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4.4 SHORTEST PATHS

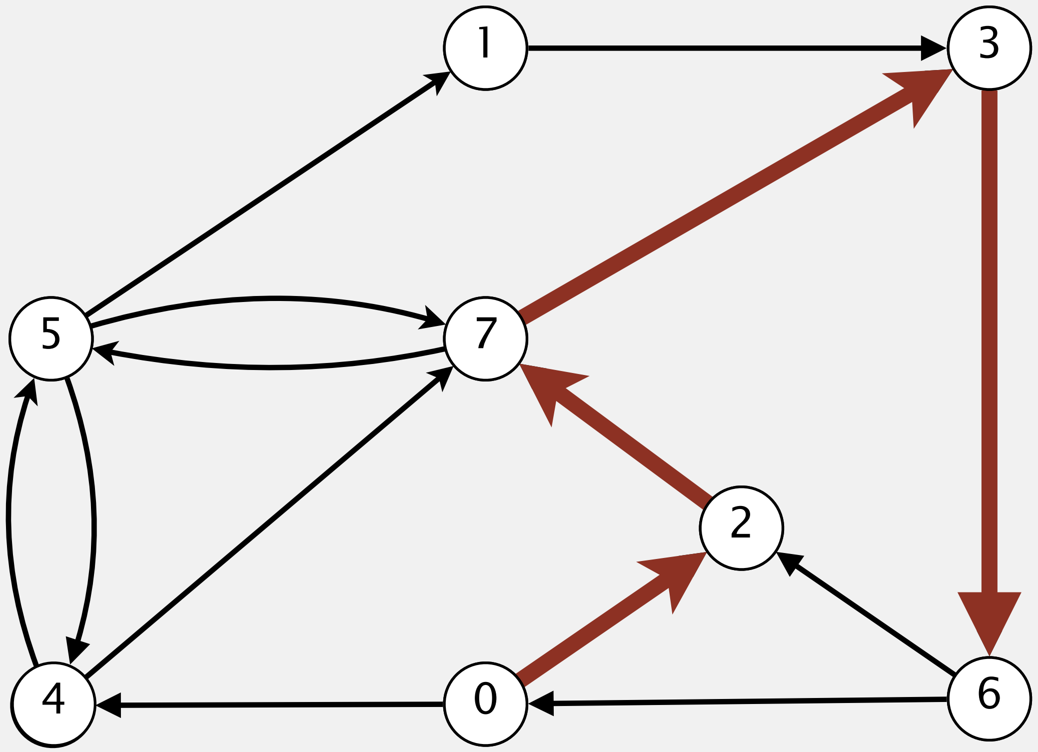
- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

edge-weighted digraph

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

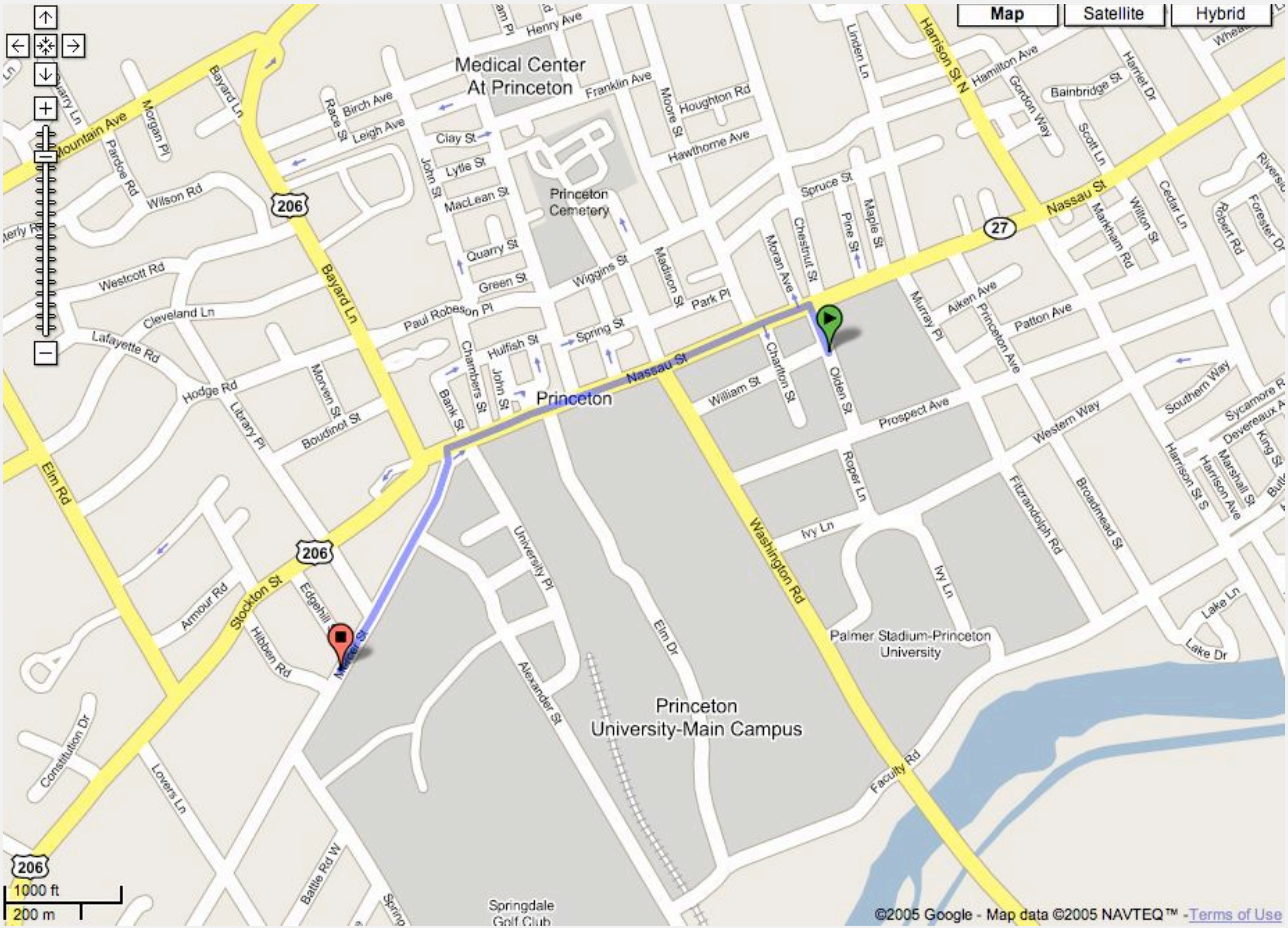


shortest path from 0 to 6

```
0->2  0.26
2->7  0.34
7->3  0.39
3->6  0.52
```

0.26 + 0.34 + 0.39 + 0.52 = 1.51

Google maps



Map
Satellite
Hybrid
Medical Center
At Princeton
Henry Ave
Franklin Ave
Birch Ave
Leigh Ave
Clay St
Lytle St
MacLean St
Quarry St
Green St
Paul Robeson Pl
Hulfish St
Spring St
Wiggins St
Park Pl
Princeton
Cemetery
Houghton Rd
Hawthorne Ave
Moore St
Madison St
Spruce St
Chestnut St
Moran Ave
Pine St
Maple St
Linden Ln
Harrison St N
Hamilton Ave
Gordon Way
Bainbridge St
Harriet Dr
Scott Ln
Nassau St
27
Markham Rd
Wilton St
Cedar Ln
Robert Rd
Forester Dr
Mountain Ave
Morgan Pl
Pardoe Rd
Wilson Rd
Bayard Ln
206
Westcott Rd
Cleveland Ln
Lafayette Rd
Hodge Rd
Library Pl
Morven St
Boudinot St
Bank St
Chambers St
John St
Princeton
Nassau St
William St
Charlton St
Olden St
Murray Pl
Aiken Ave
Princeton Ave
Patton Ave
Prospect Ave
Western Way
Southern Way
Sycamore R
Devereaux A
King St
Marshall St
Harrison Ave
Harrison St S
Roper Ln
Ivy Ln
Fitzrandolph Rd
Broadmead St
Washington Rd
University Pl
Elm Rd
Armour Rd
Stockton St
Edgehill
Hibben Rd
Mercer St
Alexander St
Elm Dr
Palmer Stadium-Princeton
University
Princeton
University-Main Campus
Lake Ln
Lake Dr
Faculty Rd
Constitution Dr
Lovers Ln
Battle Rd W
Springdale
Golf Club
1000 ft
200 m
©2005 Google - Map data ©2005 NAVTEQ™ - Terms of Use

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source-sink: from one vertex s to another t.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

which variant?

Simplifying assumption. Each vertex is reachable from s.

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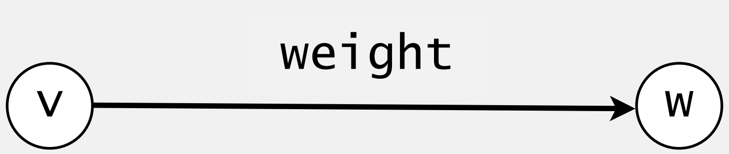
4.4 Shortest Paths

- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- *negative weights*

Weighted directed edge API

`public class DirectedEdge`		
	`DirectedEdge(int v, int w, double weight)`	*weighted edge v→w*
`int`	`from()`	*vertex v*
`int`	`to()`	*vertex w*
`double`	`weight()`	*weight of this edge*
`String`	`toString()`	*string representation*



Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

`from()` and `to()` replace `either()` and `other()`

Edge-weighted digraph API

	public class EdgeWeightedDigraph	
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges adjacent from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

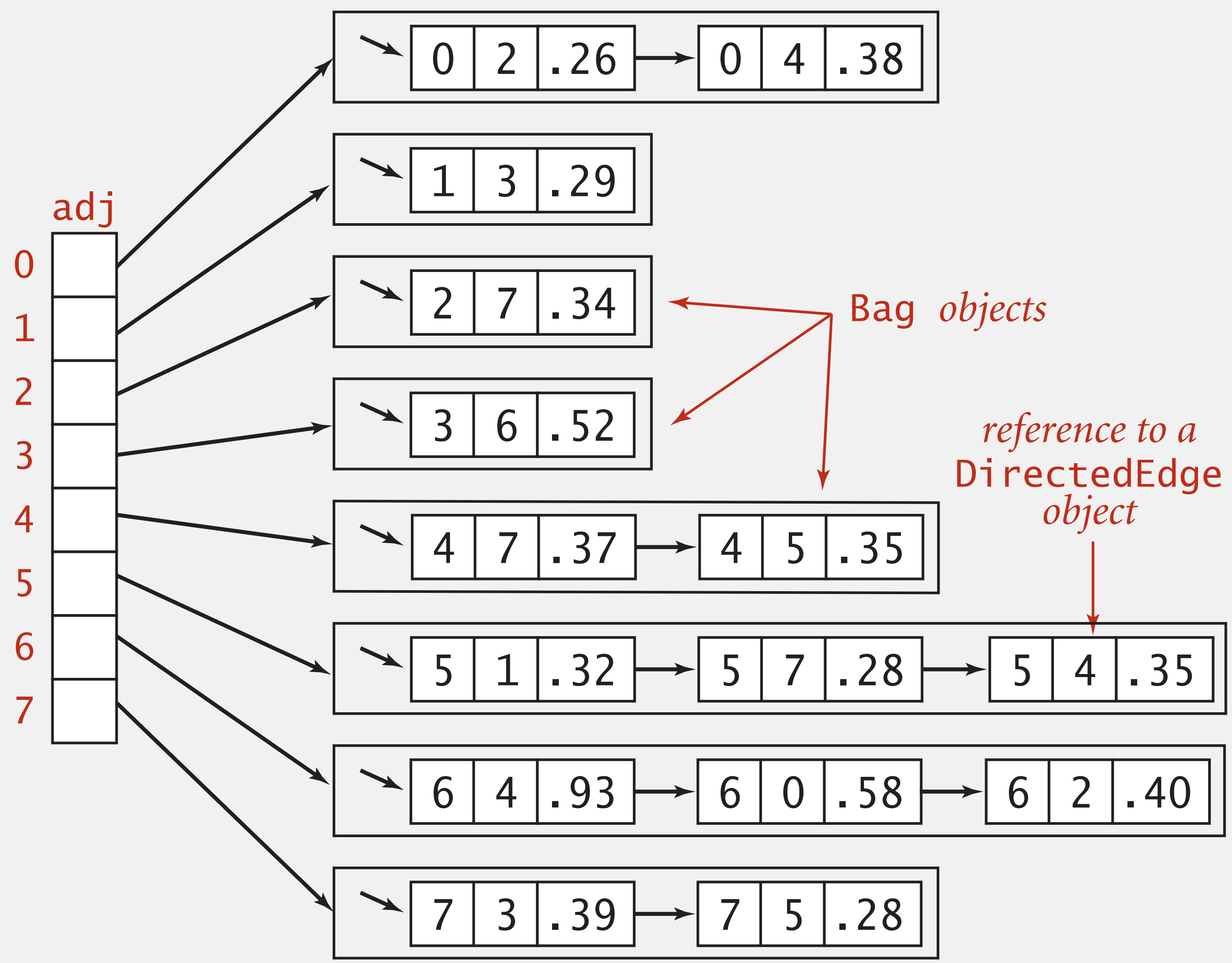
Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

