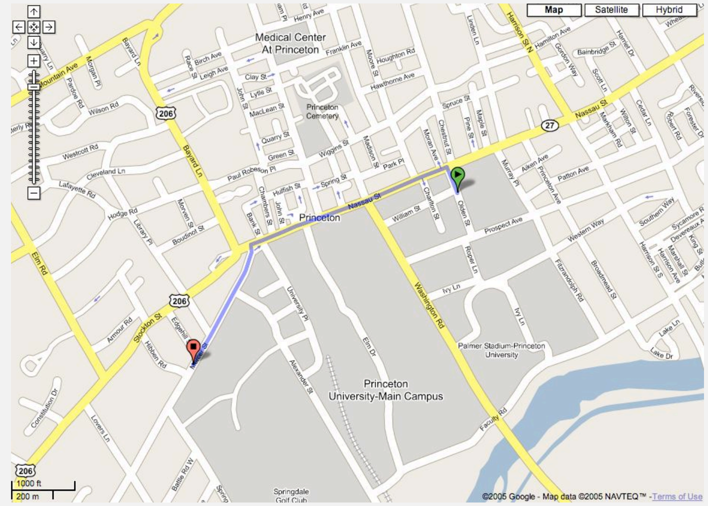
4.4 Shortest Paths

- Dijkstra's algorithm
- implementation
- acyclic networks
- negative weights

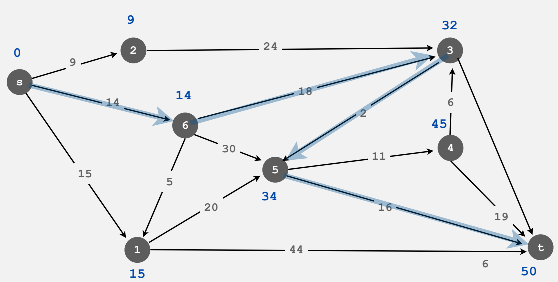
Reference: Algorithms in Java, 4th edition, Section 4.4

Google maps



Shortest paths in a weighted digraph

Given a weighted digraph `G`, find the shortest directed path from `s` to `t`.



shortest path: s→6→3→5→t
cost: 14 + 18 + 2 + 16 = 50

Shortest path versions

Which vertices?

- From one vertex to another.
- From one vertex to every other.
- Between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Arbitrary weights.
- Euclidean weights.

Cycles?

Early history of shortest paths algorithms

Shimbel (1955). Information networks.

Ford (1956). RAND, economics of transportation.

Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, Seitz (1957).
Combat Development Dept. of the Army Electronic Proving Ground.

Dantzig (1958). Simplex method for linear programming.

Bellman (1958). Dynamic programming.

Moore (1959). Routing long-distance telephone calls for Bell Labs.

Dijkstra (1959). Simpler and faster version of Ford's algorithm.

Shortest path applications

- Maps.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Subroutine in advanced algorithms.
- Routing of telecommunications messages.
- Approximating piecewise linear functions.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

- **Dijkstra's algorithm**
- implementation
- acyclic networks
- negative weights

Edsger W. Dijkstra: select quotes

" The question of whether computers can think is like the question of whether submarines can swim. "

" Do only what only you can do. "

" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "

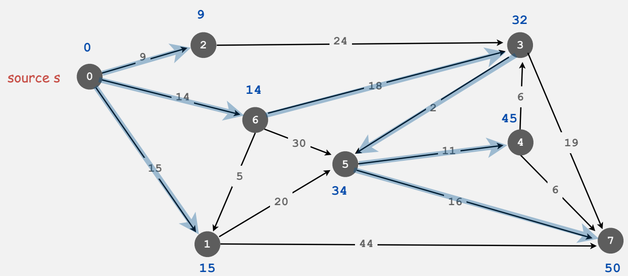
Edger Dijkstra
Turing award 1972

Single-source shortest-paths

Input. Weighted digraph `G`, source vertex `s`.

Goal. Find shortest path from `s` to every other vertex.

Observation. Use parent-link representation to store shortest path tree.

