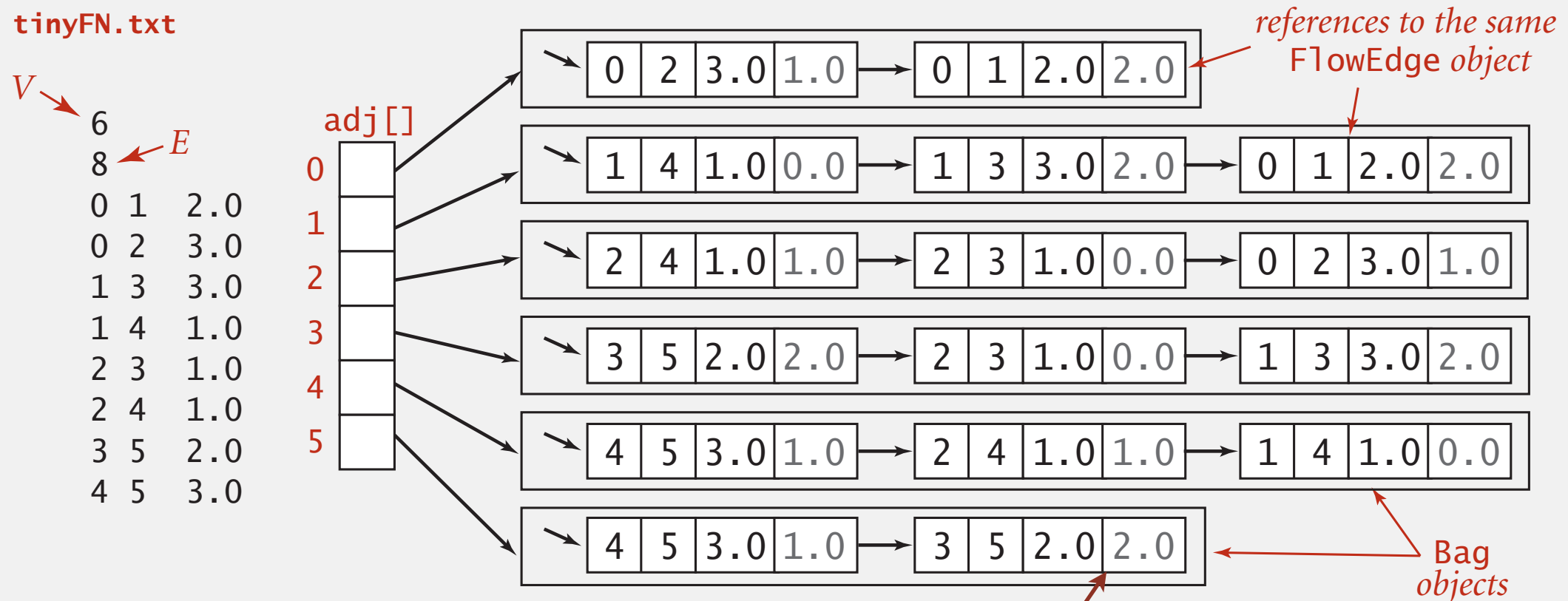
same as EdgeWeightedGraph, but adjacency lists of FlowEdges instead of Edges

add forward edge
add backward edge

Flow network: adjacency-lists representation

Maintain vertex-indexed array of `FlowEdge` lists (use `Bag` abstraction).

tinyFN.txt

V → 6
8 ← *E*
0 1 2.0
0 2 3.0
1 3 3.0
1 4 1.0
2 3 1.0
2 4 1.0
3 5 2.0
4 5 3.0

adj[]

| 0 |
| 1 |
| 2 |
| 3 |
| 4 |
| 5 |

references to the same FlowEdge object

0 2 3.0 1.0 → 0 1 2.0 2.0

1 4 1.0 0.0 → 1 3 3.0 2.0 → 0 1 2.0 2.0

2 4 1.0 1.0 → 2 3 1.0 0.0 → 0 2 3.0 1.0

3 5 2.0 2.0 → 2 3 1.0 0.0 → 1 3 3.0 2.0

4 5 3.0 1.0 → 2 4 1.0 1.0 → 1 4 1.0 0.0

4 5 3.0 1.0 → 3 5 2.0 2.0

Bag objects

Note. Adjacency list includes edges with 0 residual capacity.

