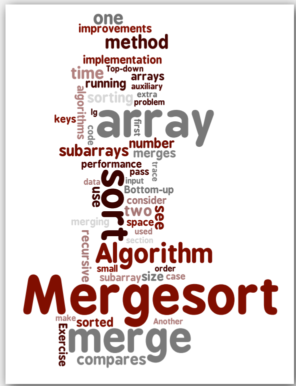
2.2 Mergesort



- mergesort
- bottom-up mergesort
- sorting complexity
- comparators
- stability

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. ← today

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort. ← next lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

- mergesort
- bottom-up mergesort
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- comparators
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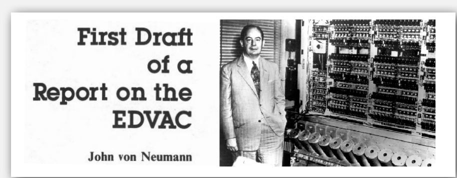
Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview



Merging

Q. How to combine two sorted subarrays into a sorted whole.

A. Use an auxiliary array.

	k	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]	i	j	aux[0]	aux[1]	aux[2]	aux[3]	aux[4]	aux[5]	aux[6]	aux[7]	aux[8]	aux[9]
input		E	E	G	M	R	A	C	E	R	T			-	-	-	-	-	-	-	-	-	-
copy		E	E	G	M	R	A	C	E	R	T			E	E	G	M	R	A	C	E	R	T
												0	5										
	0	A										0	6	E	E	G	M	R	A	C	E	R	T
	1	A	C									0	7	E	E	G	M	R		C	E	R	T
	2	A	C	E								1	7	E	E	G	M	R			E	R	T
	3	A	C	E	E							2	7		E	G	M	R			E	R	T
	4	A	C	E	E	E						2	8			G	M	R			E	R	T
	5	A	C	E	E	E	G					3	8			G	M	R				R	T
	6	A	C	E	E	E	G	M				4	8				M	R				R	T
	7	A	C	E	E	E	G	M	R			5	8					R				R	T
	8	A	C	E	E	E	G	M	R	R		5	9									R	T
	9	A	C	E	E	E	G	M	R	R	T	6	10										T
merged result		A	C	E	E	E	G	M	R	R	T												

Abstract in-place merge trace

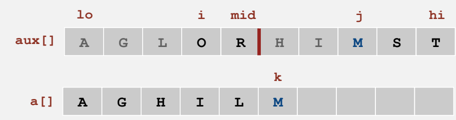
Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);     // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);   // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)                                          copy
      aux[k] = a[k];

   int i = lo, j = mid+1;                                                 merge
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);      // postcondition: a[lo..hi] sorted
}
```



Assertions

Assertion. Statement to test assumptions about your program.

- Helps detect logic bugs.
- Documents code.

Java assert statement. Throws an exception unless boolean condition is true.

```
assert isSorted(a, lo, hi);
```

Can enable or disable at runtime. ⇒ No cost in production code.

```
java -ea MyProgram   // enable assertions
java -da MyProgram   // disable assertions (default)
```

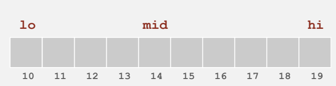
Best practices. Use to check internal invariants. Assume assertions will be disabled in production code (e.g., don't use for external argument-checking).

Mergesort: Java implementation

```
public class Merge
{
   private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort (a, aux, lo, mid);
      sort (a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort trace