V → 8
E → 15

```
4 5  0.35
5 4  0.35
4 7  0.37
5 7  0.28
7 5  0.28
5 1  0.32
0 4  0.38
0 2  0.26
7 3  0.39
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```



Edge-weighted digraph: adjacency-lists implementation in Java

Same as `EdgeWeightedGraph` except replace `Graph` with `Digraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w to only v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

	`public class SP`	
	`SP(EdgeWeightedDigraph G, int s)`	*shortest paths from s in graph G*
`double`	`distTo(int v)`	*length of shortest path from s to v*
`Iterable <DirectedEdge>`	`pathTo(int v)`	*shortest path from s to v*
`boolean`	`hasPathTo(int v)`	*is there a path from s to v?*

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4.4 Shortest Paths

- APIs
- **shortest-paths properties**
- Dijkstra's algorithm
- edge-weighted DAGs
- negative weights

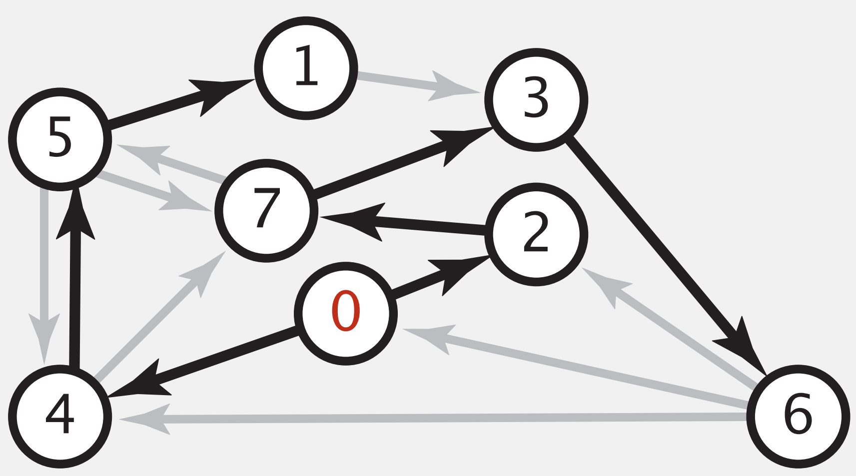
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.



shortest-paths tree from 0

	edgeTo[]		distTo[]
0	null		0
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.37	0.97
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.49
7	2->7	0.34	0.60

parent-link representation

Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.

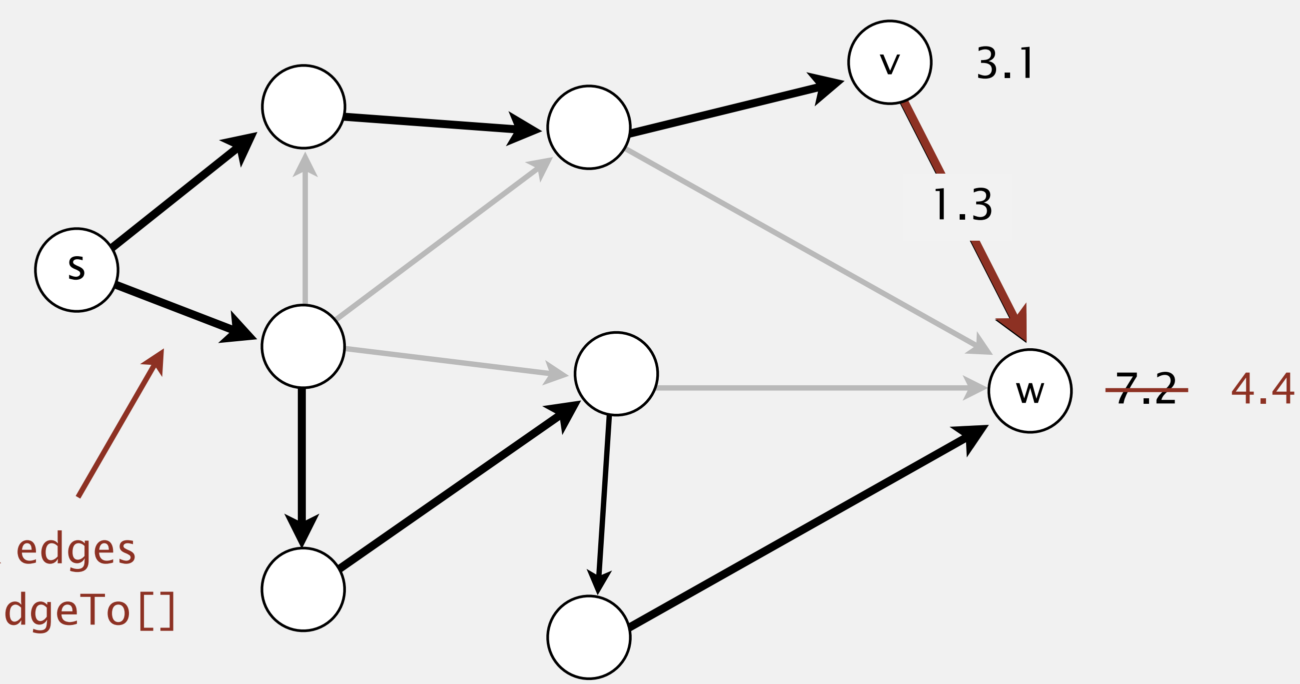
```
public double distTo(int v)
{  return distTo[v];  }

public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- distTo[v] is length of shortest known path from s to v.
- distTo[w] is length of shortest known path from s to w.
- edgeTo[w] is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v,
  update both distTo[w] and edgeTo[w].

v→w successfully relaxes



black edges
are in edgeTo[]

Edge relaxation

Relax edge $e = v \rightarrow w$.

- distTo[v] is length of shortest known path from s to v.
- distTo[w] is length of shortest known path from s to w.
- edgeTo[w] is last edge on shortest known path from s to w.
- If e = v→w gives shorter path to w through v, update both distTo[w] and edgeTo[w].

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```

Shortest-paths optimality conditions

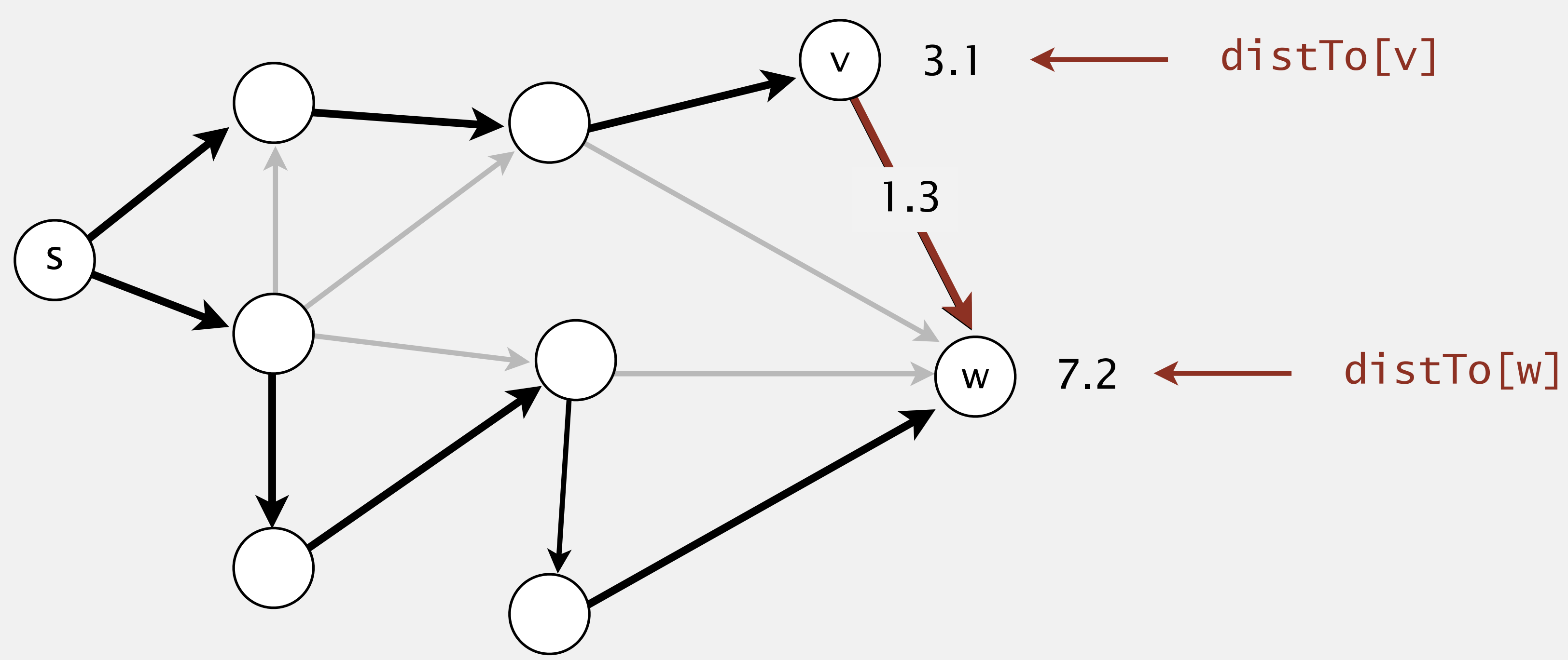
Proposition. Let G be an edge-weighted digraph.

Then distTo[] are the shortest path distances from s iff:

- distTo[s] $=$ 0.
- For each vertex v, distTo[v] is the length of some path from s to v.
- For each edge e = v→w, distTo[w] ≤ distTo[v] + e.weight().

Pf. ⇐ [necessary]

- Suppose that distTo[w] > distTo[v] + e.weight() for some edge e = v→w.
- Then, e gives a path from s to w (through v) of length less than distTo[w].



Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph.
Then `distTo[]` are the shortest path distances from `s` iff:

- `distTo[s]` $= 0$.
- For each vertex `v`, `distTo[v]` is the length of some path from `s` to `v`.
- For each edge $e = v \rightarrow w$, $\texttt{distTo[w]} \le \texttt{distTo[v]} + \texttt{e.weight()}$.

Pf. $\Rightarrow$ [sufficient]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = w$ is a shortest path from s to w.
- Then,

$$\begin{aligned}
\texttt{distTo}[v_1] &\le \texttt{distTo}[v_0] + e_1\texttt{.weight()} \\
\texttt{distTo}[v_2] &\le \texttt{distTo}[v_1] + e_2\texttt{.weight()} \\
&\ldots \\
\texttt{distTo}[v_k] &\le \texttt{distTo}[v_{k-1}] + e_k\texttt{.weight()}
\end{aligned}$$

e_i = i^{th} edge on shortest path from s to w

- Add inequalities; simplify; and substitute $\texttt{distTo}[v_0] = \texttt{distTo[s]} = 0$:

$$\texttt{distTo[w]} = \texttt{distTo}[v_k] \le e_1\texttt{.weight()} + e_2\texttt{.weight()} + \ldots + e_k\texttt{.weight()}$$

weight of some path from s to w

weight of shortest path from s to w

- Thus, `distTo[w]` is the weight of shortest path to w. ▪

Generic shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:

- **Relax any edge.**

Proposition. Generic algorithm computes SPT (if it exists) from `s`.

Pf sketch.

- `distTo[v]` is always the length of a simple path from `s` to `v`.
- Each successful relaxation decreases `distTo[v]` for some `v`.
- `distTo[v]` can decrease at most a finite number of times. ▪

Generic shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:

- **Relax any edge.**

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman–Ford algorithm (no negative cycles).

Shortest paths: quiz 1

Let `e` = `v→w` be an edge with weight `17.0`. Suppose that `distTo[v] = ∞` and `distTo[w] = 15.0`. Which is the value of `distTo[w]` after calling `relax(e)`?

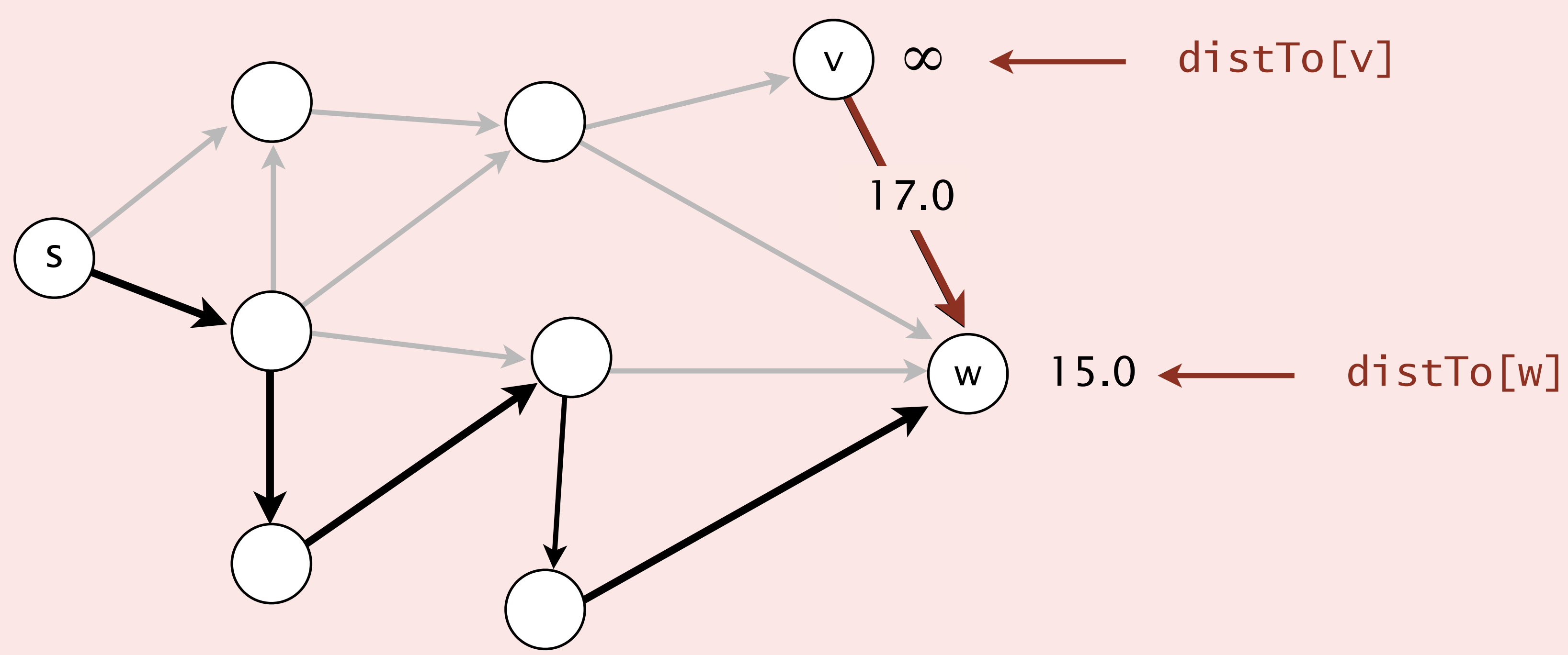
A. The program will throw a `java.lang.RuntimeException`.

B. `15.0`

C. `17.0`

D. $+\infty$

E. *I don't know.*



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4.4 Shortest Paths

- APIs
- shortest-paths properties
- *Dijkstra's algorithm*
- edge-weighted DAGs
- negative weights

Edsger W. Dijkstra: select quotes

" Do only what only you can do. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "

" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "

```
⍎'⎕',∈Nρ⊂S←'←⎕←(3=T)∨M∧2=T←⊃+/(V⌽''⊂M),(V⊖''⊂M),(V,⌽V)⌽''(V,V←1 ¯1)⊖''⊂M'
```

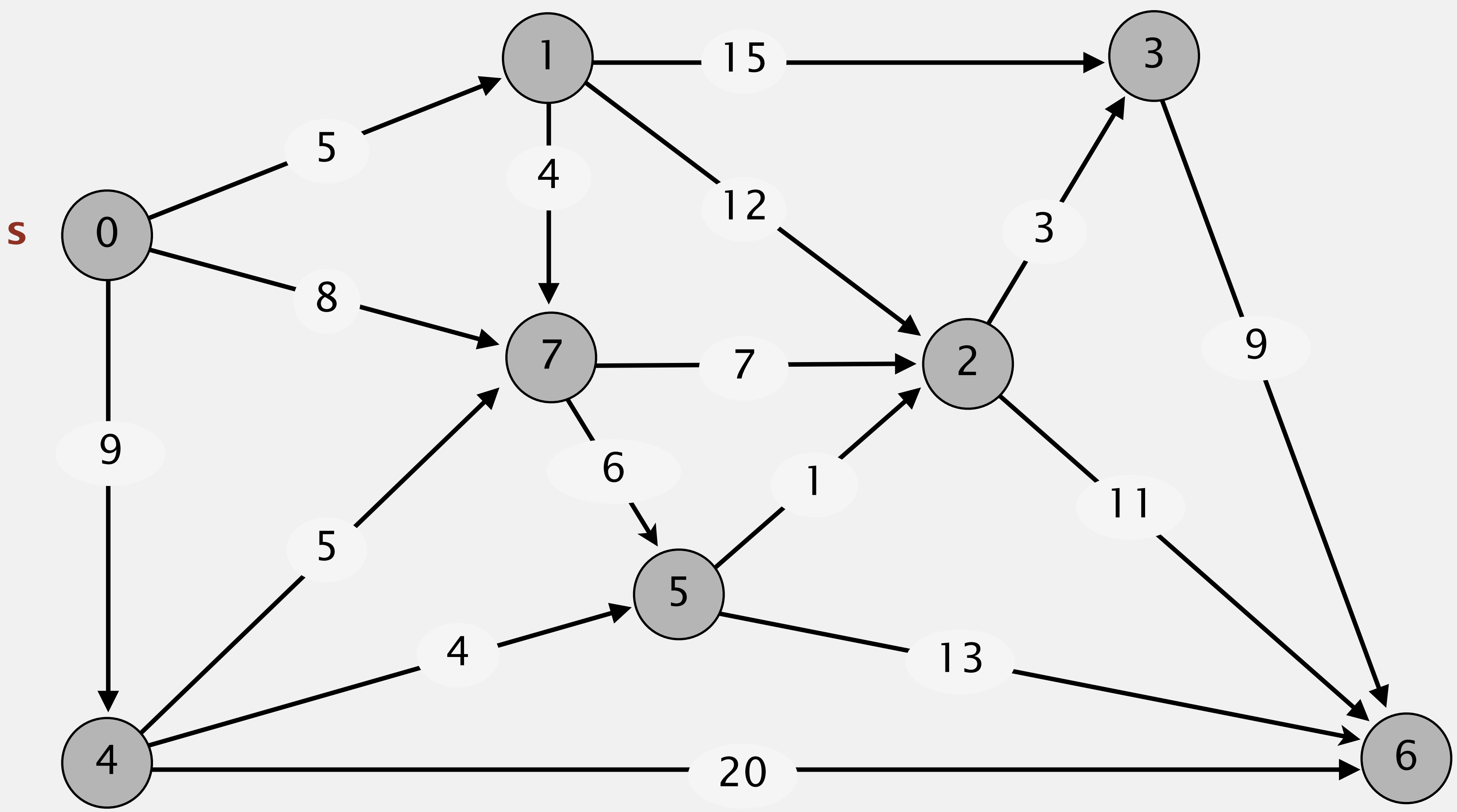
http://catpad.net/michael/apl

Edsger W. Dijkstra
Turing award 1972

Edsger W. Dijkstra: select quotes

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.