(residual network is represented implicitly)

Finding a shortest augmenting path (cf. breadth-first search)

```
private boolean hasAugmentingPath(FlowNetwork G, int s, int t)
{
    edgeTo = new FlowEdge[G.V()];
    marked = new boolean[G.V()];

    Queue<Integer> queue = new Queue<Integer>();
    queue.enqueue(s);
    marked[s] = true;
    while (!queue.isEmpty() && !marked[t])
    {
        int v = queue.dequeue();

        for (FlowEdge e : G.adj(v))
        {
            int w = e.other(v);
            if (!marked[w] && (e.residualCapacityTo(w) > 0) )
            {
                edgeTo[w] = e;
                marked[w] = true;
                queue.enqueue(w);
            }
        }
    }

    return marked[t];
}
```

can stop BFS as soon as
augmenting path is discovered

found path from s to w
in the residual network?

save last edge on path to w;
mark w;
add w to the queue

is t reachable from s in residual network?

Ford–Fulkerson: Java implementation

```java
public class FordFulkerson
{
    private boolean[] marked;    // true if s->v path in residual network
    private FlowEdge[] edgeTo;    // last edge on s->v path
    private double value;         // value of flow

    public FordFulkerson(FlowNetwork G, int s, int t)
    {
        value = 0.0;
        while (hasAugmentingPath(G, s, t))
        {
            double bottle = Double.POSITIVE_INFINITY;
            for (int v = t; v != s; v = edgeTo[v].other(v))
                bottle = Math.min(bottle, edgeTo[v].residualCapacityTo(v));

            for (int v = t; v != s; v = edgeTo[v].other(v))
                edgeTo[v].addResidualFlowTo(v, bottle);

            value += bottle;
        }
    }

    private boolean hasAugmentingPath(FlowNetwork G, int s, int t)
    {  /* See previous slide. */  }

    public double value()
    {  return value;  }

    public boolean inCut(int v)
    {  return marked[v];  }
}
```

compute `edgeTo[]` and `marked[]`

compute bottleneck capacity

augment flow

update value of flow

is v reachable from s in residual network?