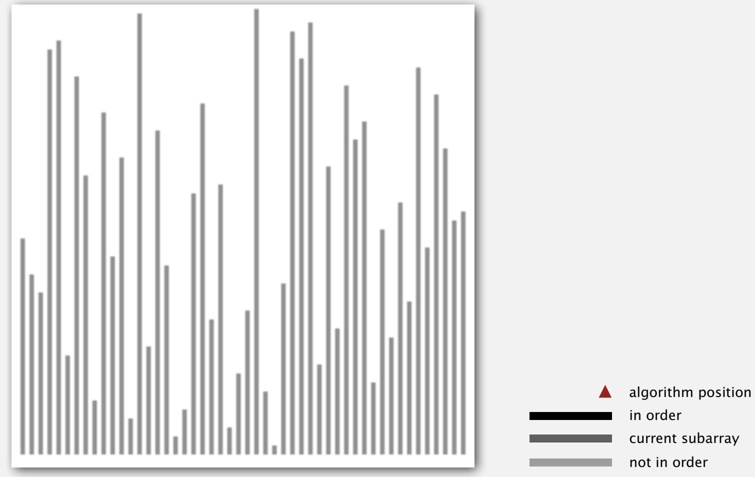
```
                                          a[]
          lo      hi       0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
                           M  E  R  G  E  S  O  R  T  E  X  A  M  P  L  E
     merge(a,  0,  0,  1)  E  M  R  G  E  S  O  R  T  E  X  A  M  P  L  E
     merge(a,  2,  2,  3)  E  M  G  R  E  S  O  R  T  E  X  A  M  P  L  E
   merge(a,  0,  1,  3)    E  G  M  R  E  S  O  R  T  E  X  A  M  P  L  E
     merge(a,  4,  4,  5)  E  G  M  R  E  S  O  R  T  E  X  A  M  P  L  E
     merge(a,  6,  6,  7)  E  G  M  R  E  S  O  R  T  E  X  A  M  P  L  E
   merge(a,  4,  5,  7)    E  G  M  R  E  O  R  S  T  E  X  A  M  P  L  E
 merge(a,  0,  3,  7)      E  E  G  M  O  R  R  S  T  E  X  A  M  P  L  E
     merge(a,  8,  8,  9)  E  E  G  M  O  R  R  S  E  T  X  A  M  P  L  E
     merge(a, 10, 10, 11)  E  E  G  M  O  R  R  S  E  T  A  X  M  P  L  E
   merge(a,  8,  9, 11)    E  E  G  M  O  R  R  S  A  E  T  X  M  P  L  E
     merge(a, 12, 12, 13)  E  E  G  M  O  R  R  S  A  E  T  X  M  P  L  E
     merge(a, 14, 14, 15)  E  E  G  M  O  R  R  S  A  E  T  X  M  P  E  L
   merge(a, 12, 13, 15)    E  E  G  M  O  R  R  S  A  E  T  X  E  L  M  P
 merge(a,  8, 11, 15)      E  E  G  M  O  R  R  S  A  E  E  L  M  P  T  X
merge(a,  0,  7, 15)       A  E  E  E  E  G  L  M  M  O  P  R  R  S  T  X
```

Trace of merge results for top-down mergesort

result after recursive call

Mergesort animation

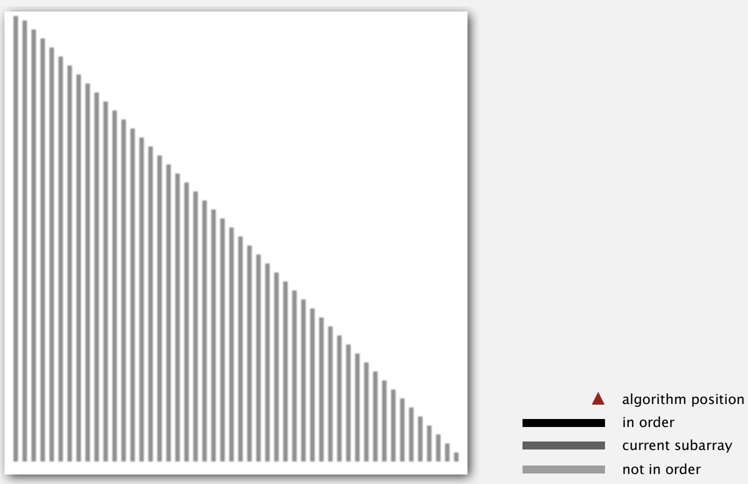
50 random elements



http://www.sorting-algorithms.com/merge-sort

Mergesort animation

50 reverse-sorted elements



http://www.sorting-algorithms.com/merge-sort

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort: number of compares and array accesses

Proposition. Mergesort uses at most $N \lg N$ compares and $6 N \lg N$ array accesses to sort any array of size N.

Pf sketch. The number of compares $C(N)$ and array accesses $A(N)$ to mergesort an array of size N satisfies the recurrences:

$$C(N) \le C(\lfloor N/2 \rfloor) + C(\lceil N/2 \rceil) + N \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

left half — right half — merge

$$A(N) \le A(\lfloor N/2 \rfloor) + A(\lceil N/2 \rceil) + 6N \quad \text{for } N > 1, \text{ with } A(1) = 0.$$

We solve the simpler divide-and-conquer recurrence when N is a power of 2.