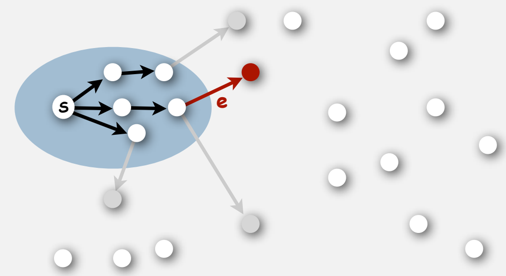


	0	1	2	3	4	5	6	7
distTo[v]	0	15	9	32	45	34	14	50
edgeTo[v]	–	0→1	0→2	6→3	5→4	3→5	0→6	5→7
marked[v]	T	T	T	T	T	T	T	T

Dijkstra's algorithm

Start with vertex s and greedily grow tree T

- find cheapest path ending in an edge e with exactly one endpoint in T
- add e to T
- continue until no edges leave T

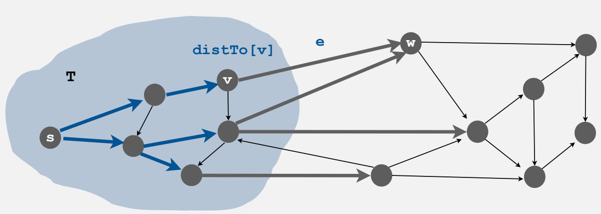


Dijkstra's algorithm

Initialize T to `s`, `distTo[s]` to `0`.

Repeat until T contains all vertices reachable from `s`:

- find edge `e` with `v` in T and `w` not in T that minimizes `distTo[v] + e.weight()`

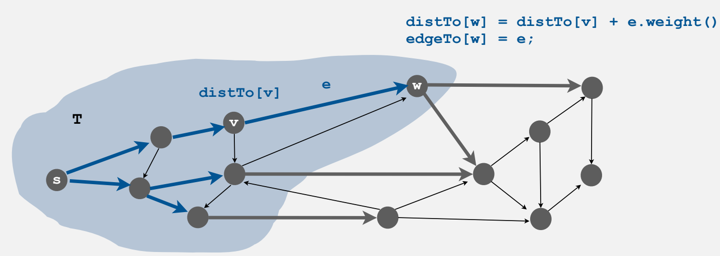


Dijkstra's algorithm

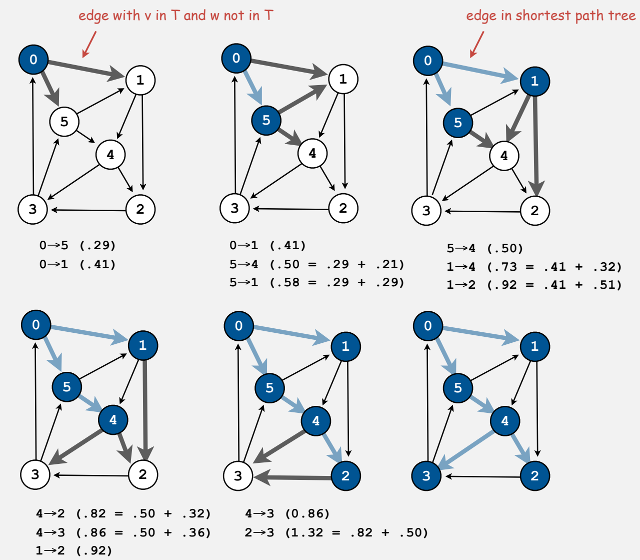
Initialize T to `s`, `distTo[s]` to `0`.

Repeat until T contains all vertices reachable from `s`:

- find edge `e` with `v` in T and `w` not in T that minimizes `distTo[v] + e.weight()`
- set `distTo[w] = distTo[v] + e.weight()` and `edgeTo[w] = e`
- add `w` to T



Dijkstra's algorithm example



0→1	.41
0→5	.29
1→2	.51
1→4	.32
2→3	.50
3→0	.45
3→5	.38
4→2	.32
4→3	.36
5→1	.29
5→4	.21

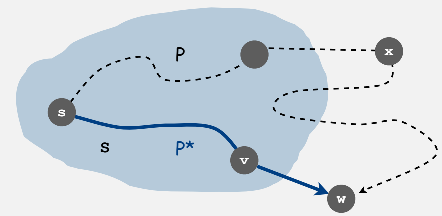
Dijkstra's algorithm: correctness proof

Invariant. For `v` in T, `distTo[v]` is the length of the shortest path from `s` to `v`.