6.4 MAXIMUM FLOW

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

Maxflow and mincut applications

Maxflow/mincut is a widely applicable problem-solving model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.
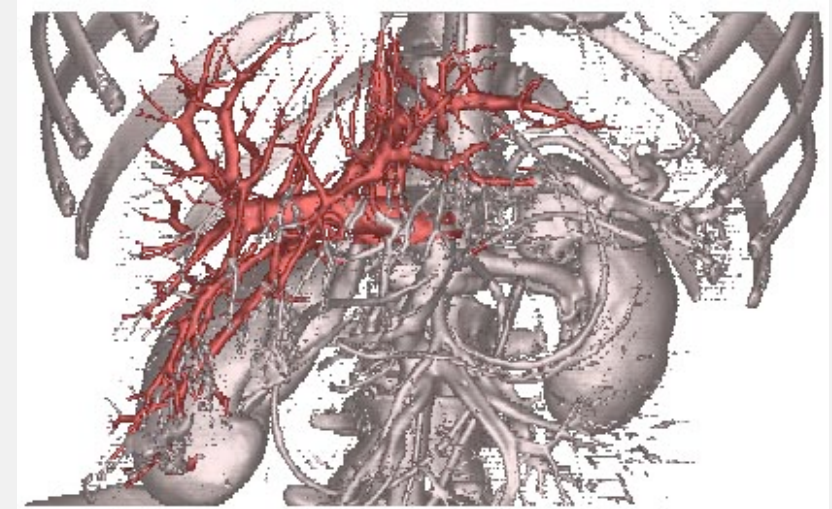
liver and hepatic vascularization segmentation
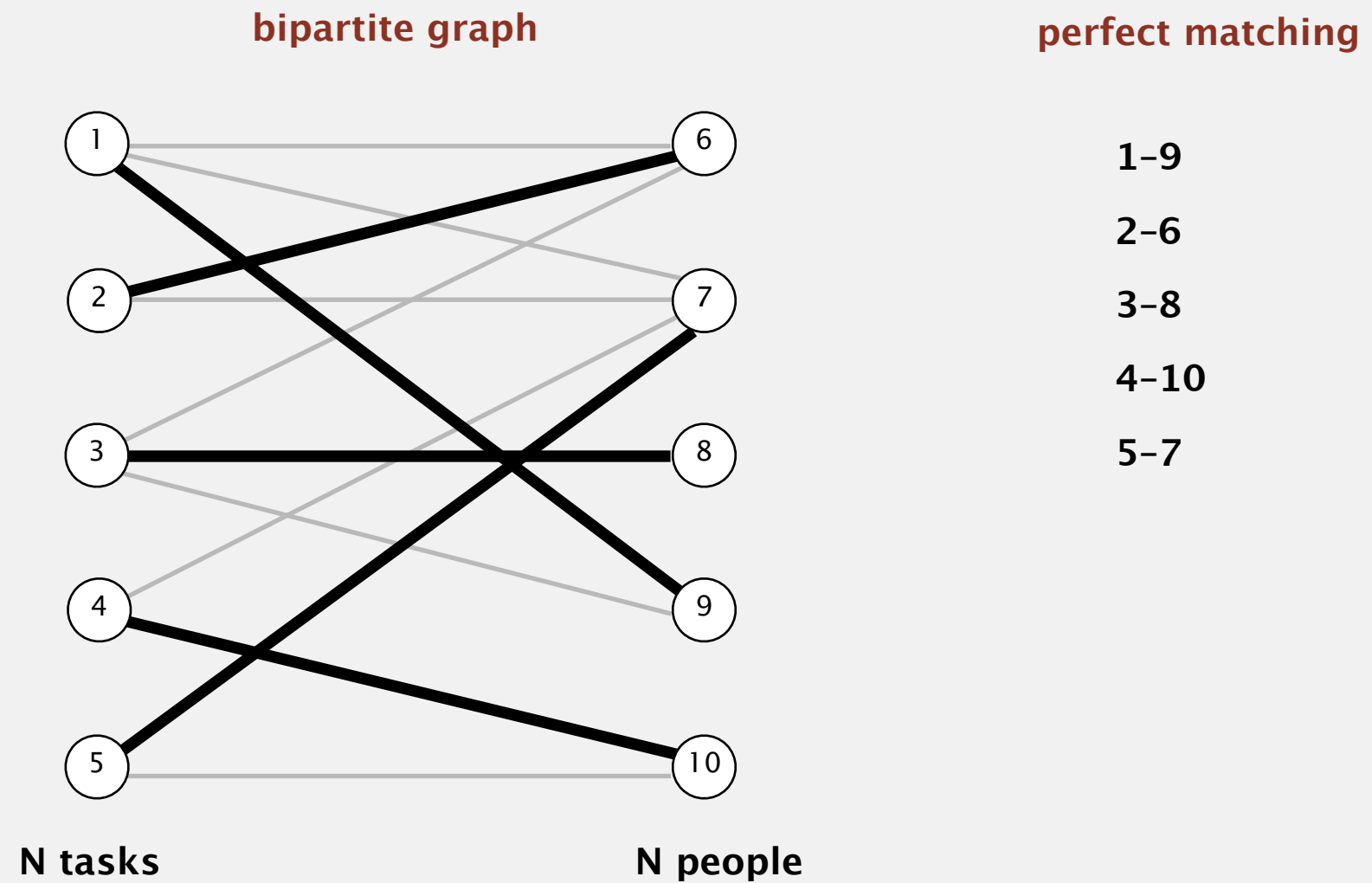
Bipartite matching problem

Problem. Given N people and N tasks, assign the tasks to people so that:

- Every task is assigned to a qualified person.
- Every person is assigned to exactly one task.

Bipartite matching problem

Problem. Given a bipartite graph, find a perfect matching.

bipartite graph

perfect matching

1–9

2–6

3–8

4–10

5–7

N tasks

N people

Network flow formulation of bipartite matching

- Create s, t, one vertex for each task, and one vertex for each person.
- Add edge from s to each task (of capacity 1).
- Add edge from each person to t (of capacity 1).
- Add edge from task to qualified person (of infinite capacity).
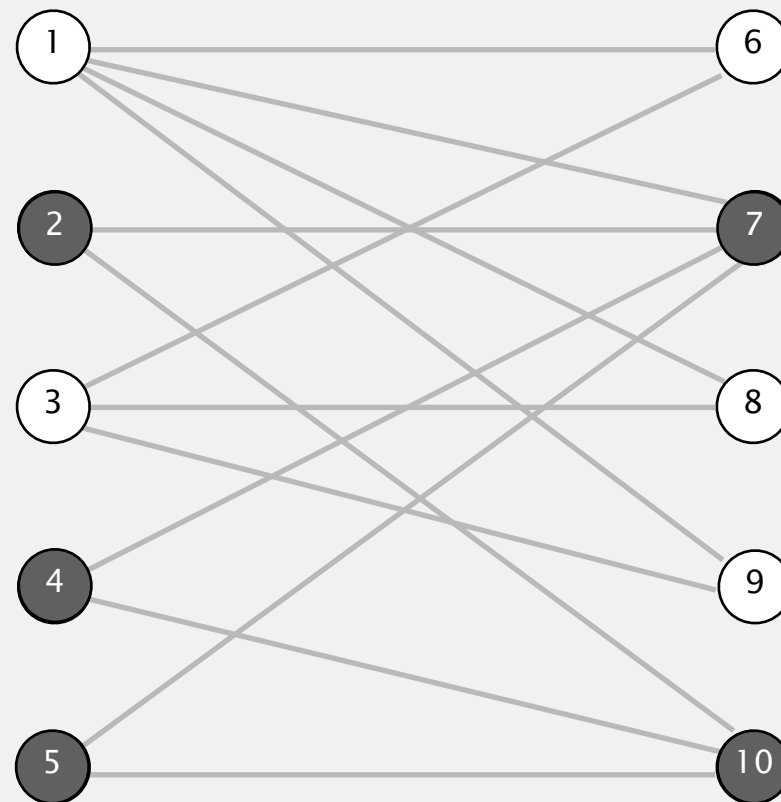
flow network