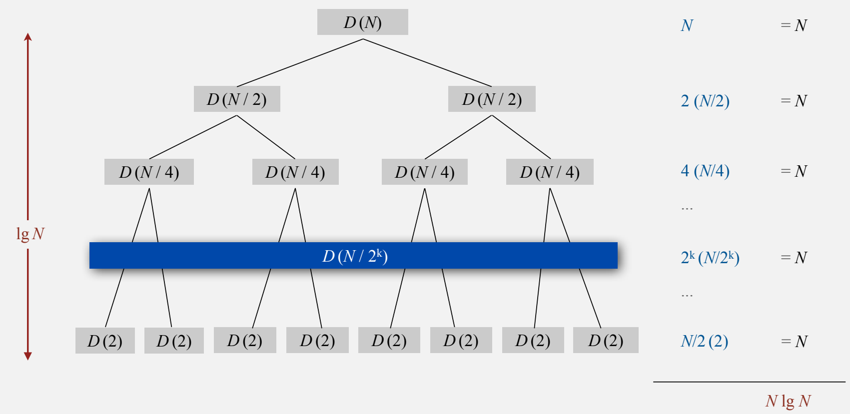
$$D(N) = 2\,D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

result holds for all N (see COS 340)

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



Divide-and-conquer recurrence: proof by expansion

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

$D(N) = 2\,D(N/2) + N$	given
$D(N)/N = 2\,D(N/2)/N + 1$	divide both sides by N
$= D(N/2)/(N/2) + 1$	algebra
$= D(N/4)/(N/4) + 1 + 1$	apply to first term
$= D(N/8)/(N/8) + 1 + 1 + 1$	apply to first term again
. . .	
$= D(N/N)/(N/N) + 1 + 1 + \ldots + 1$	stop applying, D(1) = 0
$= \lg N$	

Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 3. [assuming N is a power of 2]

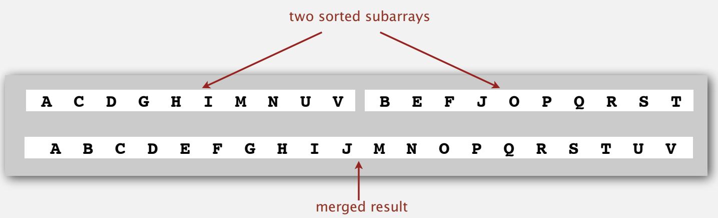
- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$D(2N) = 2\,D(N) + 2N$	given
$= 2\,N \lg N + 2N$	inductive hypothesis
$= 2\,N (\lg (2N) - 1) + 2N$	algebra
$= 2\,N \lg (2N)$	QED

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N.

Pf. The array `aux[]` needs to be of size N for the last merge.



Def. A sorting algorithm is in-place if it uses $O(\log N)$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge for the bored. In-place merge. [Kronrod, 1969]

Mergesort: practical improvements

Use insertion sort for small subarrays.

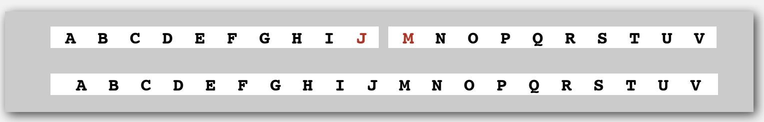
- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 elements.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1) Insertion.sort(a, lo, hi);
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```


Mergesort: practical improvements

Stop if already sorted.

- Is biggest element in first half ≤ smallest element in second half?
- Helps for partially-ordered arrays.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   if (!less(a[mid+1], a[mid])) return;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```


Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)             aux[k] = a[j++];
      else if (j > hi)              aux[k] = a[i++];
      else if (less(a[j], a[i]))    aux[k] = a[j++];    ← merge from a[] to aux[]
      else                          aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(aux, a, lo, mid, hi);
}
```

switch roles of aux[] and a[]

Mergesort: practical improvements

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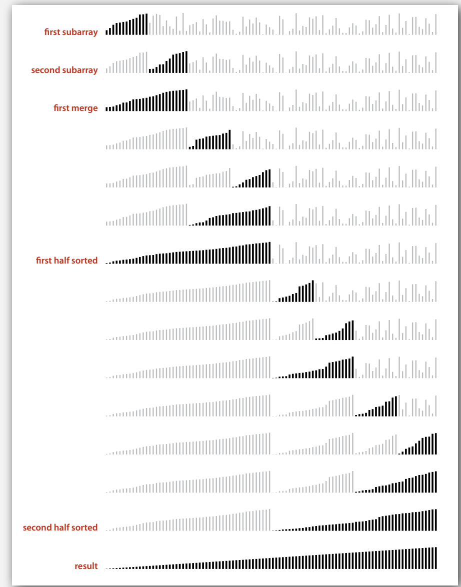
Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

Ex. See `MergeX.java` or `Arrays.sort()`.

Mergesort visualization



Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ...

	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom line. No recursion needed!

Bottom-up mergesort: Java implementation

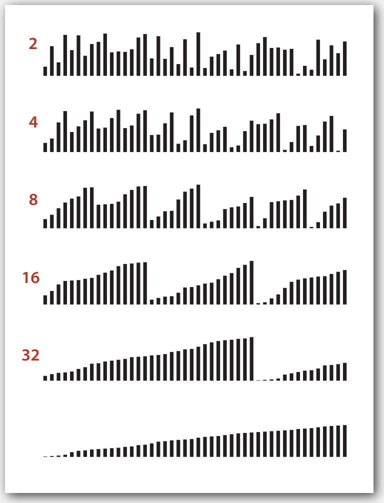
```
public class MergeBU
{
   private static Comparable[] aux;

   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

Bottom line. Concise industrial-strength code, if you have the space.

Bottom-up mergesort: visual trace



Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X.

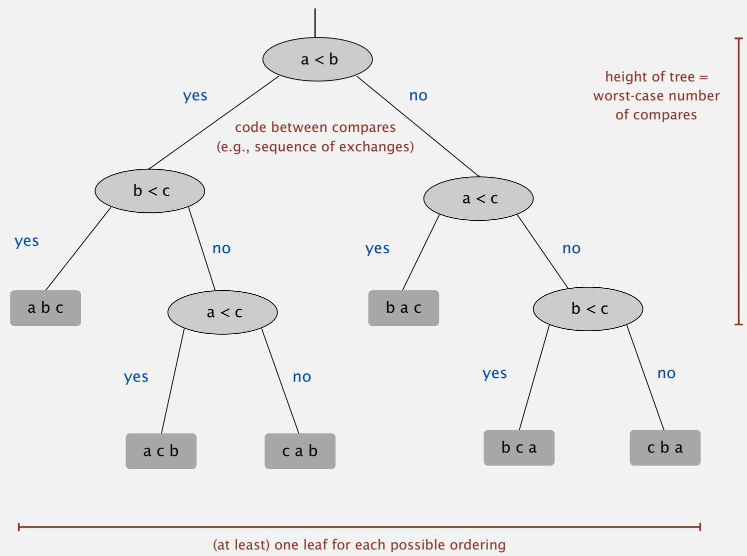
Model of computation. Specify allowable operations.
Cost model. Focus on fundamental operations.
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best cost guarantee for X.

lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree. ← can access information only through compares (e.g., our Java sorting framework)
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$???
- Optimal algorithm: mergesort ???

Decision tree (for 3 distinct elements a, b, and c)

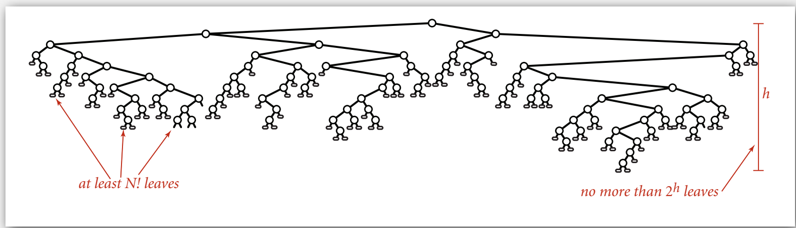


Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.



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- Assume array consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.

$$2^h \geq \#\text{ leaves} \geq N!$$
$$\Rightarrow h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

Complexity of sorting

Model of computation. Specify allowable operations.
Cost model. Focus on fundamental operations.
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best cost guarantee for X.