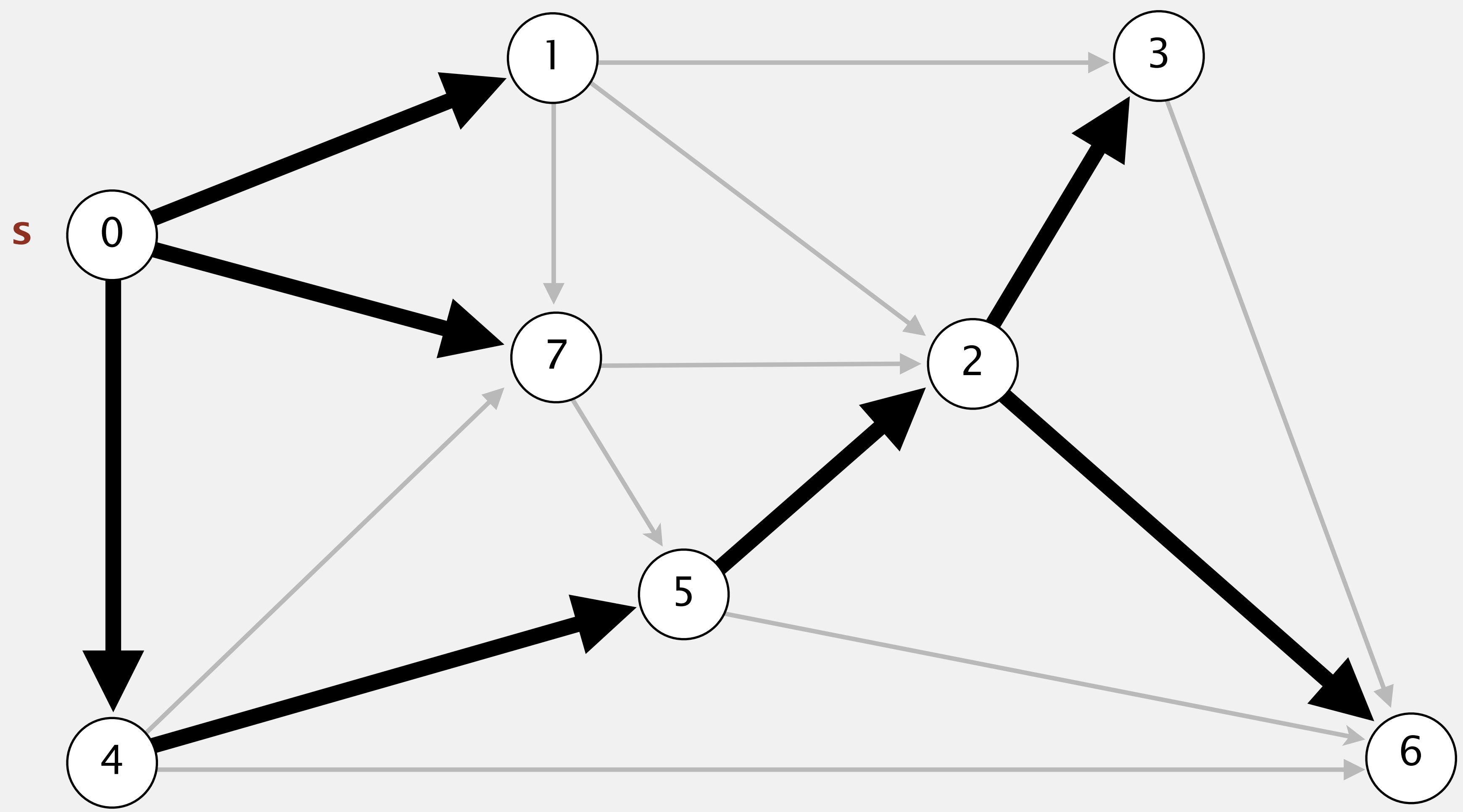


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

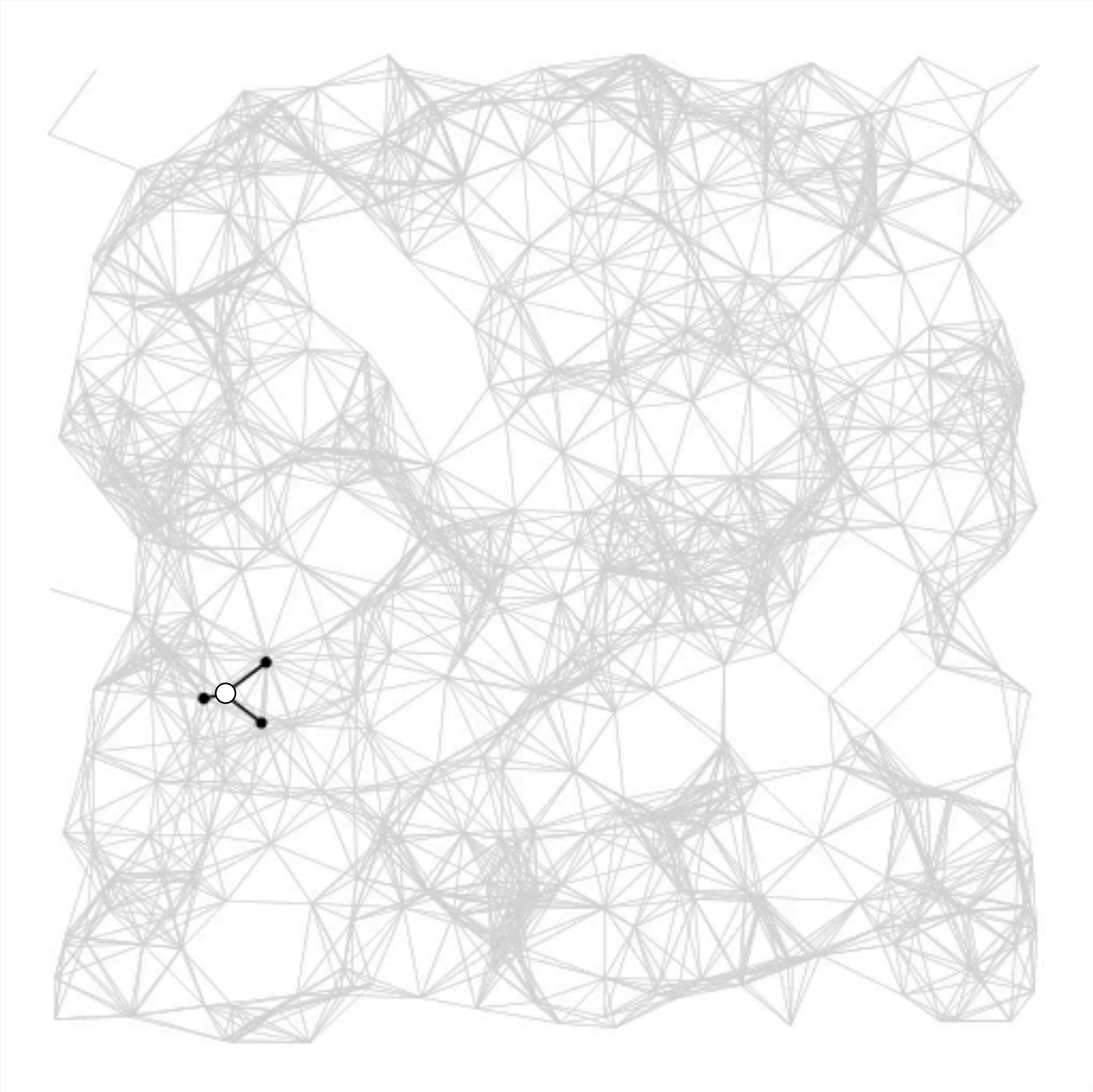
- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



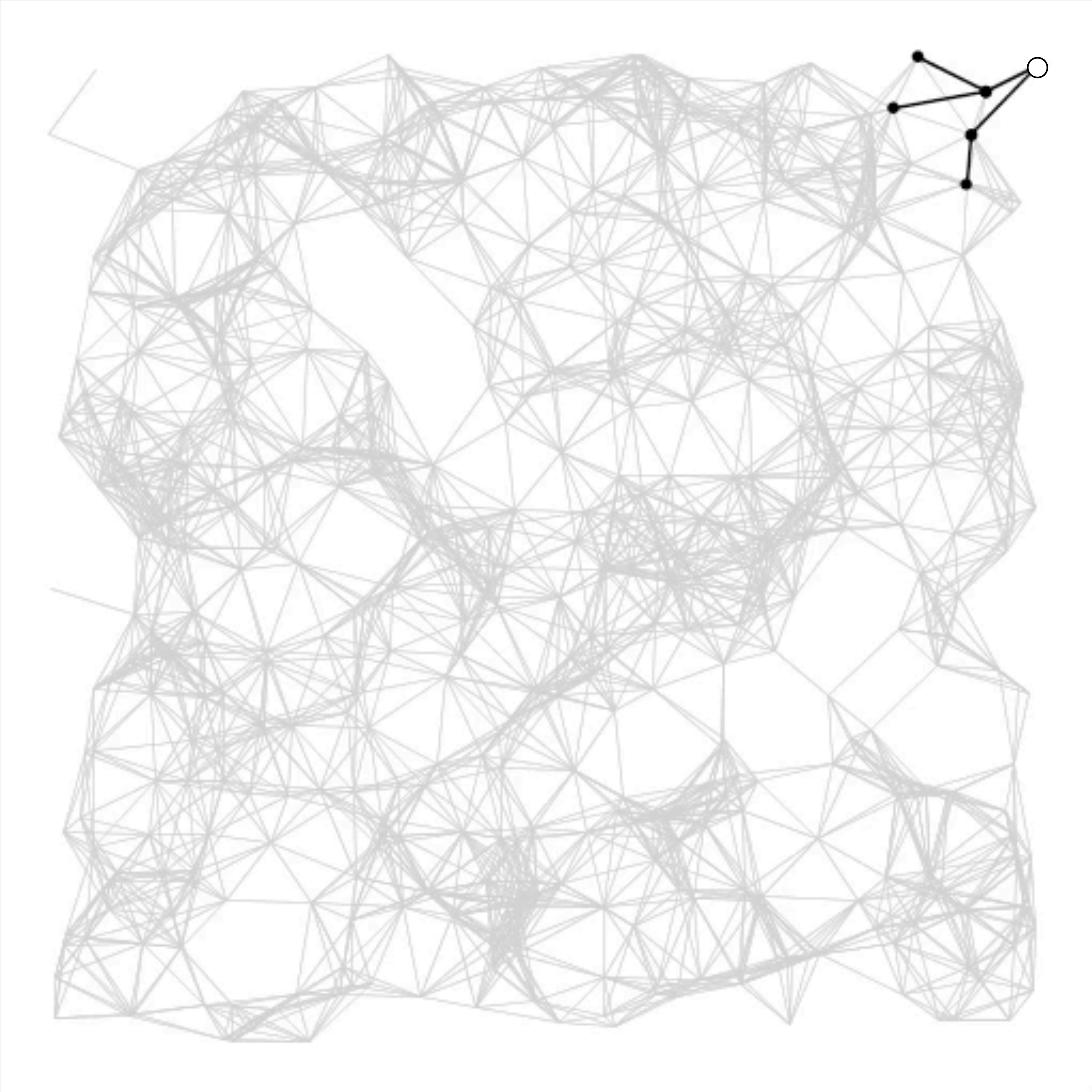
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof 1

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge e = v→w is relaxed exactly once (when vertex v is relaxed), leaving distTo[w] ≤ distTo[v] + e.weight().
- Inequality holds until algorithm terminates because:
  - distTo[w] cannot increase ← distTo[] values are monotone decreasing
  - distTo[v] will not change ← we choose lowest distTo[] value at each step (and edge weights are nonnegative)

when relaxing v

if u has not yet been relaxed,

then distTo[u] ≥ distTo[v]

- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
          int v = pq.delMin();
          for (DirectedEdge e : G.adj(v))
             relax(e);
      }
    }
 }
```

relax vertices in order of distance from s

Dijkstra's algorithm: Java implementation

```java
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ

Shortest paths: quiz 2

What is the order of growth of the running time of Dijkstra's algorithm when using a binary heap for the priority queue?