N tasks   N people
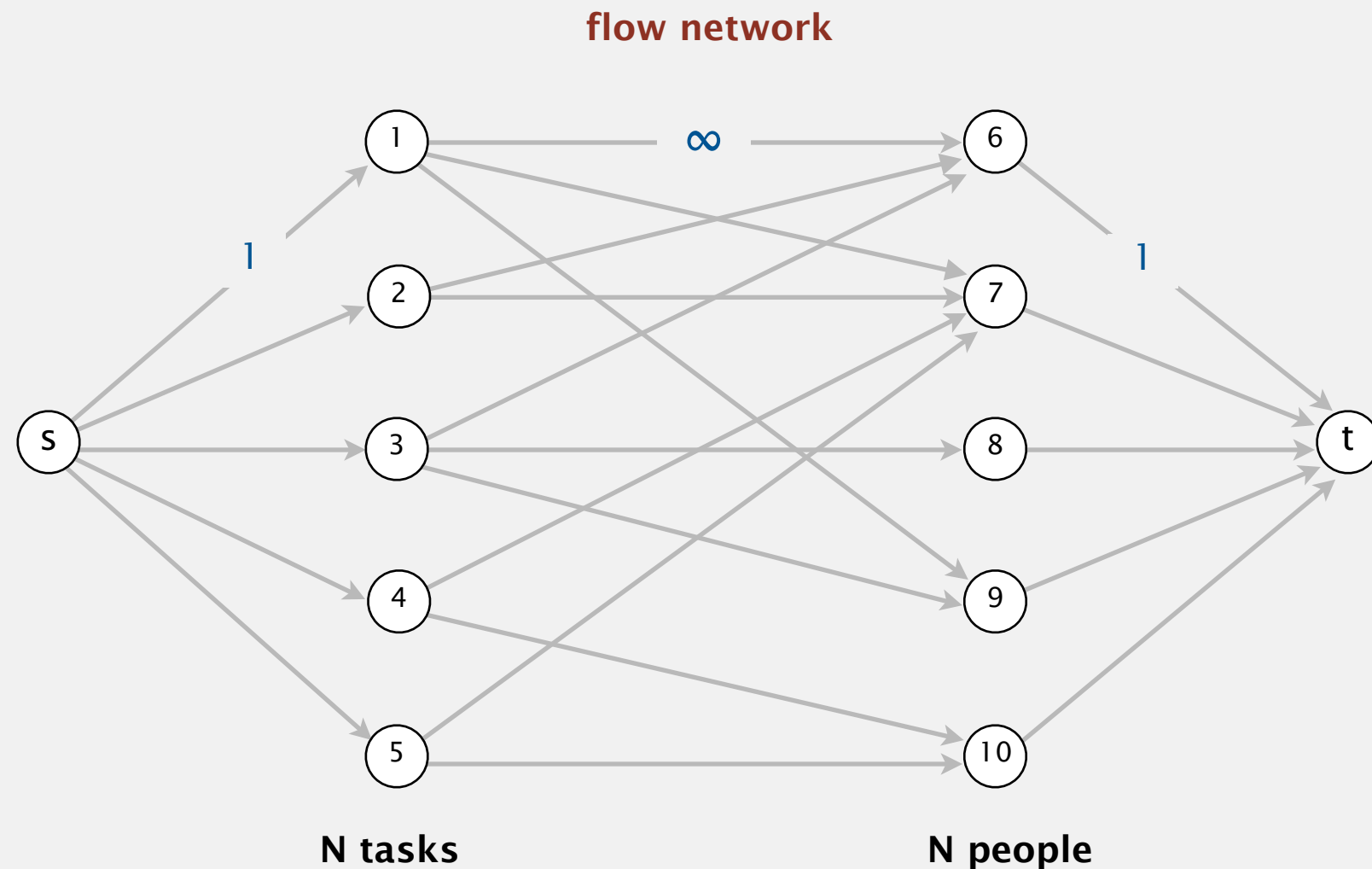
Network flow formulation of bipartite matching