Pf. (by induction on |T|)

- Let `w` be next vertex added to T.
- Let P* be the `s` → `w` path through `v`.
- Consider any other `s` → `w` path P, and let `x` be first node on path outside T.
- P is already as long as P* as soon as it reaches `x` by greedy choice.
- Thus, `distTo[w]` is the length of the shortest path from `s` to `w`.

assuming that edge weights are nonnegative



- Dijkstra's algorithm
- **implementation**
- acyclic networks
- negative weights

Weighted digraph API

Nomenclature reset: "Weighted directed graph" = "Network"

`public class DirectedEdge`		
	`DirectedEdge(int v, int w, double weight)`	*create a weighted edge v→w*
`int`	`from()`	*vertex v*
`int`	`to()`	*vertex w*
`double`	`weight()`	*the weight*

`public class Network`		*weighted digraph data type*
	`Network(int V)`	*create an empty digraph with V vertices*
	`Network(In in)`	*create a digraph from input stream*
`void`	`addEdge(DirectedEdge e)`	*add a weighted edge from v to w*
`Iterable<DirectedEdge>`	`adj(int v)`	*return an iterator over edges leaving v*
`int`	`V()`	*return number of vertices*
`int`	`E()`	*return number of edges*
`Iterable<DirectedEdge>`	`edges()`	*return an iterator over all the network's edges*

Network: adjacency-lists implementation in Java

```
public class Network
{
   private final int V;
   private final Bag<Edge>[] adj;

   public Network(int V)
   {
      this.V = V;
      adj = (Bag<DirectedEdge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }

   public int V()
   {  return V;  }
}
```

similar to edge-weighted undirected graph, but only add edge to v's adjacency set