A. $V + E$

B. $V \log E$

C. $E \log V$

D. $E \log E$

E. *I don't know.*

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	1 †	$\log V$ †	1 †	$E + V \log V$

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

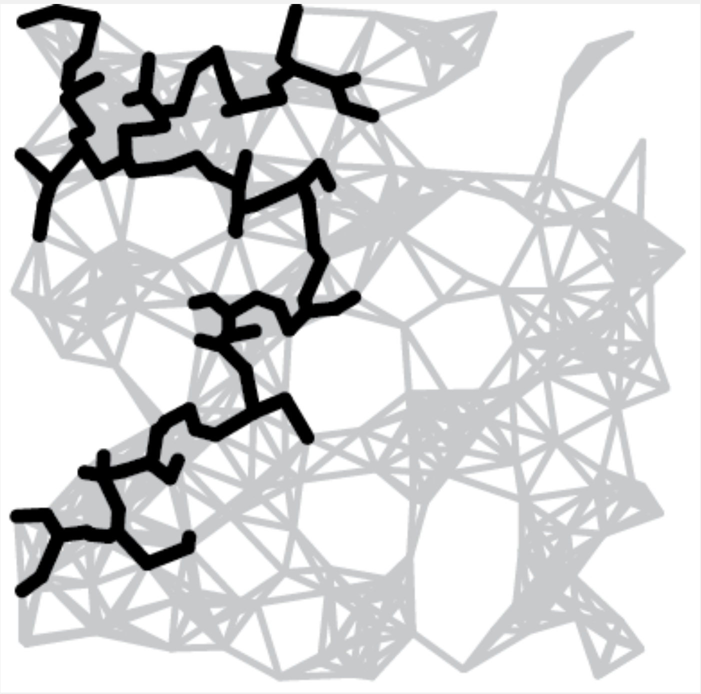
Computing a spanning tree in a graph

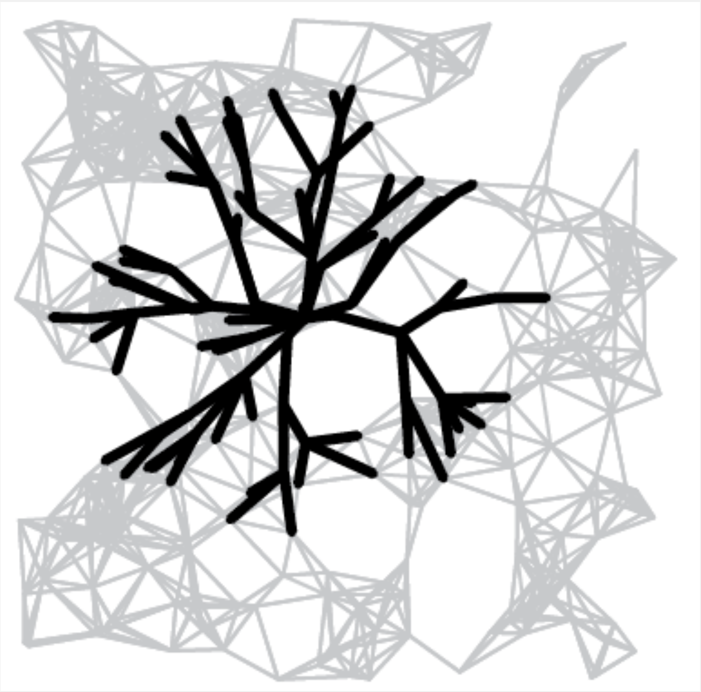
Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a spanning tree.

Main distinction: rule used to choose next vertex for the tree.

- Prim: Closest vertex to the tree (via an undirected edge).
- Dijkstra: Closest vertex to the source (via a directed path).





Note: DFS and BFS are also in this family of algorithms.

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4.4 SHORTEST PATHS

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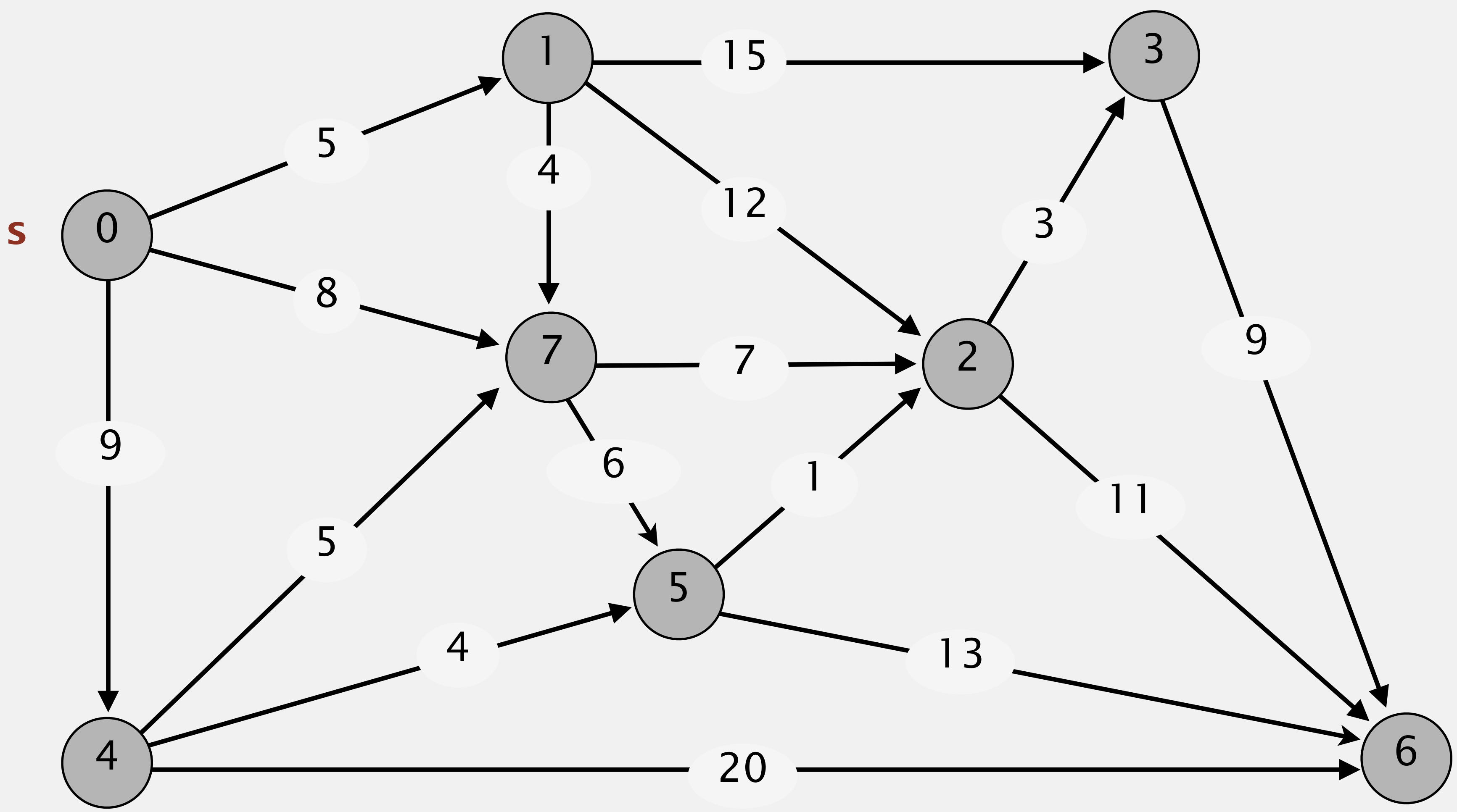
- shortest-paths properties
- Dijkstra's algorithm
- **edge-weighted DAGs**
- negative weights

Acyclic edge-weighted digraphs

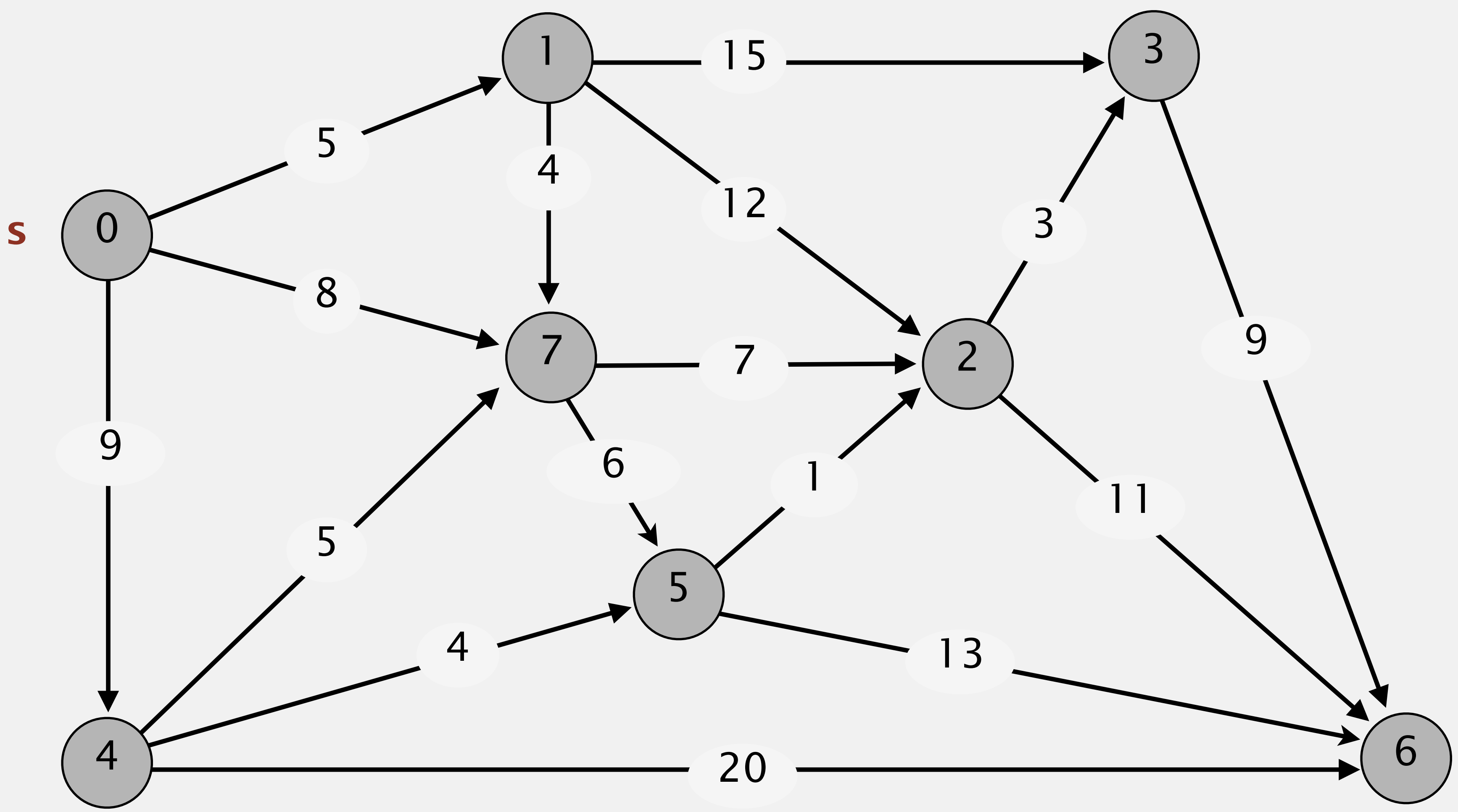
Q. Suppose that an edge-weighted digraph has no directed cycles.
Is it easier to find shortest paths than in a general digraph?



A. Yes!

Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.

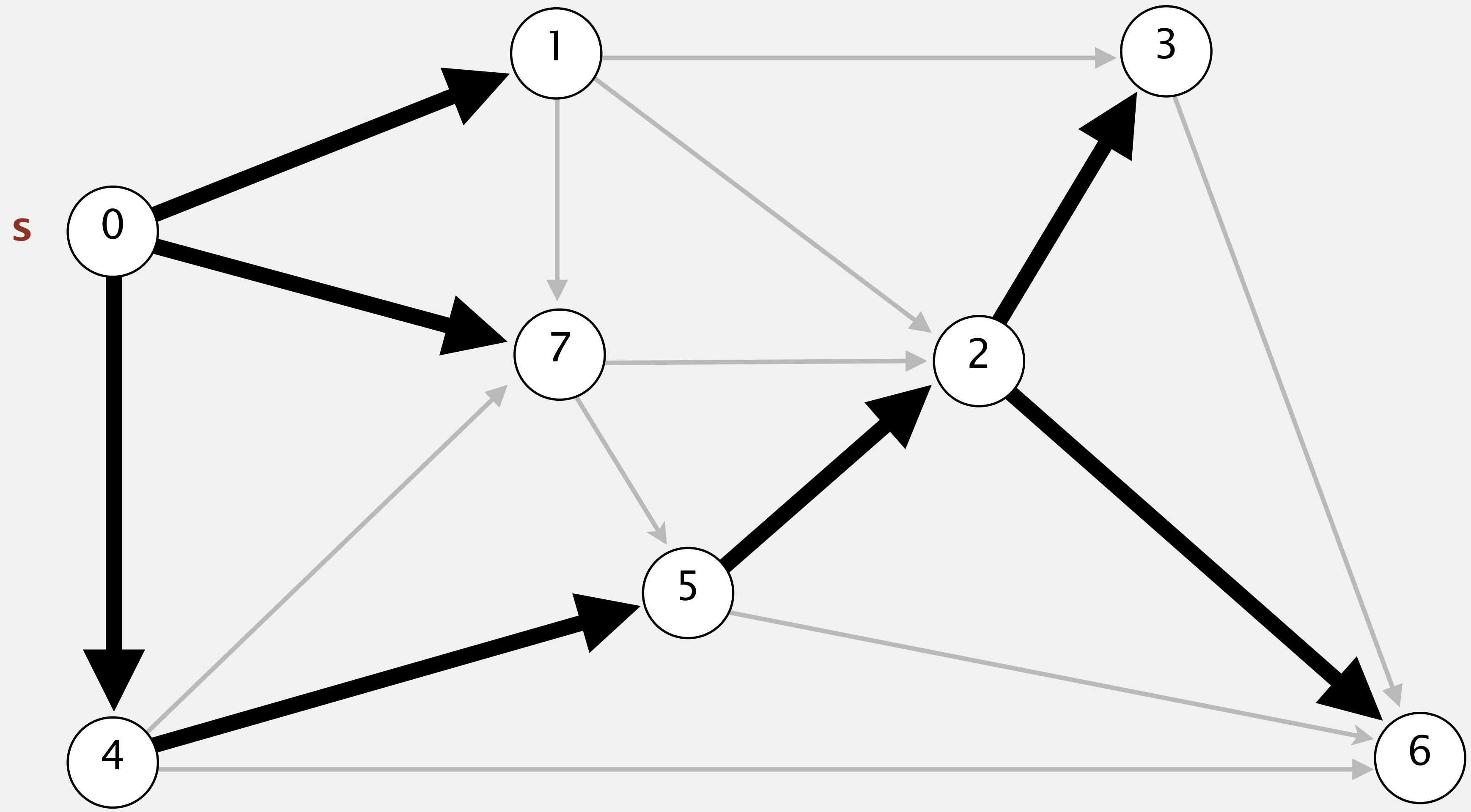


an edge-weighted DAG

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.



0 1 4 7 5 2 3 6

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Shortest paths: quiz 3

What is the order of growth of the running time of the topological sort algorithm for computing shortest paths in an edge-weighted DAG?

A. V

B. E

C. $V + E$

D. $V \log E$

E. *I don't know.*

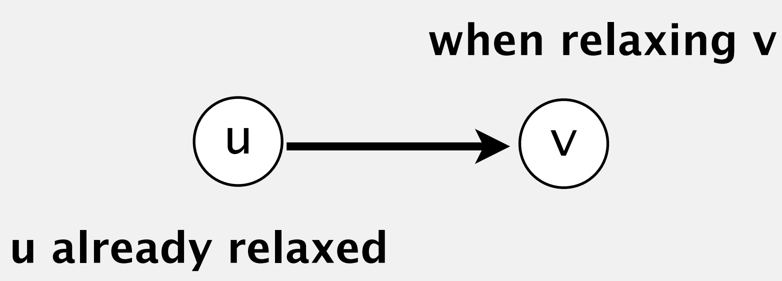
Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes the SPT in any edge-weighted DAG.

edge weights can be negative!

Pf.