1–1 correspondence between perfect matchings in bipartite graph and integer-valued maxflows of value N in flow network.

flow network

N tasks   **N people**
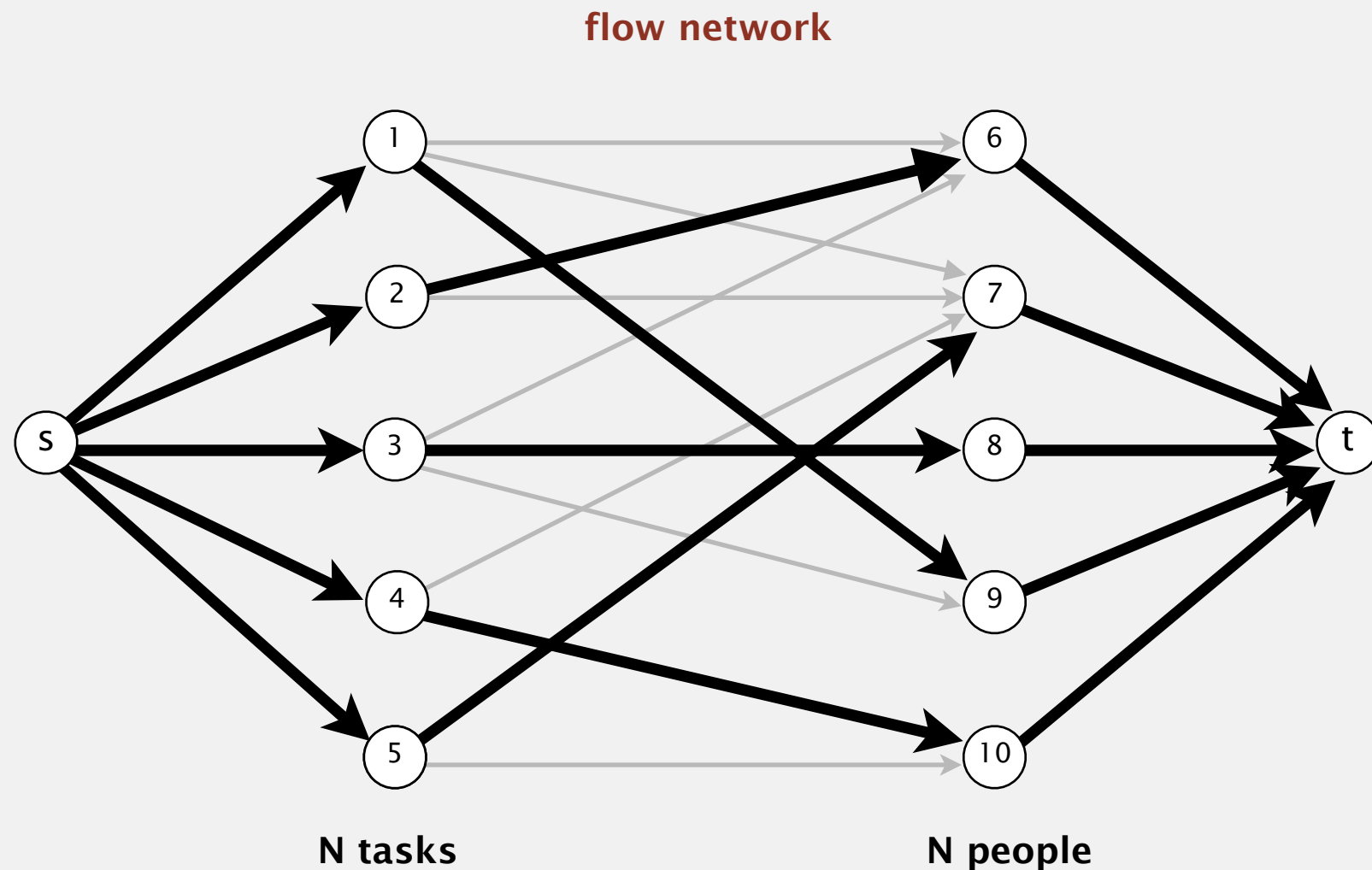
What the mincut tells us