Weighted directed edge: implementation in Java

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight;  }

}
```

similar to `Edge` for undirected weighted graphs, but simpler

`from()` and `to()` replace `either()` and `other()`

Shortest path data type

Design pattern.

- `DijkstraSPT` class is a `Network` client.
- Client query methods return distance and path iterator.

`public class DijkstraSPT`	
`DijkstraSPT(Network G, int s)`	*shortest path from s in graph G*
`double distTo(int v)`	*length of shortest path from s to v*
`Iterable <DirectedEdge> pathTo(int v)`	*shortest path from s to v*

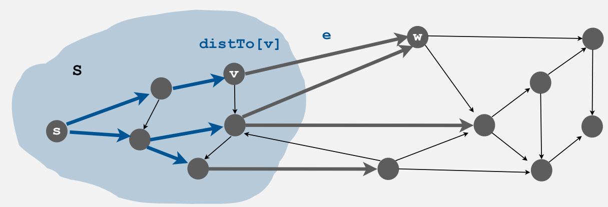
```
In in = new In("network.txt");
Network G = new Network(in);
int s = 0, t = G.V() - 1;
DijkstraSPT spt = new DijkstraSPT(G, s);
StdOut.println("distance = " + spt.distTo(t));
for (DirectedEdge e : spt.pathTo(t))
    StdOut.println(e);
```


Dijkstra implementation challenge

Find edge `e` with `v` in S and `w` not in S that minimizes `distTo[v] + e.weight()`.

How difficult?

- Intractable.
- O(E) time. ← try all edges
- O(V) time.
- O(log E) time. ← Dijkstra with a binary heap
- O(log* E) time.
- Constant time.



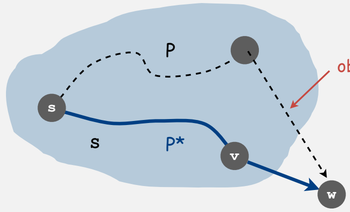
Lazy vs. eager implementation

Issue:

- PQ contains edges from a vertex v in S to a vertex w not in S.
- Adding w to the tree requires adding its incident edges to PQ.
- Some edges on the PQ become obsolete.

Obsolete edge:

- An edge that will never be added to the tree



Lazy approach

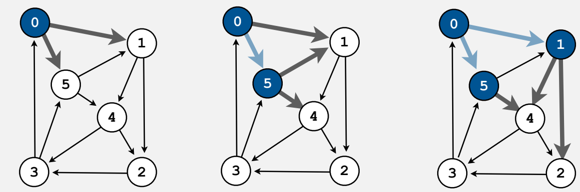
- Leave obsolete edges on PQ
- Check for obsolescence when removing

test for obsolescence: are both vertices on the tree?

Eager approach

- Remove obsolete edges from PQ (need more sophisticated PQ)
- only need one edge per vertex

Lazy Dijkstra's algorithm example



0→5 (.29)
0→1 (.41)

priority queue

0→1 (.41)
5→4 (.50 = .29 + .21)
5→1 (.58 = .29 + .29)

5→4 (.50)
1→4 (.73 = .41 + .32)
1→2 (.92 = .41 + .51)

0→1	.41
0→5	.29
1→2	.51
1→4	.32
2→3	.50
3→0	.45
3→5	.38
4→2	.32
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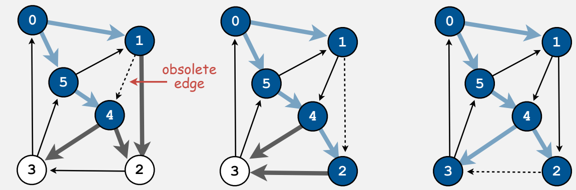


1→4 (.73)
4→2 (.82 = .50 + .32)
4→3 (.86 = .50 + .36)
1→2 (.92)

4→3 (0.86)
1→2 (.92)
2→3 (1.32 = .82 + .50)

1→2 (.92)
2→3 (1.32)

Lazy implementation of Dijkstra's algorithm

```
import java.util.Comparator;

public class LazyDijkstraSPT
{
   private boolean[] marked;
   private double[] distTo;
   private DirectedEdge[] edgeTo;
   private MinPQ<DirectedEdge> pq;

   private class ByDistanceFromSource implements Comparator<DirectedEdge>
   {
      public int compare(DirectedEdge e, DirectedEdge f)
      {
         double x = distTo[e.from()] + e.weight();
         double y = distTo[f.from()] + f.weight();
         if      (x < y) return -1;
         else if (x > y) return +1;
         else            return  0;
      }
   }

   public LazyDijkstra(Network G, int s)
   {
      marked = new boolean[G.V()];
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new MinPQ<DirectedEdge>(new ByDistanceFromSource());
      dijkstra(G, s);
   }
}
```

NOT a static method

compare edges in pq by `distTo[v] + e.weight()`

Lazy implementation of Dijkstra's algorithm

```
private void dijkstra(Network G, int s)
{
   visit(G, s);
   while (!pq.isEmpty())
   {
      DirectedEdge e = pq.delMin();
      int v = e.from(), w = e.to();
      if (marked[w]) continue;
      distTo[w] = e;
      distTo[w] = distTo[v] + e.weight();
      visit(G, w);
   }
}
```

edge is obsolete

found shortest path to w