- Each edge `e = v→w` is relaxed exactly once (when vertex `v` is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no vertex pointing to v will be relaxed after v is relaxed



- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

topological order

Content-aware resizing

Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

http://www.youtube.com/watch?v=vIFCV2spKtg

Content-aware resizing

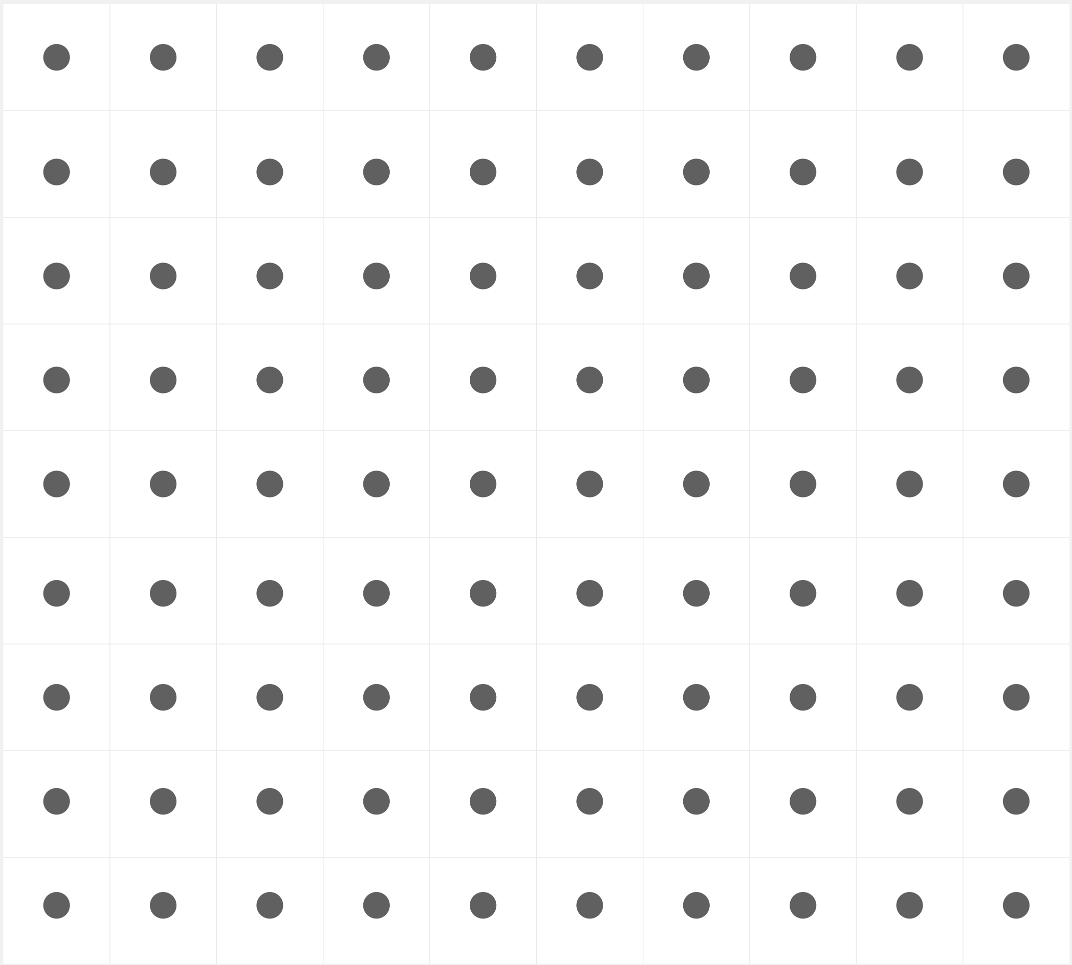
Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop, Imagemagick, GIMP, ...

Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = "energy function" of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
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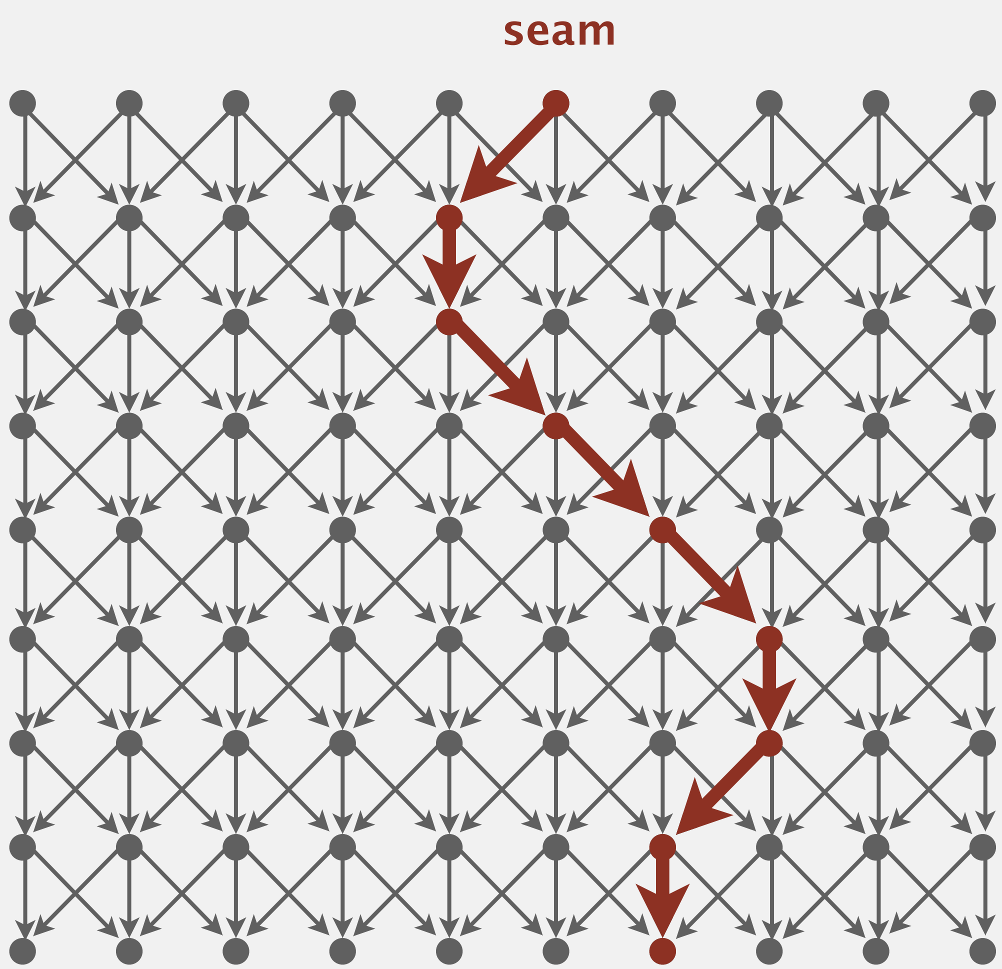
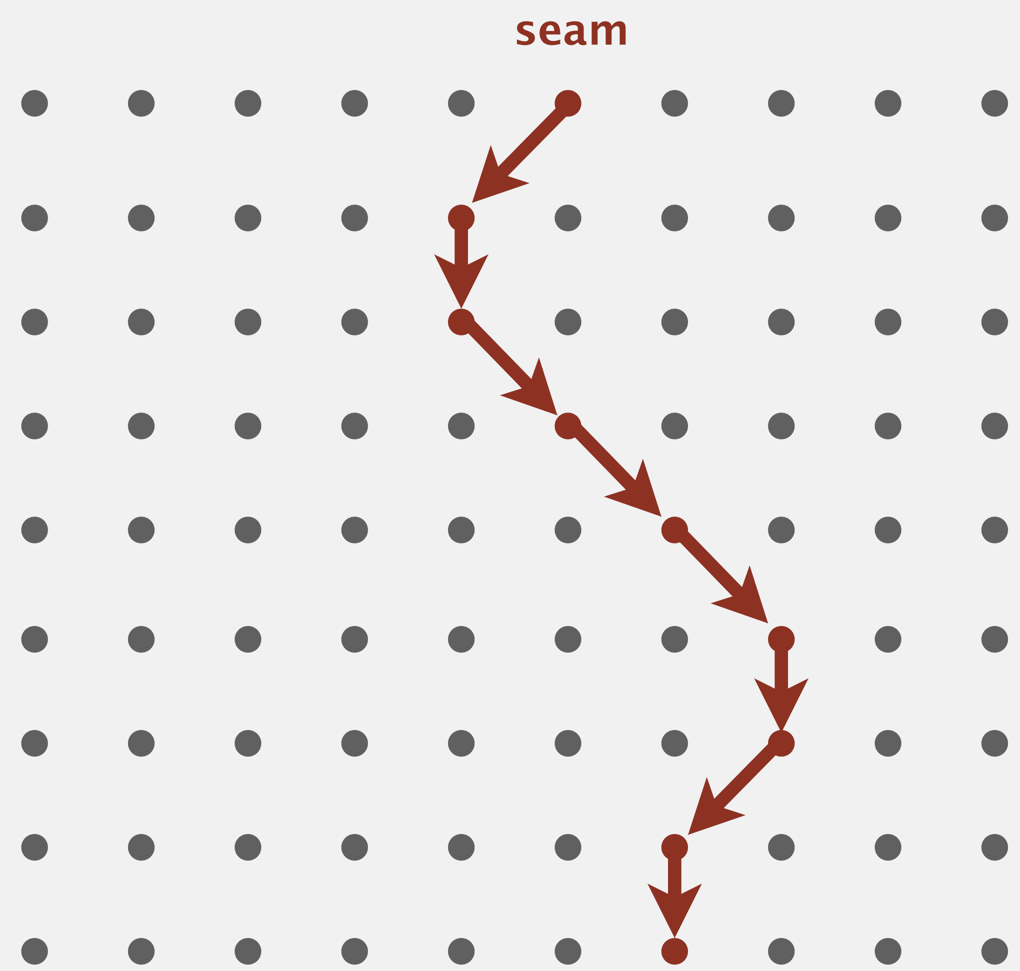
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To remove vertical seam:

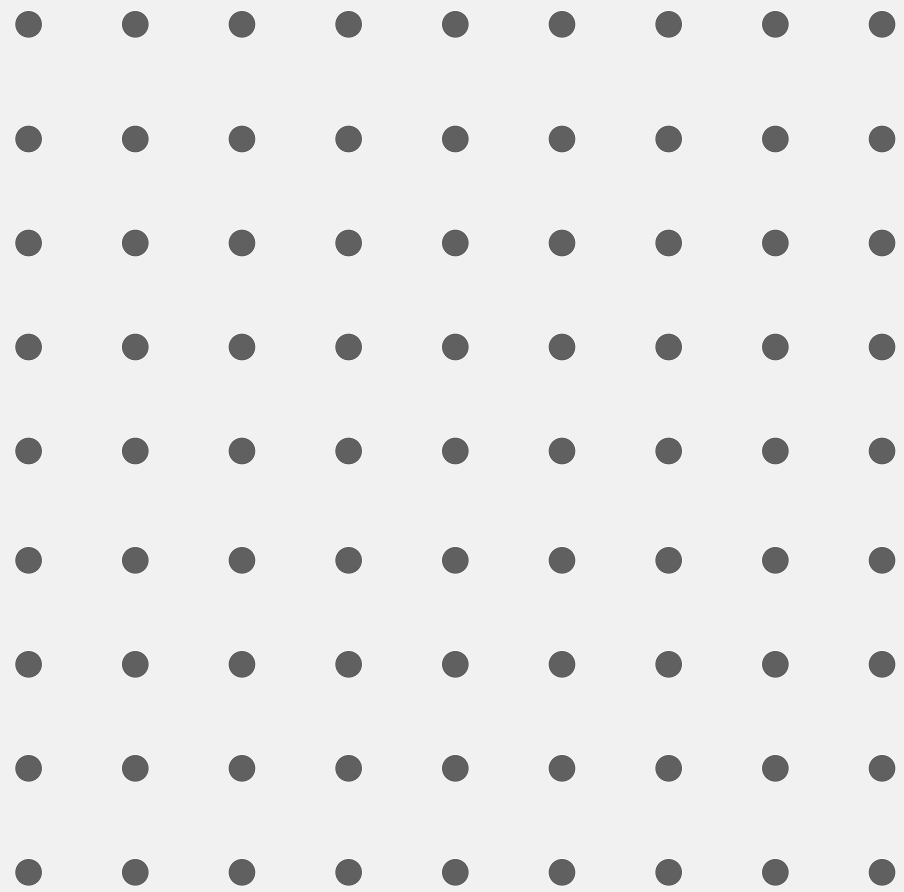
- Delete pixels on seam (one in each row).



Content-aware resizing

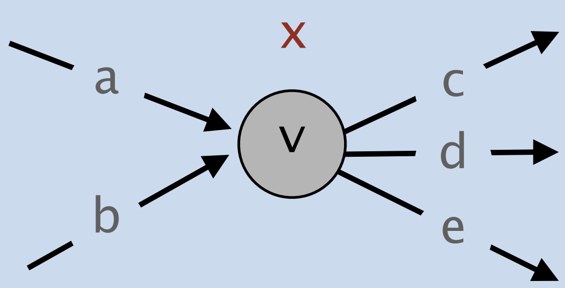
To remove vertical seam:

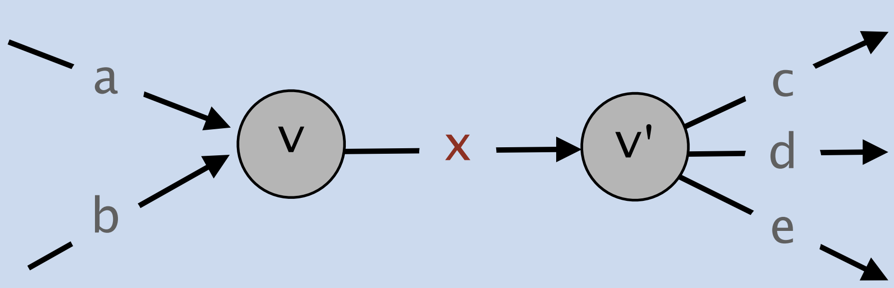
- Delete pixels on seam (one in each row).