Goal. When no perfect matching, explain why.

no perfect matching exists

S = { 2, 4, 5 }
T = { 7, 10 }

tasks in S
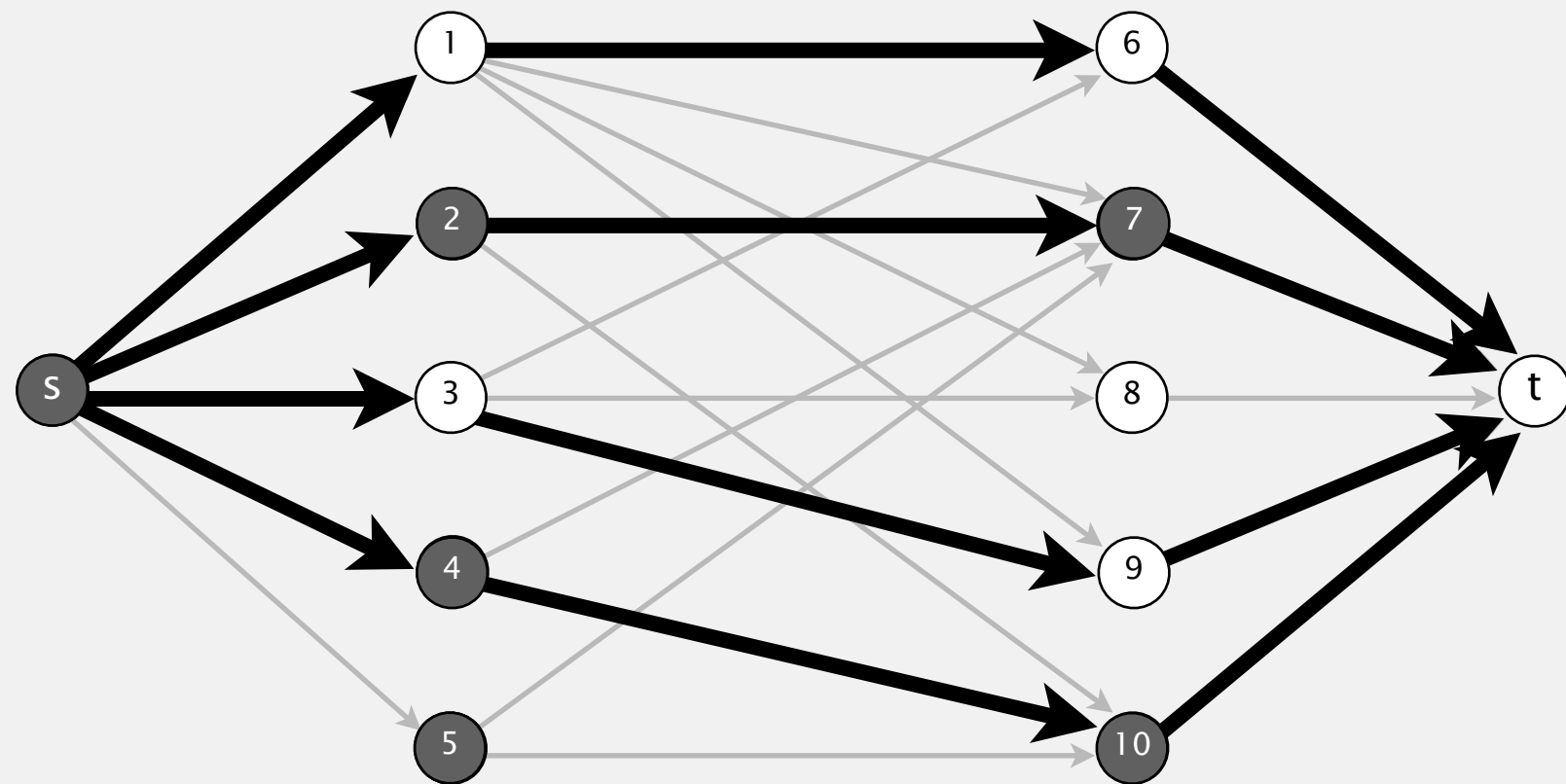can be matched
only to
people in T

| S | > | T |

Mincut. Consider mincut (A, B).

- Let S = tasks on s side of cut.

- Let T = people on s side of cut.

- Fact: $|S| > |T|$; tasks in S can be matched only to people in T.