```
private void visit(Network G, int v)
{
   marked[v] = true;
   for (DirectedEdge e : G.adj(w))
      if (!marked[e.to()]) pq.insert(e);
}
```

add all edges v->w to pq, provided w not already in S

Dijkstra's algorithm running time

Proposition. Dijkstra's algorithm computes shortest paths in O(E log E) time.

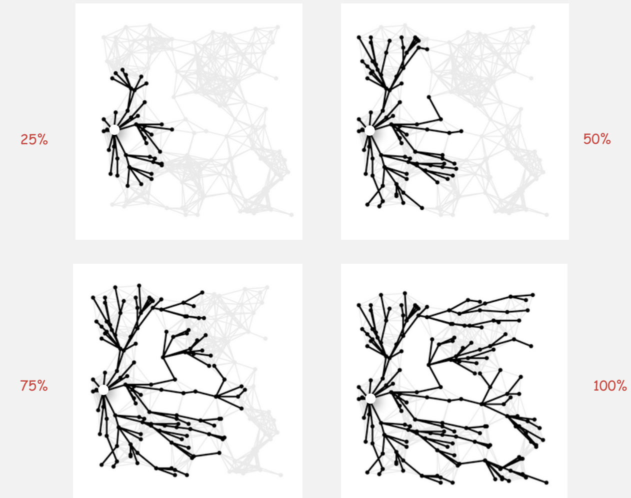
Pf.

operation	frequency	time per op
delete min	E	log E
insert	E	log E

Improvements.
- Eagerly eliminate obsolete edges from PQ.
- Maintain on PQ at most one edge incident to each vertex v not in T
  ⇒ at most V edges on PQ.
- Use fancier priority queue: best in theory yields O(E + V log V).

Shortest path trees

Remark. Dijkstra examines vertices in increasing distance from source.



Priority-first search

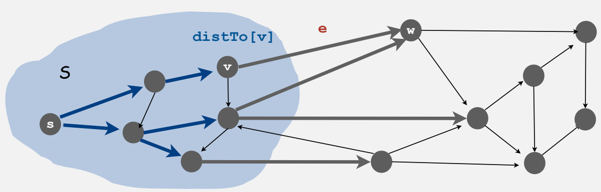
Insight. All of our graph-search methods are the same algorithm!
- Maintain a set of explored vertices S.
- Grow S by exploring edges with exactly one endpoint leaving S.

DFS. Take edge from vertex which was discovered most recently.

BFS. Take edge from vertex which was discovered least recently.

Prim. Take edge of minimum weight.

Dijkstra. Take edge to vertex that is closest to s.



Challenge. Express this insight in reusable Java code.

Priority-first search: application example

Shortest s-t paths in Euclidean graphs (maps)
- Vertices are points in the plane.
- Edge weights are Euclidean distances.

A sublinear algorithm.
- Assume graph is already in memory.
- Start Dijkstra at s.
- Stop when you reach t.

Even better: exploit geometry
- For edge v→w, use weight d(v, w) + d(w, t) – d(v, t).
- Proof of correctness for Dijkstra still applies.
- In practice only O(V $^{1/2}$) vertices examined.
- Special case of A* algorithm

Euclidean distance

[Practical map-processing programs precompute many of the paths.]

- Dijkstra's algorithm
- implementation
- **acyclic networks**
- negative weights

Acyclic networks

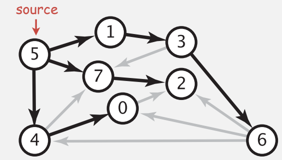
Suppose that a network has no cycles.

Q. Is it easier to find shortest paths than in a general network?

A. Yes!

A. AND negative weights are no problem

5->4	0.35
4->7	0.37
5->7	0.28
5->1	0.32
4->0	0.38
0->2	0.26
3->7	0.39
1->3	0.29
7->2	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



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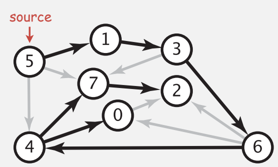
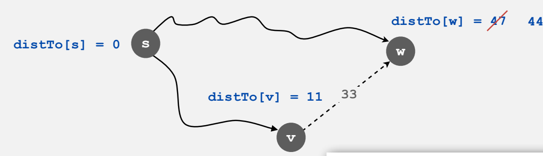
6->4	-0.93



A key operation

Relax edge `e` from `v` to `w`.

- `distTo[v]` is length of some path from `s` to `v`.
- `distTo[w]` is length of some path from `s` to `w`.
- If `v→w` gives a shorter path to `w` through `v`, update `distTo[w]` and `edgeTo[w]`.