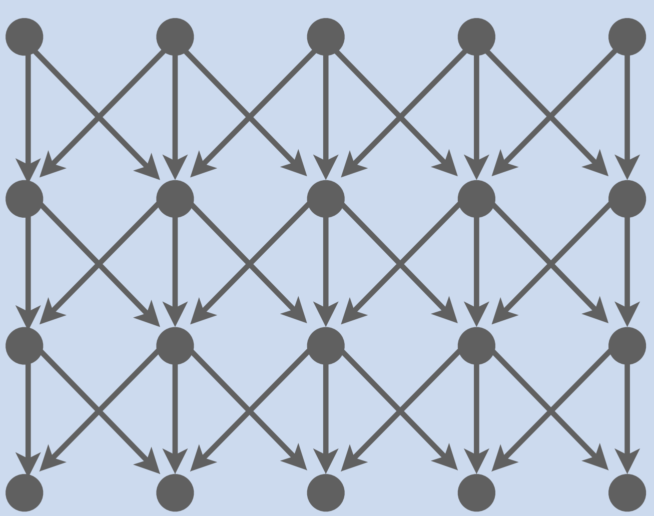
Shortest Path Variants in a Digraph

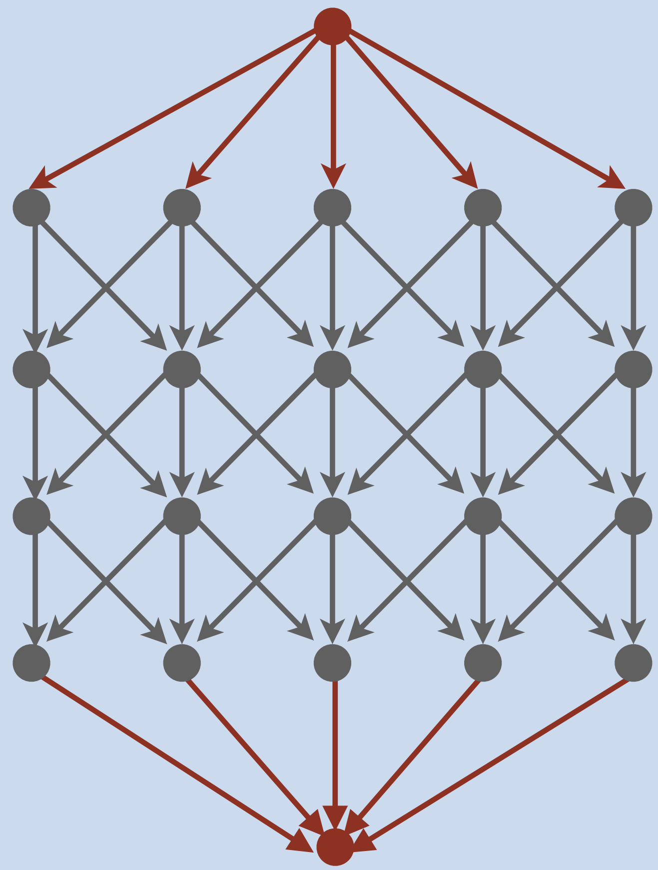
Q1. How to model vertex weights (along with edge weights)?





Q2. How to model multiple sources and sinks?





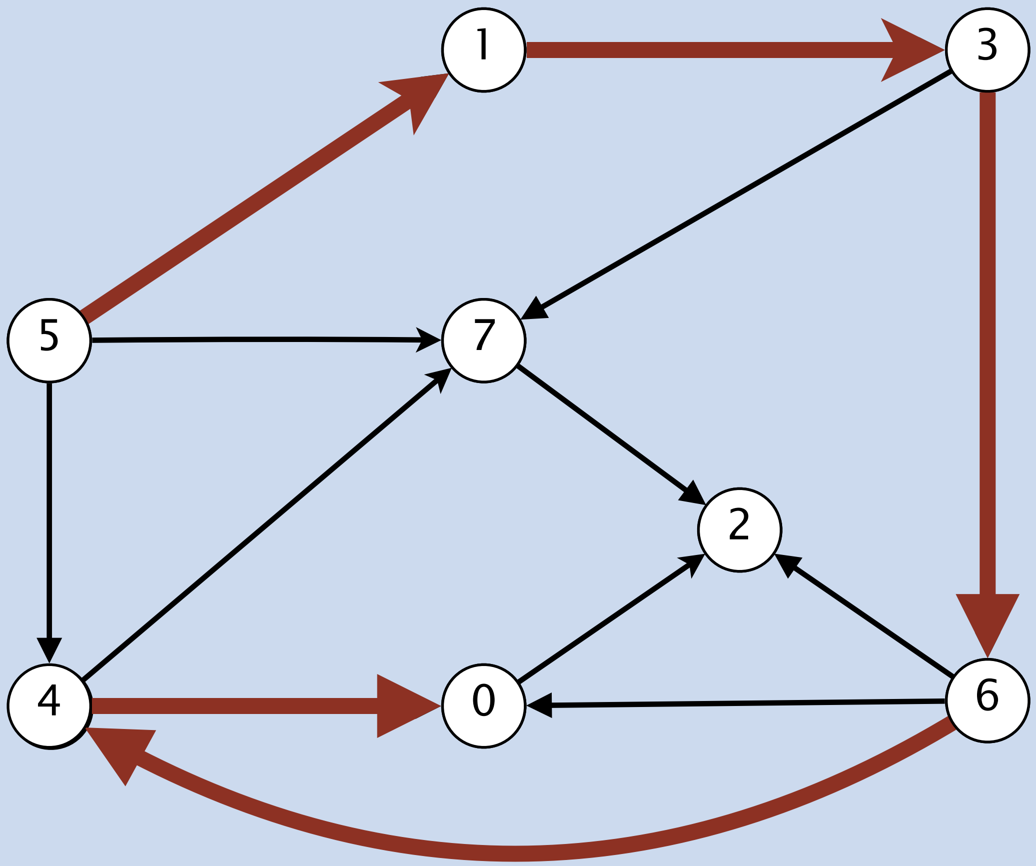
Longest Path in a DAG

Challenge. Given an edge-weight DAG, find the longest path from s to

Warning. Problem in digraphs is NP-COMPLETE.

longest paths input

```
5->4  0.35
4->7  0.37
5->7  0.28
5->1  0.32
4->0  0.38
0->2  0.26
3->7  0.39
1->3  0.29
7->2  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



longest path from 5 to 0

(0.32 + 0.29 + 0.52 + 0.93 + 0.38 = 2.44)

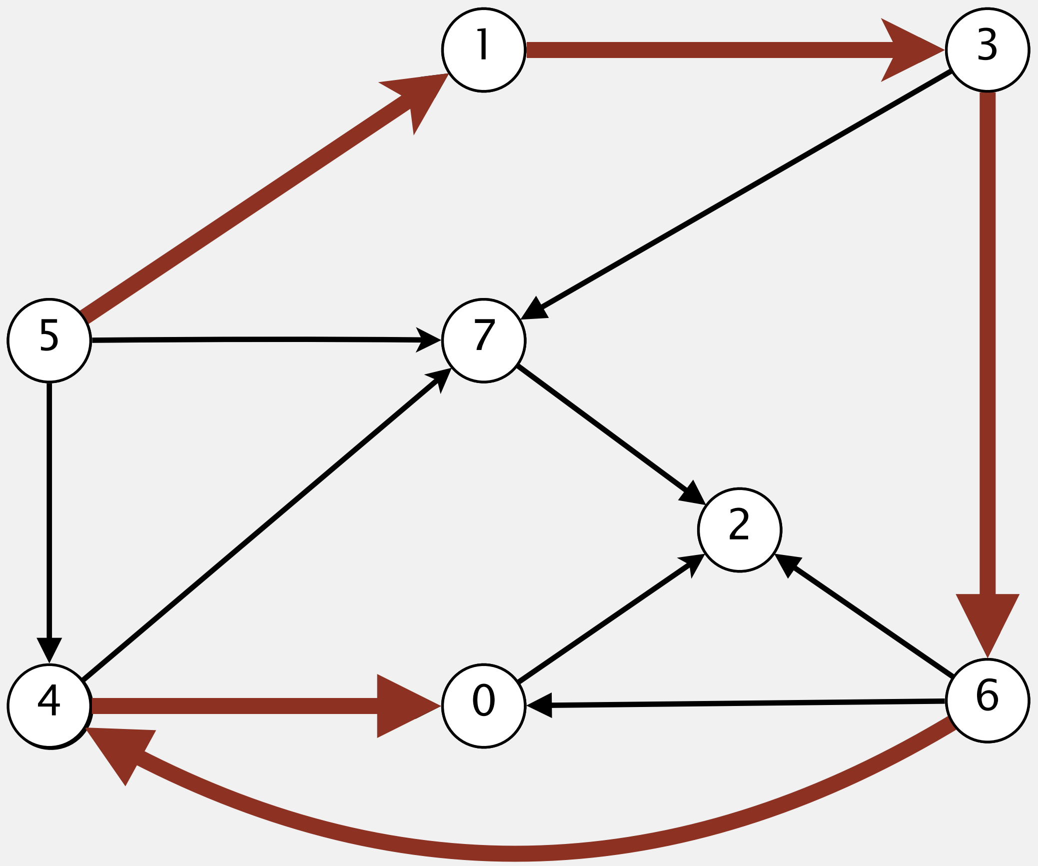
Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.

equivalent: reverse direction of inequality in `relax()`

longest paths input		shortest paths input	
5->4	0.35	5->4	-0.35
4->7	0.37	4->7	-0.37
5->7	0.28	5->7	-0.28
5->1	0.32	5->1	-0.32
4->0	0.38	4->0	-0.38
0->2	0.26	0->2	-0.26
3->7	0.39	3->7	-0.39
1->3	0.29	1->3	-0.29
7->2	0.34	7->2	-0.34
6->2	0.40	6->2	-0.40
3->6	0.52	3->6	-0.52
6->0	0.58	6->0	-0.58
6->4	0.93	6->4	-0.93



longest path from 5 to 0

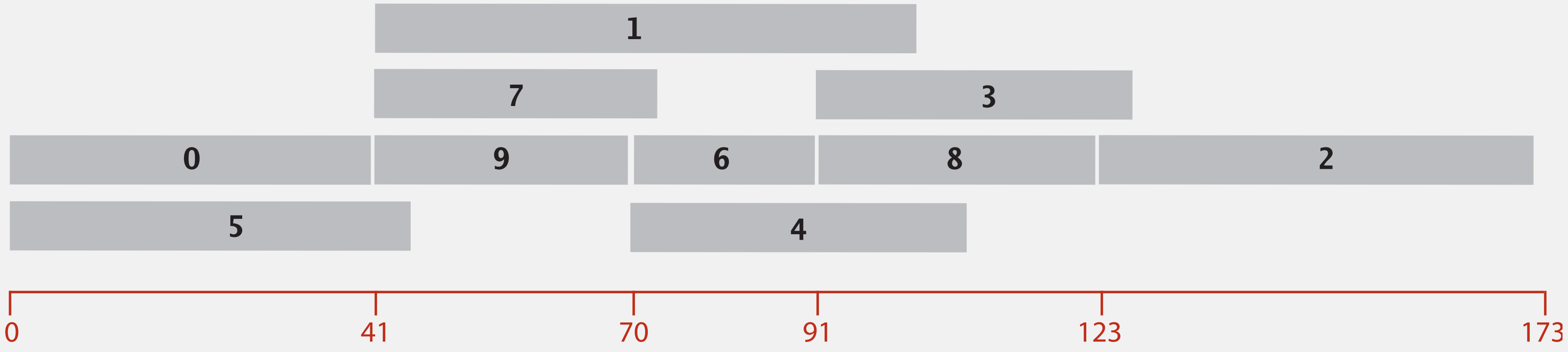
(0.32 + 0.29 + 0.52 + 0.93 + 0.38 = 2.44)

Key point. Topological sort algorithm works even with negative weights.

Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

job	*duration*	*must complete before*
0	41.0	1 7 9
1	51.0	2
2	50.0	
3	36.0	
4	38.0	
5	45.0	
6	21.0	3 8
7	32.0	3 8
8	32.0	2
9	29.0	4 6