S = { 2, 4, 5 }
T = { 7, 10 }

tasks in S
can be matched
only to
people in T

| S | > | T |

no perfect matching exists

Bottom line. When no perfect matching, mincut explains why.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i		team	wins	losses	to play	ATL	PHI	NYM	WAS
0		Atlanta	83	71	8	–	1	6	1
1		Philly	80	79	3	1	–	0	2
2		New York	78	78	6	6	0	–	0
3		Washington	77	82	3	1	2	0	–

Washington is mathematically eliminated.

- Washington finishes with ≤ 80 wins.
- Atlanta already has 83 wins.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i		team	wins	losses	to play	ATL	PHI	NYM	WAS
0		Atlanta	83	71	8	–	1	6	1
1		Philly	80	79	3	1	–	0	2
2		New York	78	78	6	6	0	–	0
3		Washington	77	82	3	1	2	0	–

Philadelphia is mathematically eliminated.
- Philadelphia finishes with ≤ 83 wins.
- Either New York or Atlanta will finish with ≥ 84 wins.

Observation. Answer depends not only on how many games already won and left to play, but on whom they're against.

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i		team	wins	losses	to play	NYY	BAL	BOS	TOR	DET
0		New York	75	59	28	–	3	8	7	3
1		Baltimore	71	63	28	3	–	2	7	4
2		Boston	69	66	27	8	2	–	0	0
3		Toronto	63	72	27	7	7	0	–	0
4		Detroit	49	86	27	3	4	0	0	–

AL East (August 30, 1996)

Detroit is mathematically eliminated.

- Detroit finishes with ≤ 76 wins.
- Wins for R = { NYY, BAL, BOS, TOR } = 278.
- Remaining games among { NYY, BAL, BOS, TOR } = $3 + 8 + 7 + 2 + 7 = 27$.
- Average team in R wins $305/4 = 76.25$ games.

Baseball elimination problem: maxflow formulation

Intuition. Remaining games flow from s to t.

games left
between 1 and 2

team 2 can still win
this many more games

g_{12}

∞

∞

$w_4 + r_4 - w_2$

team vertices
(each team other than 4)

game vertices
(each pair of teams other than 4)
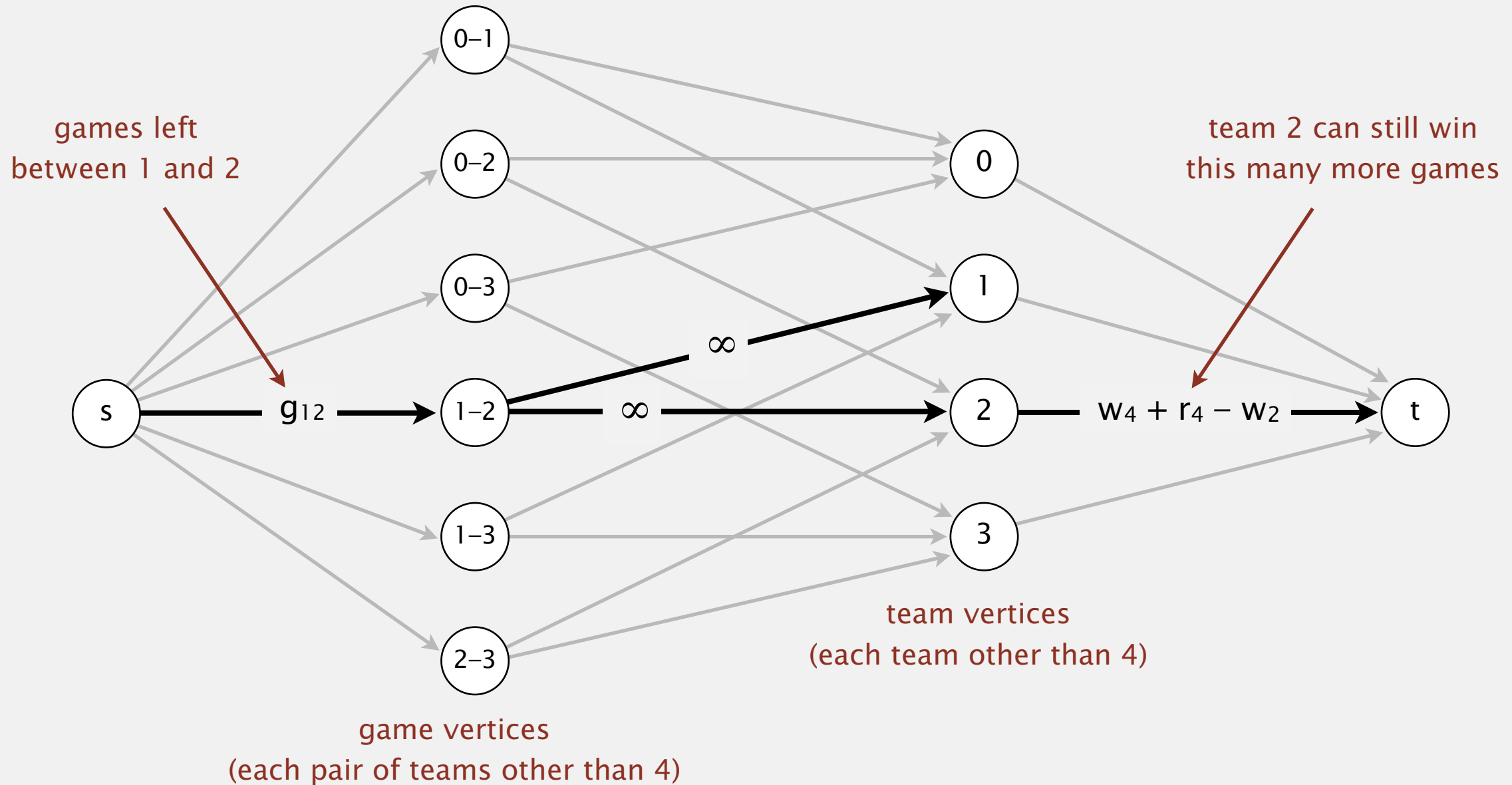
Fact. Team 4 not eliminated iff all edges pointing from s are full in maxflow.

