Homework 2

Out: *Mar 1* Due: *Mar 9*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince the grader. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Write a linear program for finding the largest sphere contained inside a given polyhedron $\{x : Ax \le b\}$ and explain why it is correct. What does it mean if this program is infeasible or unbounded?

§2 Suppose we are given points $(x_1, y_1), \ldots (x_n, y_n)$ in the plane and want to fit a line $y = ax+b$ to them. There are various notions of what a good line is; the most common one (linear regression) seeks to minimize the ℓ_2^2 error:

$$\varepsilon_2(a, b) = \sum_i (ax_i - b - y_i)^2$$

and can be solved using linear algebra. Write linear programs which compute lines minimizing the ℓ_1 and ℓ_∞ error, namely

$$\varepsilon_1(a, b) = \sum_i |ax_i - b - y_i|,$$

$$\varepsilon_\infty(a, b) = \max_i |ax_i - b - y_i|.$$

What is the interpretation of dual feasible solutions for these programs?

§3 Helly's theorem states that for any finite set of closed convex sets $K_1, \ldots, K_m$ in $\mathbf{R}^n$, if every $n + 1$ of them have a nonempty intersection then they all have a nonempty intersection. Use duality to prove Helly's theorem for the special case of halfspaces $K_j = \{x : \langle x, a_j\rangle \le b_j\}$.

§4 (Reducing to full-dimensionality in the Ellipsoid method.) Consider the feasibility linear program

$$P = \exists ?x : \quad Ax \le b,$$

and the 'thickened' program

$$P_\epsilon = \exists ?x : \quad Ax \le b + \epsilon\mathbf{1}.$$

Clearly if P is feasible then P_ϵ is also feasible. Show that the converse is true for sufficiently small ϵ, and derive a bound on ϵ in terms of the bit complexity of the inputs (assume a_{ij}, b_j are integers of size at most 2^L.)

§5 (von Neumann's Minmax Theorem) A *two-player zero sum game* consists of two players min and max, who play strategies from finite sets $[n]$ and $[m]$ respectively, and a payoff matrix $A = (a_{ij})_{i\in[n],j\in[m]}$ which defines the outcome of the game for every pair (i,j). If max reveals her strategy before min and each player tries to optimize her outcome, then that outcome will be exactly:

$$\max_{j\in[m]} \min_{i\in[n]} a_{ij}.$$

On the other hand, if min goes first, then the outcome will be

$$\min_{i\in[n]} \max_{j\in[m]} a_{ij},$$

which may be different.

Now consider the setting where min and max are allowed to play *mixed* strategies, which are probability distributions over $[n]$ and $[m]$ respectively. A mixed strategy $x \in \Delta_n = \{x \in \mathbf{R}^n_+ : \sum_i x_i = 1\}$ may be interpreted as a randomized rule in which min chooses $i \in [n]$ with probability x_i. The expected outcome for a pair of mixed strategies (x, y) may be written as

$$\mathbf{E}_{i\sim x_i} \mathbf{E}_{j\sim y_j} a_{ij} = y^T Ax.$$

An *equilibrium* is a pair of mixed strategies for which the order in which they are announced does not matter as long as the random choice is made simultaneously for both players, i.e.:

$$\min_{x\in\Delta_n} \max_{y\in\Delta_m} y^T Ax = \max_{y\in\Delta_m} \min_{x\in\Delta_n} y^T Ax.$$

Show that an equilibrium always exists and give a polynomial time algorithm for computing it in the case where A has integer entries.

§6 Given a directed graph $G = (V, E)$ with edge capacities $c_e \geq 0$, and two distinguished vertices s and t, the minimum cut problem seeks to find a subset of E of minimum total weight which disconnects s and t. Show that the following linear programming relaxation of this problem always has an integral optimum with $x_e \in \{0, 1\}$.

$$\min \sum_e x_e c_e$$

$$\begin{array}{ll} \forall e \in E & x_e \geq 0 \\ \forall \text{s-t paths } p & \sum_{e\in p} x_e \geq 1. \end{array}$$

(hint: You need to show that for every cost function c_e and feasible solution x, there is an integral feasible solution x' with $\sum_e c_e x'_e \leq \sum_e c_e x_e$. Use the edge lengths x_e to find a useful embedding of the vertices of the graph.)