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Complexity results in context

Other operations? Mergesort is optimal with respect to number of compares (e.g., but not to number of array accesses).

Space?

- Mergesort is not optimal with respect to space usage.
- Insertion sort, selection sort, and shellsort are space-optimal.

Challenge. Find an algorithm that is both time- and space-optimal. [stay tuned]

Lessons. Use theory as a guide.

Ex. Don't try to design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares.

Complexity results in context (continued)

Lower bound may not hold if the algorithm has information about:

- The initial order of the input.
- The distribution of key values.
- The representation of the keys.

Partially-ordered arrays. Depending on the initial order of the input, we may not need $N \lg N$ compares.

insertion sort requires only N-1 compares if input array is sorted

Duplicate keys. Depending on the input distribution of duplicates, we may not need $N \lg N$ compares.

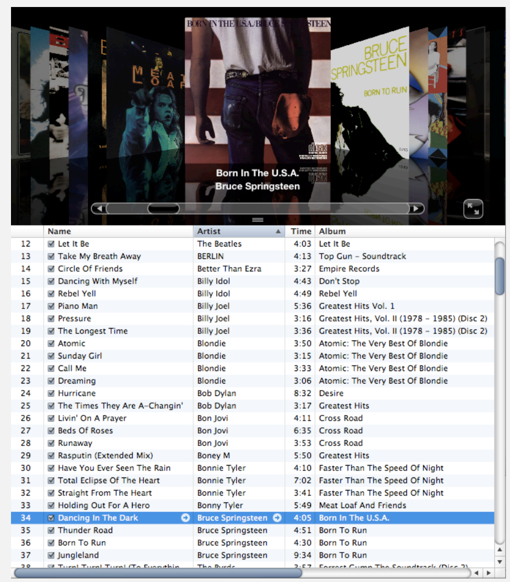
stay tuned for 3-way quicksort

Digital properties of keys. We can use digit/character compares instead of key compares for numbers and strings.

stay tuned for radix sorts

- mergesort
- bottom-up mergesort
- sorting complexity
- **comparators**
- stability

Sort by artist name



Sort by song name

Natural order

Comparable interface: sort uses type's natural order.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }
   …
   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

Generalized compare

Comparable interface: sort uses type's natural order.

Problem 1. May want to use a non-natural order.

Problem 2. Desired data type may not come with a "natural" order.

Ex. Sort strings by:

- Natural order. `Now is the time`
- Case insensitive. `is Now the time`
- Spanish. `café cafetero cuarto churro nube ñoño`
- British phone book. `McKinley Mackintosh`

pre-1994 order for digraphs ch and ll and rr

```
String[] a;
...
Arrays.sort(a);
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
Arrays.sort(a, Collator.getInstance(Locale.SPANISH));
```

import `java.text.Collator;`

Comparators

Solution. Use Java's `Comparator` interface.

```
public interface Comparator<Key>
{
   public int compare(Key v, Key w);
}
```

Remark. `compare()` must implements a total order like `compareTo()`.

Advantages. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

- Can add any number of new orders to a data type.
- Can add an order to a library data type with no natural order.

Comparator example

Reverse order. Sort an array of strings in reverse order.

comparator implementation

```
public class ReverseOrder implements Comparator<String>
{
   public int compare(String a, String b)
   {
      return b.compareTo(a);
   }
}
```

client

```
   ...
   Arrays.sort(a, new ReverseOrder());
   ...
```


Sort implementation with comparators

To support comparators in our sort implementations:

- Use `Object` instead of `Comparable`.
- Pass `Comparator` to `sort()` and `less()`.
- Use it in `less()`.

Ex. Insertion sort.

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{  return c.compare(v, w) < 0;   }

private static void exch(Object[] a, int i, int j)
{  Object swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```


Generalized compare

Comparators enable multiple sorts of a single array (by different keys).