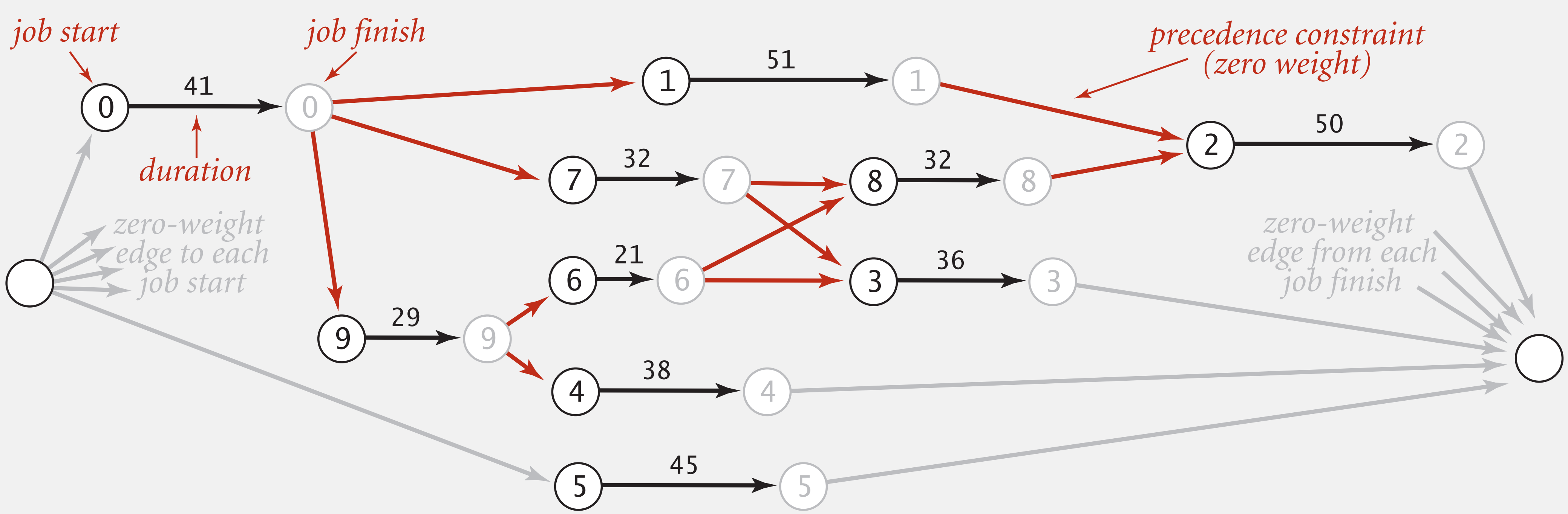
Parallel job scheduling solution

Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

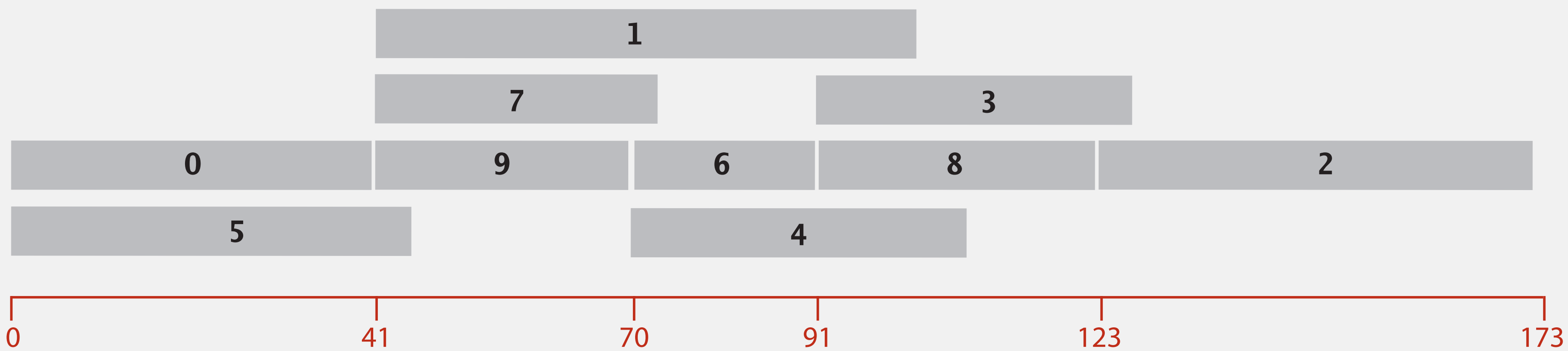
- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

job	duration	must complete before
0	41.0	1 7 9
1	51.0	2
2	50.0	
3	36.0	
4	38.0	
5	45.0	
6	21.0	3 8
7	32.0	3 8
8	32.0	2
9	29.0	4 6

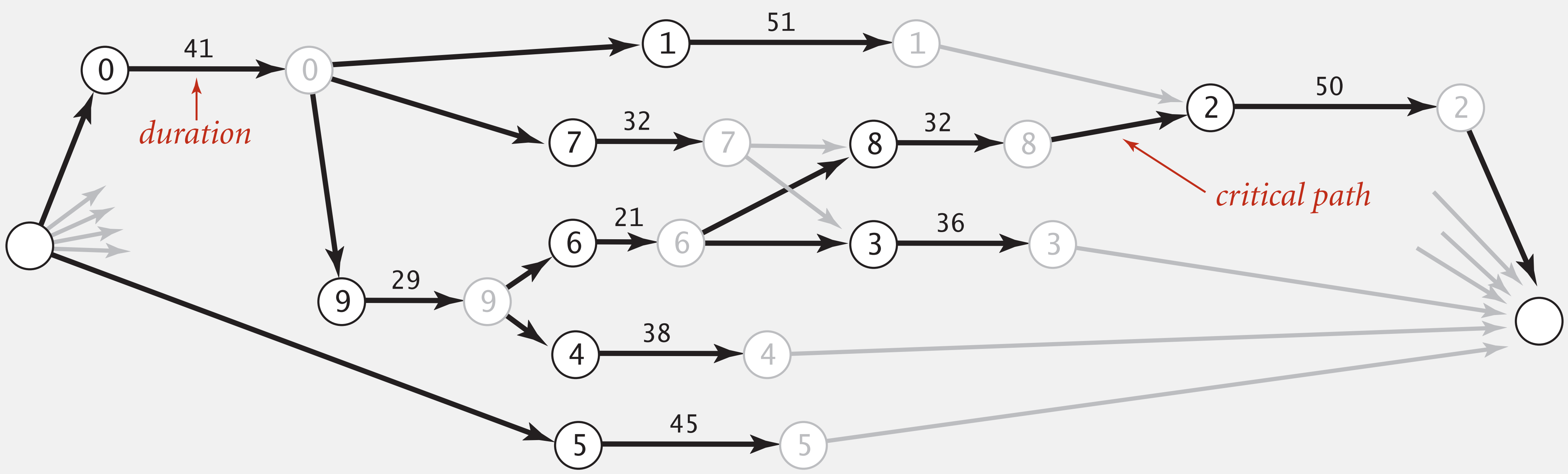


Critical path method

CPM. Use longest path from the source to schedule each job.



Parallel job scheduling solution



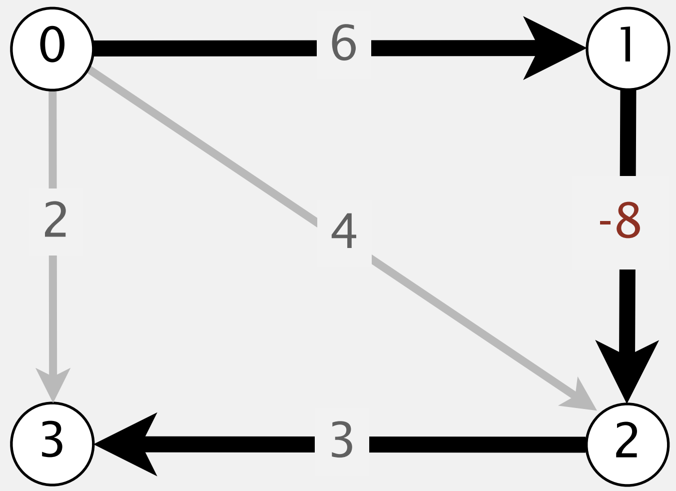
http://algs4.cs.princeton.edu

4.4 Shortest Paths

- *APIs*
- *shortest-paths properties*
- *Dijkstra's algorithm*
- *edge-weighted DAGs*
- ***negative weights***

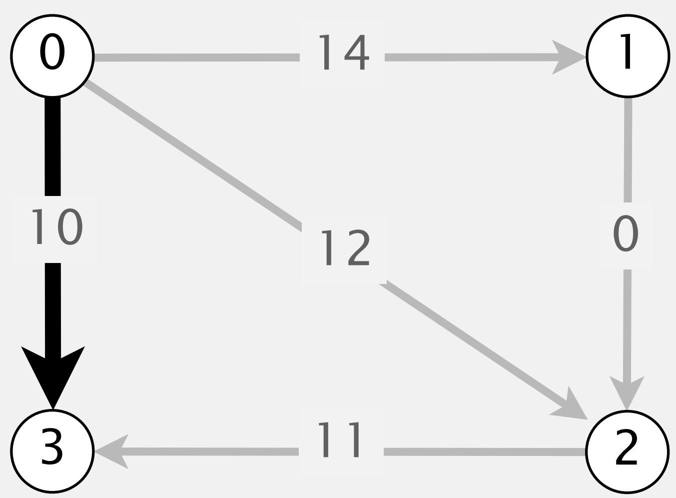
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects the vertices in the order 0, 3, 2, 1
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 8 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

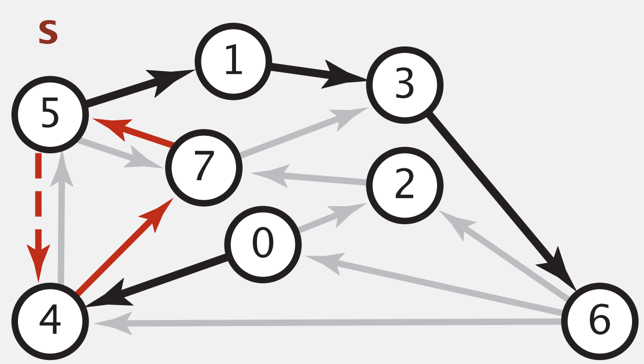
Conclusion. Need a different algorithm.

Negative cycles

A negative cycle is a directed cycle whose sum of edge weights is negative.

digraph

4->5	0.35
5->4	-0.66
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



negative cycle (-0.66 + 0.37 + 0.28)

5->4->7->5

shortest path from 0 to 6

0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

Bellman–Ford algorithm

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:

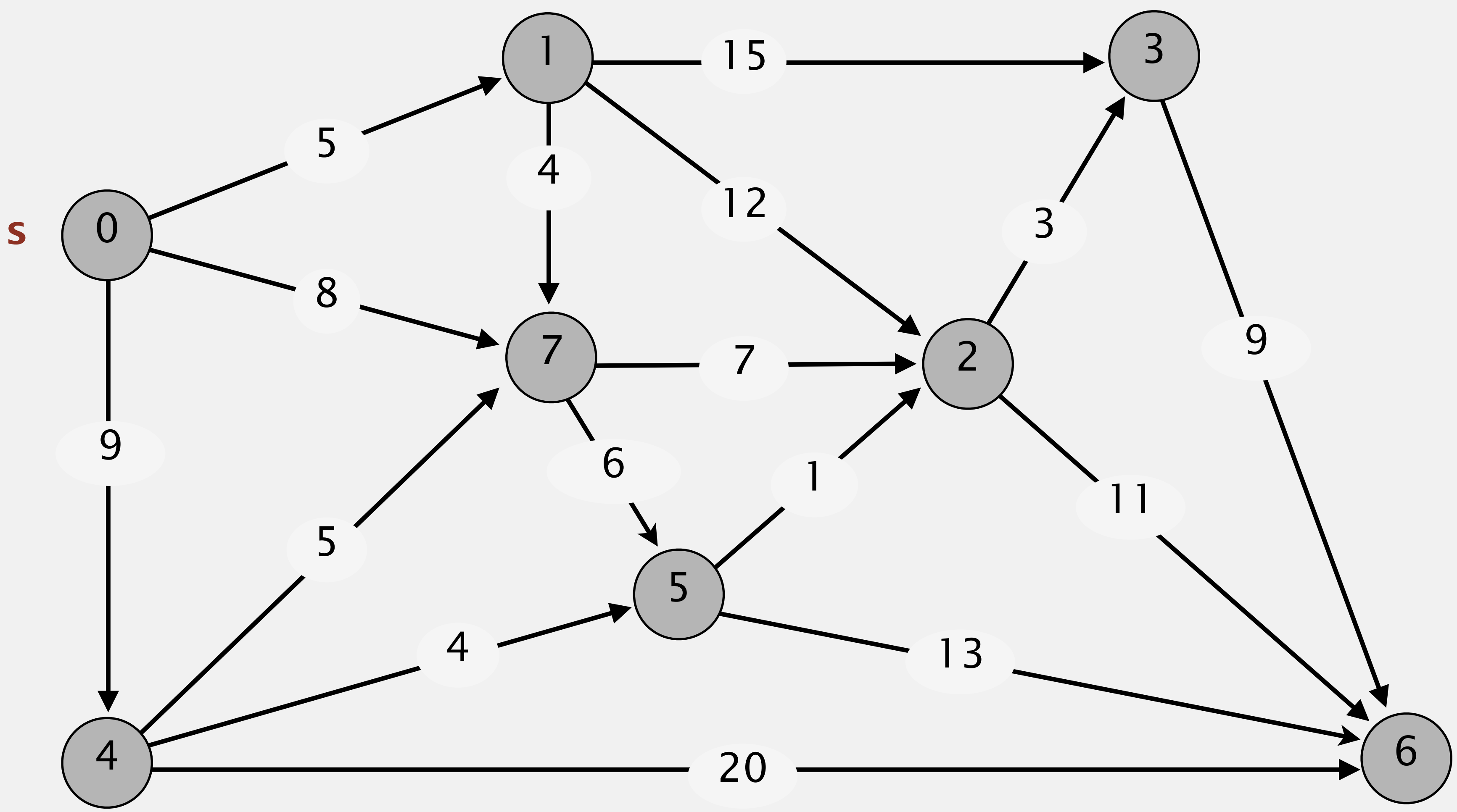
- **Relax each edge.**

