PRINCETON UNIVERSITY SPR'12	COS 521: ADVANCED ALGORITHMS

Homework 4

Out: *Apr 14* Due: *May 1*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince the grader. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Use semidefinite programming to give a better approximation to MAX-2SAT than the 3/4-approximation we gave via linear programming. Ideally your approximation ratio should exceed 0.8

§2 We are given an undirected graph $G = (V, E)$ with capacities c_e on edges, and m pairs of (source, sink) pairs $(s_1, t_1), \ldots (s_m, t_m)$ with $s_i, t_i \in V$. An *integral flow* between a source sink pair (s, t) is a single path connecting s and t and a quantity q indicating the amount of flow shipped on that path (the difference between this and the usual notion of flow is that all the flow is on one path). Let M be the largest number for which it is possible to M units of integral flow between all the (s_i, t_i) pairs simultaneously without violating the edge capacities. Give a $O(\log n)$ approximation for M using linear programming. (hint: randomized rounding).

§3 Every positive semidefinite matrix can be written as a sum of outer products $A = XX^T = \sum_i x_i x_i^T$ where x_i are the columns of X. The Laplacian matrix of a $d-$regular graph is defined to be $L = dI - A$ where A is the adjacency matrix (recall that in the Cheeger lecture, we dealt with the *normalized Laplacian* $\frac{1}{d}L = I - \frac{1}{d}A$). Write L as a sum of outer products. What is the appropriate generalization to irregular graphs? Weighted graphs? What is the nullspace of a Laplacian matrix?

§4 Use the Chebyshev iteration to show that the diameter of an undirected unweighted graph is at most $O(\sqrt{\kappa(L)} \log n)$ where $\kappa(L) = \frac{\lambda_{max}(L)}{\lambda_{min}(L)}$ is the condition number of the Laplacian (ignoring the zero eigenvalue). (hint: to bound the distance between vertices i, j, consider the number of iterations that the method needs to solve $Lx = e_i - e_j$ where e_i, e_j are standard basis vectors).

§5 The *spectral norm* of a symmetric matrix may be defined equivalently as

$$\|A\| = \max_i |\lambda_i(A)| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq q} \frac{x^T A x}{x^T x} = \max_{x,y \neq 0} \frac{x^T A y}{\|x\| \|y\|}.$$

Show that these definitions are equivalent.

§6 Suppose A is the adjacency matrix of a $d-$regular graph $G = (V, E)$ with second eigenvalue λ_2. Prove that for any two subsets $S, T \subset V$ of the vertices,

$$\left| E(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda_2(A) \cdot \sqrt{|S||T|},$$

where $E(S,T)$ denotes the number of edges with one endpoint in S and one in T. (hint: apply the last characterization of the spectral norm in the last lecture to indicator vectors of S and T).

This statement is known as the *expander mixing lemma* and tells us that in a graph with small λ_2 (i.e., large spectral gap $d - \lambda_2$), the number of edges between any two sets is close to what we would expect in a random graph with density d/n.

§7 The Chebyshev method takes $O(\sqrt{\kappa(A)}\log(1/\epsilon)$ iterations to solve $Ax = b$, where each iteration requires a multiplication by A. The preconditioned Chebyshev method solves instead the system $P^{-1}Ax = P^{-1}b$ for some invertible P called a *preconditioner*, and takes $O(\sqrt{\kappa(P^{-1}A)}\log(1/\epsilon))$ iterations with each iteration now requiring multiplication by both P^{-1} and A.

Suppose we are interested in solving $L_G x = b$ for L_G the Laplacian of an undirected graph $G = (V, E)$ with m edges. We will show that this can be done in time $O(m^{3/2}\log^c n)$ by taking the preconditioner $P = L_T$ to be the Laplacian of a suitably chosen tree.

- The *Moore-Penrose pseudoinverse* of a matrix $A = \sum_i \lambda_i u_i u_i^T$ with eigenvalues λ_i and eigenvectors u_i is defined to be

$$A^+ = \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} u_i u_i^T.$$

Show that for a square symmetric matrix A the pseudoinverse satisfies

$$AA^+ = A^+A = \sum_{\lambda_i \neq 0} u_i u_i^T,$$

the projection onto the range of A. In particular, A^+ acts as an inverse if we restrict attention to vectors in range(A), and as long as we are dealing with such vectors we may take A to be invertible.

In particular for the special case of a Laplacian, L^+ is the inverse on all vectors orthogonal to the all 1's vector.

- Let

$$A \preceq B$$

denote

$$x^T A x \leq x^T B x \qquad \forall x \in \mathbf{R}^n.$$

Let $e = (u, v)$ be an edge connecting two vertices $u, v \in V$, and let P be a path connecting the same two vertices. Prove that

$$L_e \preceq \text{length(P)} L_P,$$

where L_e and L_P are the Laplacians of the edge and path respectively. (hint: use induction and Cauchy-Schwarz.)

- Suppose $T \subset G$ is a spanning tree of G. Show that

$$L_T \preceq L_G.$$

  Now for any pair u, v define the stretch $\mathrm{str}_T(u, v)$ to be the length of the unique path between u and v in T. Use the previous inequality to show that

$$L_G \preceq \left(\sum_{uv \in E} \mathrm{str}_T(u, v) \right) \cdot L_T.$$

  (hint: use the outer product expansion to write L_G as a sum of Laplacians of edges.)

- Show that for any invertible A, B, the condition number satisfies:

$$\kappa(AB^{-1}) = \kappa(BA^{-1}) = \left(\max_{x \neq 0} \frac{x^T A x}{x^T B x} \right) \cdot \left(\max_{y \neq 0} \frac{y^T B y}{y^T A y} \right).$$

  This quantity is sometimes called the *relative condition number* and measures the maximum multiplicative distortion between the quadratic forms of A and B. Conclude that $A \preceq B \preceq \kappa \cdot B$ implies that $\kappa(AB^{-1}) \leq \kappa$ for any symmetric invertible matrices A, B.

  More generally, show that as long as A and B (not necessarily invertible) have the same nullspace, the same conclusion holds for $\kappa(AB^{+})$, where we only consider vectors orthogonal to the nullspace.

- Conclude that for a spanning tree $T \subset G$,

$$\kappa(L_G L_T^{+}) \leq \left(\sum_{uv \in E} \mathrm{str}_T(u, v) \right),$$

  where the condition number is the ratio of the largest to the smallest eigenvalue, ignoring the 0 eigenvalue since all vectors are orthogonal to the 1's vector.

- Show that for any tree T and any vector y orthogonal to the all $1s$ vector, the inversion $L_T^{+} y$ can be computed in $O(n)$ time.

- A result of Abraham, Bartal, and Neiman shows that every graph G contains a tree T of total stretch at most $O(m \log n \log \log n)$, and moreover such a tree can be found in $O(m \log^2 n)$ time. Assuming this result, combine everything you have shown so far to derive an $O(m^{3/2} \log^c n)$ time algorithm for solving $L_G x = b$.