Ex. Sort students by name or by section.

```
Arrays.sort(students, Student.BY_NAME);
Arrays.sort(students, Student.BY_SECT);
```

sort by name

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	2	A	991-878-4944	308 Blair
Fox	1	A	884-232-5341	11 Dickinson
Furia	3	A	766-093-9873	101 Brown
Gazsi	4	B	665-303-0266	22 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes

sort by section

Fox	1	A	884-232-5341	11 Dickinson
Chen	2	A	991-878-4944	308 Blair
Andrews	3	A	664-480-0023	097 Little
Furia	3	A	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	665-303-0266	22 Brown

Generalized compare

Ex. Enable sorting students by name or by section.

```
public class Student
{
   public static final Comparator<Student> BY_NAME = new ByName();
   public static final Comparator<Student> BY_SECT = new BySect();

   private final String name;
   private final int section;
   ...
   private static class ByName implements Comparator<Student>
   {
      public int compare(Student a, Student b)
      {  return a.name.compareTo(b.name);  }
   }

   private static class BySect implements Comparator<Student>
   {
      public int compare(Student a, Student b)
      {  return a.section - b.section;  }
   }
}
```

use this trick only if no danger of overflow

Generalized compare problem

A typical application. First, sort by name; then sort by section.

`Arrays.sort(students, Student.BY_NAME);`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	2	A	991-878-4944	308 Blair
Fox	1	A	884-232-5341	11 Dickinson
Furia	3	A	766-093-9873	101 Brown
Gazsi	4	B	665-303-0266	22 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	3	A	232-343-5555	343 Forbes

`Arrays.sort(students, Student.BY_SECT);`

Fox	1	A	884-232-5341	11 Dickinson
Chen	2	A	991-878-4944	308 Blair
Kanaga	3	B	898-122-9643	22 Brown
Andrews	3	A	664-480-0023	097 Little
Furia	3	A	766-093-9873	101 Brown
Rohde	3	A	232-343-5555	343 Forbes
Battle	4	C	874-088-1212	121 Whitman
Gazsi	4	B	665-303-0266	22 Brown

@#%&@!!. Students in section 3 no longer in order by name.

A stable sort preserves the relative order of records with equal keys.

Sorting challenge 5

Q. Which sorts are stable?
Insertion sort? Selection sort? Shellsort? Mergesort?

sorted by time		sorted by location (not stable)		sorted by location (stable)	
Chicago	09:00:00	Chicago	09:25:52	Chicago	09:00:00
Phoenix	09:00:03	Chicago	09:03:13	Chicago	09:00:59
Houston	09:00:13	Chicago	09:21:05	Chicago	09:03:13
Chicago	09:00:59	Chicago	09:19:46	Chicago	09:19:32
Houston	09:01:10	Chicago	09:19:32	Chicago	09:19:46
Chicago	09:03:13	Chicago	09:00:00	Chicago	09:21:05
Seattle	09:10:11	Chicago	09:35:21	Chicago	09:25:52
Seattle	09:10:25	Chicago	09:00:59	Chicago	09:35:21
Phoenix	09:14:25	Houston	09:01:10	Houston	09:00:13
Chicago	09:19:32	Houston	09:00:13	Houston	09:01:10
Chicago	09:19:46	Phoenix	09:37:44	Phoenix	09:00:03
Chicago	09:21:05	Phoenix	09:00:03	Phoenix	09:14:25
Seattle	09:22:43	Phoenix	09:14:25	Phoenix	09:37:44
Seattle	09:22:54	Seattle	09:10:25	Seattle	09:10:11
Chicago	09:25:52	Seattle	09:36:14	Seattle	09:10:25
Chicago	09:35:21	Seattle	09:22:43	Seattle	09:22:43
Seattle	09:36:14	Seattle	09:10:11	Seattle	09:22:54
Phoenix	09:37:44	Seattle	09:22:54	Seattle	09:36:14

no longer sorted by time — *still sorted by time*

Stability when sorting on a second key

Sorting challenge 5A

Q. Is insertion sort stable?

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

A. Yes, equal elements never more past each other.

Sorting challenge 5B

Q. Is selection sort stable?

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

A. No, long-distance exchange might move left element to the right of some equal element.

Sorting challenge 5C

Q. Is shellsort stable?

```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1;
      while (h >= 1)
      {
         for (int i = h; i < N; i++)
         {
            for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```

h	0	1	2	3	4
	B_1	B_2	B_3	B_4	A_1
4	A_1	B_2	B_3	B_4	B_1
1	A_1	B_2	B_3	B_4	B_1
	A_1	B_2	B_3	B_4	B_1

A. No. Long-distance exchanges.

Sorting challenge 5D

Q. Is mergesort stable?