Maximum flow algorithms: theory

(Yet another) holy grail for theoretical computer scientists.

year	method	worst case	discovered by
1951	**simplex**	$E^3 U$	Dantzig
1955	**augmenting path**	$E^2 U$	Ford–Fulkerson
1970	**shortest augmenting path**	E^3	Dinitz, Edmonds-Karp
1970	**fattest augmenting path**	$E^2 \log E \log(E U)$	Dinitz, Edmonds-Karp
1977	**blocking flow**	$E^{5/2}$	Cherkasky
1978	**blocking flow**	$E^{7/3}$	Galil
1983	**dynamic trees**	$E^2 \log E$	Sleator-Tarjan
1985	**capacity scaling**	$E^2 \log U$	Gabow
1997	**length function**	$E^{3/2} \log E \log U$	Goldberg-Rao
2012	**compact network**	$E^2 / \log E$	Orlin
?	?	E	?

maxflow algorithms for sparse networks with E edges, integer capacities between 1 and U

Maximum flow algorithms: practice

Warning. Worst-case order-of-growth is generally not useful for predicting or comparing maxflow algorithm performance in practice.

Best in practice. Push-relabel method with gap relabeling: $E^{3/2}$.

Computer vision. Specialized algorithms for problems with special structure.

On Implementing Push-Relabel Method for the Maximum Flow Problem

Boris V. Cherkassky[1] and Andrew V. Goldberg[2]

[1] Central Institute for Economics and Mathematics,
Krasikova St. 32, 117418, Moscow, Russia
cher@cemi.msk.su
[2] Computer Science Department, Stanford University
Stanford, CA 94305, USA
goldberg@cs.stanford.edu

Abstract. We study efficient implementations of the push-relabel method for the maximum flow problem. The resulting codes are faster than the previous codes, and much faster on some problem families. The speedup is due to the combination of heuristics used in our implementations. We also exhibit a family of problems for which the running time of all known methods seem to have a roughly quadratic growth rate.

Computational investigations of maximum flow algorithms

Ravindra K. Ahuja [a], Murali Kodialam [b], Ajay K. Mishra [c], James B. Orlin [d, *]

[a] *Department of Industrial and Management Engineering, Indian Institute of Technology, Kanpur, 208 016, India*
[b] *AT&T Bell Laboratories, Holmdel, NJ 07733, USA*
[c] *KATZ Graduate School of Business, University of Pittsburgh, Pittsburgh, PA 15260, USA*
[d] *Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Summary

Mincut problem. Find an st-cut of minimum capacity.

Maxflow problem. Find an st-flow of maximum value.

Duality. Value of the maxflow = capacity of mincut.

Proven successful approaches.
- Ford–Fulkerson (various augmenting-path strategies).
- Preflow–push (various versions).

Open research challenges.
- Practice: solve real-world maxflow/mincut problems in linear time.
- Theory: prove it for worst-case inputs.
- Still much to be learned!