```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

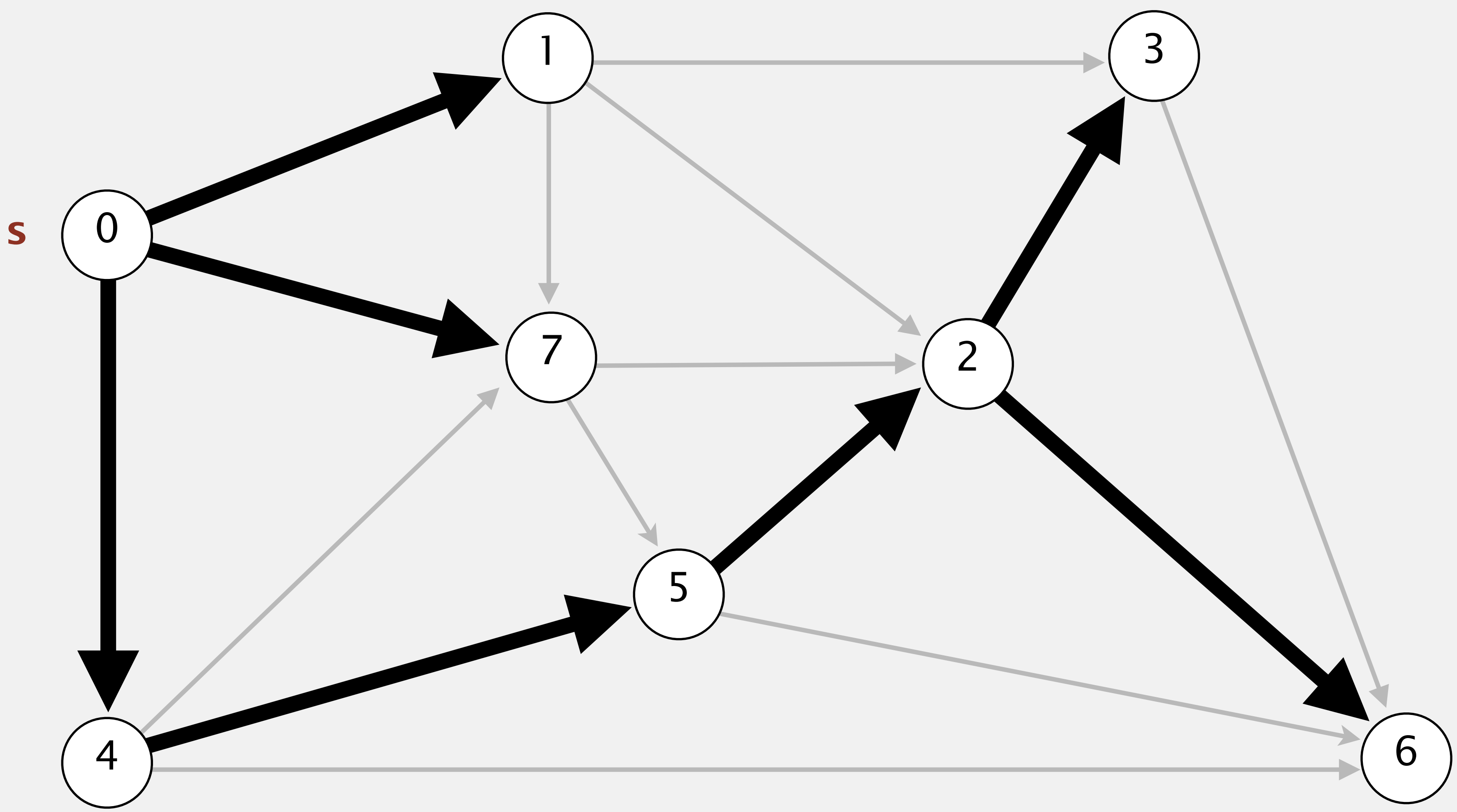


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Bellman–Ford algorithm demo

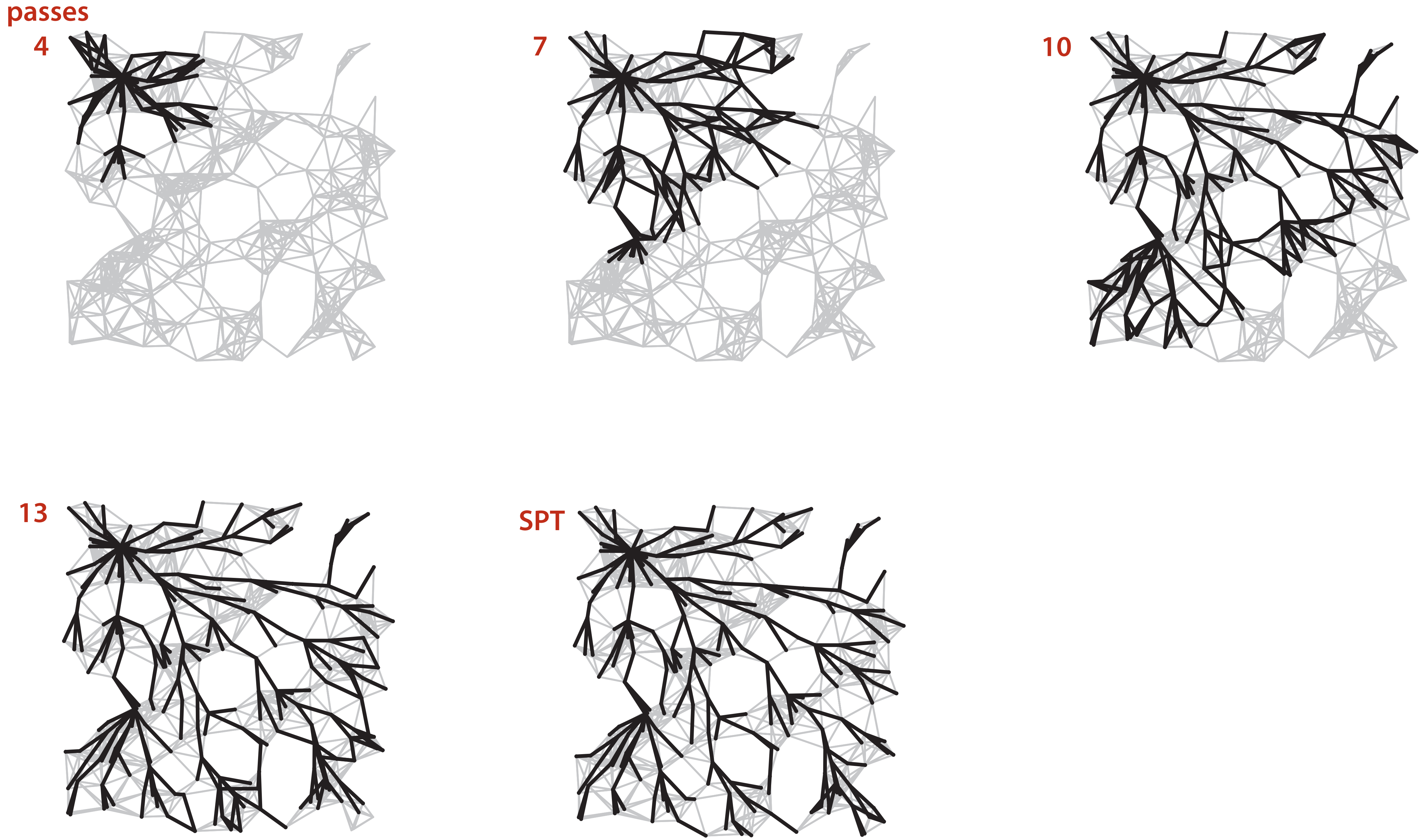
Repeat V times: relax all E edges.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Bellman–Ford algorithm: visualization



Bellman–Ford algorithm: analysis

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:

- **Relax each edge.**

Proposition. Bellman–Ford computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass i, found shortest path to each vertex v for which the shortest path from s to v contains i edges (or fewer).

Bellman–Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass `i`, no need to relax any edge adjacent from `v` in pass `i+1`.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.

- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

Single source shortest-paths implementation: cost summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	$E + V$	$E + V$	V
Dijkstra (binary heap)	no negative weights	$E \log V$	$E \log V$	V
Bellman-Ford	no negative cycles	$E V$	$E V$	V
Bellman-Ford (queue-based)		$E + V$	$E V$	V

Remark 1. Directed cycles make the problem harder.

Remark 2. Negative weights make the problem harder.

Remark 3. Negative cycles makes the problem intractable.

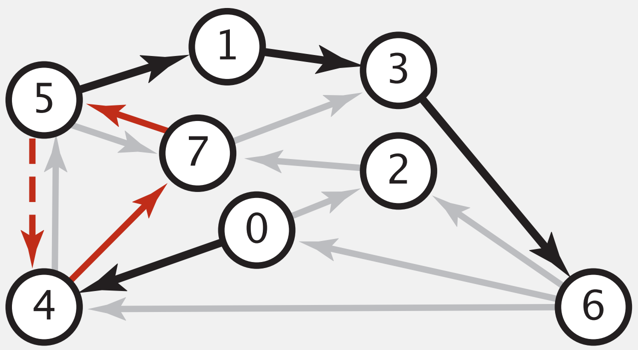
Finding a negative cycle

Negative cycle. Add two method to the API for SP.

`boolean`	`hasNegativeCycle()`	*is there a negative cycle?*
`Iterable <DirectedEdge>`	`negativeCycle()`	*negative cycle reachable from s*

digraph

```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

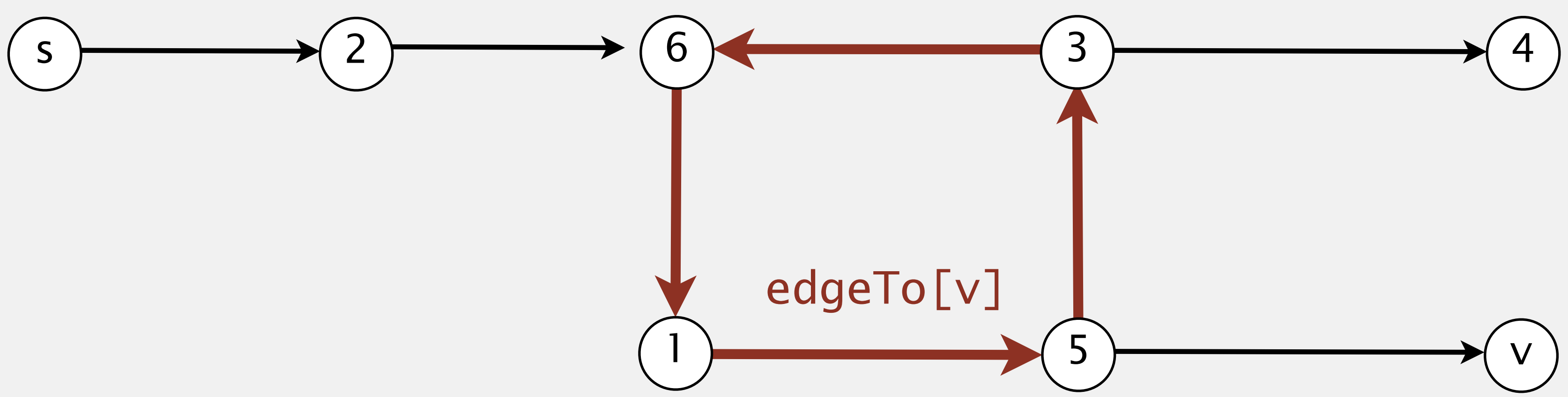


negative cycle (-0.66 + 0.37 + 0.28)

5->4->7->5

Finding a negative cycle

Observation. If there is a negative cycle, Bellman–Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If Bellman–Ford updates any vertex `v` in pass V, there exists a negative cycle (and can trace `edgeTo[v]` entries back to find one).

In practice. Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

Ex. \$1,000 $\Rightarrow$ 741 Euros $\Rightarrow$ 1,012.206 Canadian dollars $\Rightarrow$ \$1,007.14497.

$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$

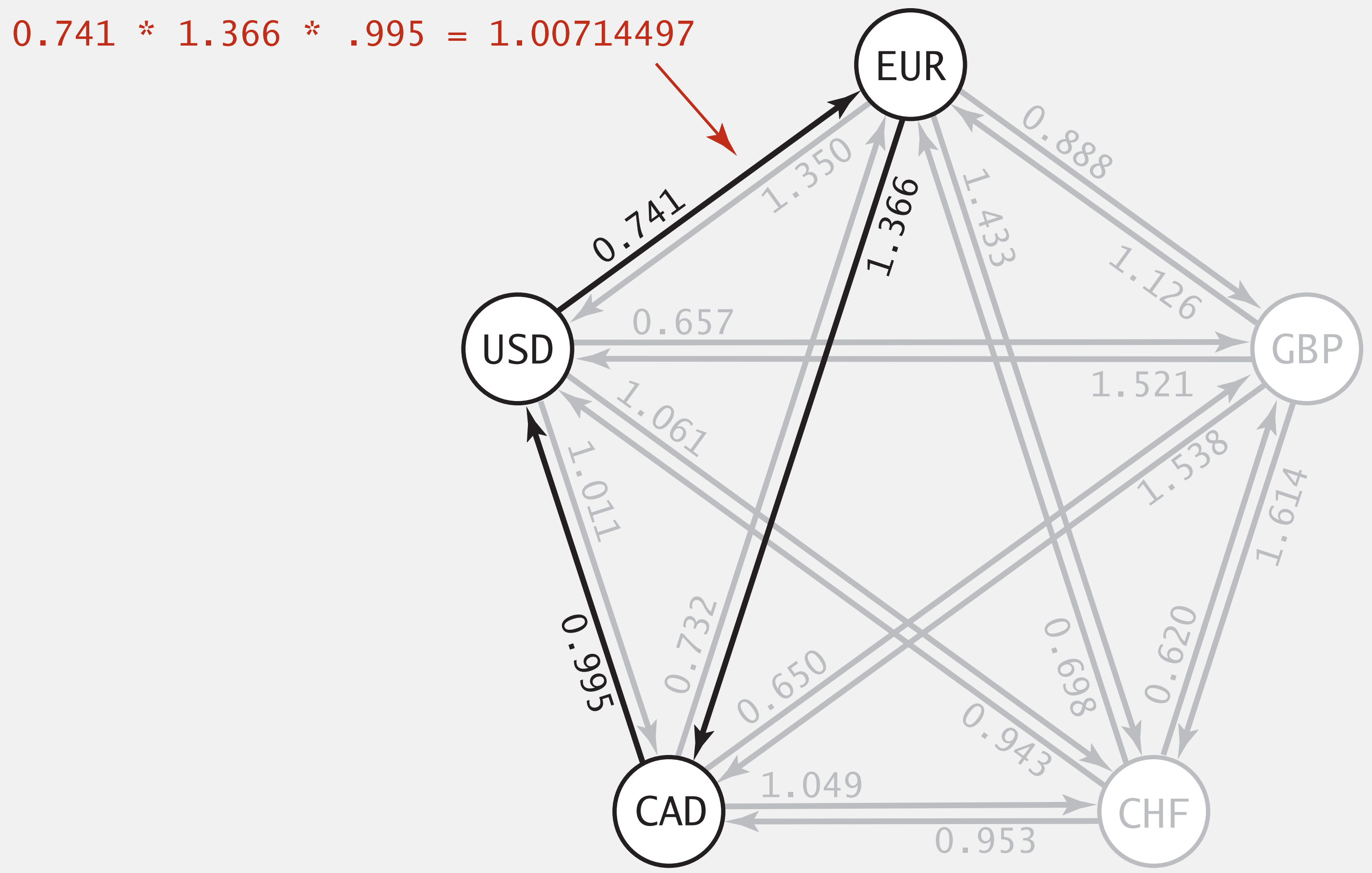
Arbitrage


THERE'S NO SUCH THING AS A FREE LUNCH
MILTON FRIEDMAN
MILK
ESSAYS ON PUBLIC POLICY

Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

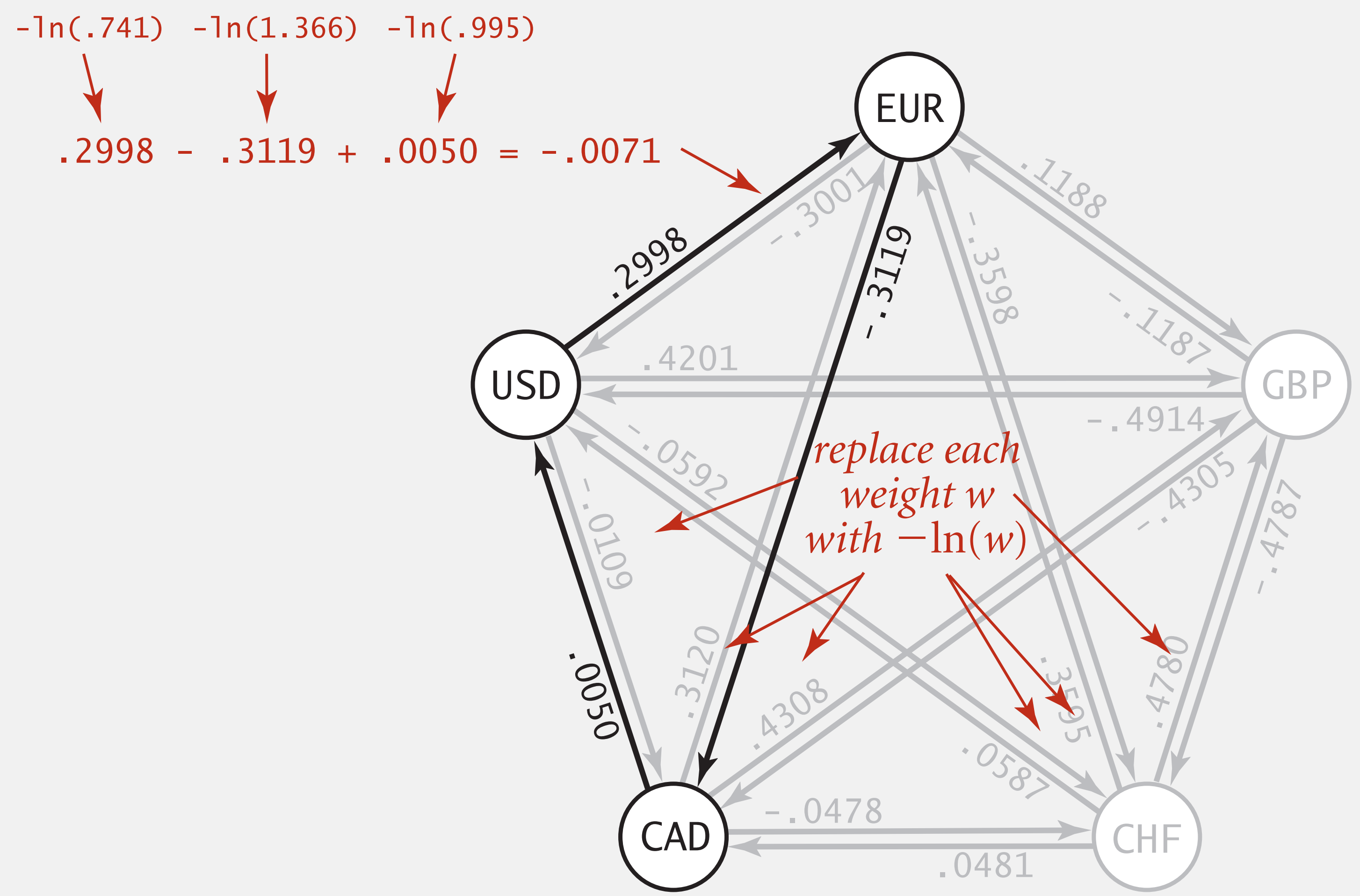


Challenge. Express as a negative cycle detection problem.

Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logarithms.

- Set weight of edge $v \rightarrow w$ to $-ln$ (exchange rate from currency v to w).
- Multiplication turns to addition; > 1 turns to < 0.
- Find a directed cycle whose sum of edge weights is < 0 (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

Shortest paths summary

Nonnegative weights.

- Arises in many application.
- Dijkstra's algorithm is nearly linear-time.

Acyclic edge-weighted digraphs.

- Arise in some applications.
- Topological sort algorithm is linear time.
- Edge weights can be negative.

Negative weights and negative cycles.

- Arise in some applications.
- Bellman–Ford is quadratic in worst case.
- If no negative cycles, can find shortest paths via Bellman–Ford.
- If negative cycles, can find one via Bellman–Ford.

Shortest-paths is a broadly useful problem-solving model.