```
public class Merge
{
   private static Comparable[] aux;
   private static void merge(Comparable[] a, int lo, int mid, int hi)
   {  /* as before */  }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, lo, mid);
      sort(a, mid+1, hi);
      merge(a, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```


Sorting challenge 5D

Q. Is mergesort stable?

	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Trace of merge results for bottom-up mergesort

A. Yes, if merge is stable.

Sorting challenge 5D (continued)

Q. Is merge stable?

```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

A. Yes, if implemented carefully (take from left subarray if equal).

Sorting challenge 5 (summary)

Q. Which sorts are stable ?

Yes. Insertion sort, mergesort.
No. Selection sort, shellsort.

Note. Need to carefully check code ("less than" vs "less than or equal to").

Postscript: optimizing mergesort (a short history)

Goal. Remove instructions from the inner loop.

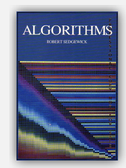
```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi )              a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];

}
```

	lo			i	mid			j		hi
aux[]	A	G	L	O	R	H	I	M	S	T

							k			
a[]	A	G	H	I	L	M				

Postscript: optimizing mergesort (a short history)

Idea 1 (1960s). Use sentinels.




```
a[M] := maxint; b[N] := maxint;
for (int i = 0, j = 0, k = 0; k < M+1; k++)
   if (less(aux[j], aux[i])) aux[k] = a[i++];
                             aux[k] = b[j++];
```

	0			i		M
a[]	A	G	L	O	R	∞

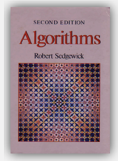
			j			N
b[]	H	I	M	S	T	∞

							k			
aux[]	A	G	H	I	L	M				

Problem 1. Still need copy.
Problem 2. No good place to put sentinels.
Problem 3. Complicates data-type interface (what is infinity for your type?)

Postscript: optimizing mergesort (a short history)

Idea 2 (1980s). Reverse copy.



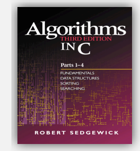

```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{
   for (int i = lo; i <= mid; i++)                         copy
      aux[i] = a[i];

   for (int j = mid+1; j <= hi; j++)                       reverse copy
      aux[j] = a[hi-j+mid+1];

   int i = lo, j = hi;
   for (int k = lo; k <= hi; k++)                          merge
      if (less(aux[j], aux[i])) a[k] = aux[j--];
      else                      a[k] = aux[i++];
}
```

Problem. Copy still in inner loop.

Postscript: optimizing mergesort (a short history)

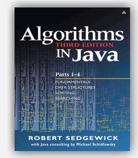
Idea 3 (1990s). Eliminate copy with recursive argument switch.

```
int mid = (lo+hi)/2;
mergesortABr(b, a, lo, mid);
mergesortABr(b, a, mid+1, hi);
mergeAB(a, lo, b, lo, mid, b, mid+1, hi);
```



Problem. Complex interactions with reverse copy.
Solution. Go back to sentinels.

`Arrays.sort()`



Sorting challenge 6

Problem. Choose mergesort for Algs 4th edition.
Recursive argument switch is out (recommended only for pros).

Q. Why not use reverse array copy?

```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{
   for (int i = lo; i <= mid; i++)
      aux[i] = a[i];

   for (int j = mid+1; j <= hi; j++)
      aux[j] = a[hi-j+mid+1];

   int i = lo, j = hi;
   for (int k = lo; k <= hi; k++)
      if (less(aux[j], aux[i])) a[k] = aux[j--];
      else                      a[k] = aux[i++];
}
```


Sorting challenge 6

Problem. Choose mergesort for Algs 4th edition.
Recursive argument switch is out (recommended only for pros)

Q. Why not use reverse array copy?

```
private static void merge(Comparable[] a, int lo, int mid, int hi)
{
   for (int i = lo; i <= mid; i++)
      aux[i] = a[i];

   for (int j = mid+1; j <= hi; j++)
      aux[j] = a[hi-j+mid+1];

   int i = lo, j = hi;
   for (int k = lo; k <= hi; k++)
      if (less(aux[j], aux[i])) a[k] = aux[j--];
      else                      a[k] = aux[i++];
}
```

A. It is not stable (!)
Solution. Back to the standard algorithm!