```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
   }
}
```

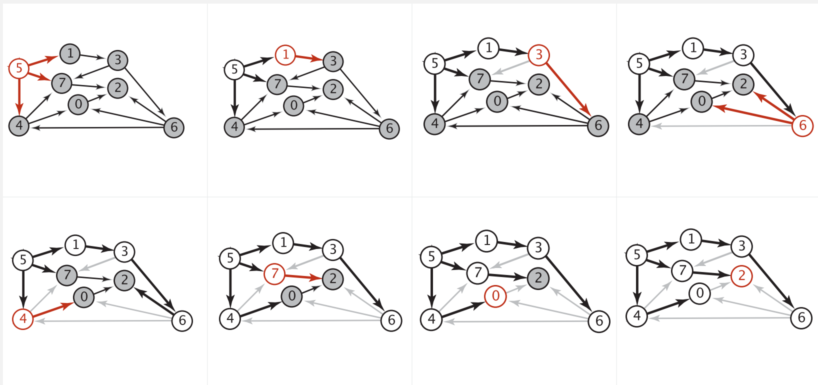
Initialization:

`distTo[s] = 0.0;`

all other `distTo[] = ∞`

Shortest paths in acyclic networks

Algorithm:
- Consider vertices in topologically sorted order
- Relax all edges incident on vertex

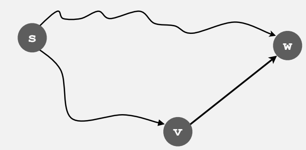


Shortest paths in acyclic networks

Algorithm:
- Consider vertices in topologically sorted order
- Relax all edges incident on vertex

Proposition. Shortest path to each vertex is known before its edges are relaxed

Proof (strong induction)
- let v->w be the last edge on the shortest path from s to w.
- v appears before w in the topological sort
  - shortest path to v is known before its edges are relaxed
  - v's edges are relaxed before w's edges are relaxed, including v->w
- therefore, shortest path to w is known before w's edges are relaxed.



Shortest paths in acyclic networks

```
public class AcyclicNetworkSPT
{
   private double[] distTo;
   private DirectedEdge[] edgeTo;
   public AcyclicNetworkSPT(Network G, int s)
   {
      distTo = new double[G.V()];
      edgeTo = new DirectedEdge[G.V()];
      for (int v = 0; v < G.V(); v++)
         distanceTo[v] = Double.POSITIVE_INFINITY;
      distanceTo[s] = 0.0;

      NetworkSort sort = new NetworkSort(G);
      for (int v : sort.topological())
         for (DirectedEdge e : G.adj(v))
            relax(e);
   }
}
```

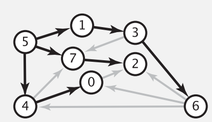
topological sort

Longest paths in acyclic networks

Algorithm:
- Negate all weights
- Find shortest path
- Negate weights in result

Equivalent: reverse sense of equality in `relax()`

5->4	0.35
4->7	0.37
5->7	0.28
5->1	0.32
4->0	0.38
0->2	0.26
3->7	0.39
1->3	0.29
7->2	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



5->4	-0.35
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4->0	-0.38
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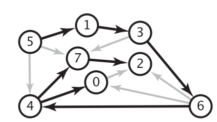
1->3	-0.29
7->2	-0.34
6->2	-0.40
3->6	-0.52
6->0	-0.58
6->4	-0.93



Note: Best known algorithm for general networks is exponential!

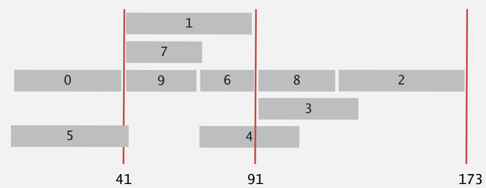
Longest paths in acyclic networks: application

Job scheduling. Given a set of jobs, with durations and precedence constraints, schedule the jobs (find a start time for each) so as to achieve the minimum completion time while respecting the constraints.

Ex:

		precedence constraints	
job	duration	count	successors
0	41.0	3	1 7 9
1	51.0	1	2
2	50.0	0	
3	36.0	0	
4	38.0	0	
5	45.0	0	
6	21.0	2	3 8
7	32.0	2	3 8
8	32.0	1	2
9	29.0	2	4 6

Solution:

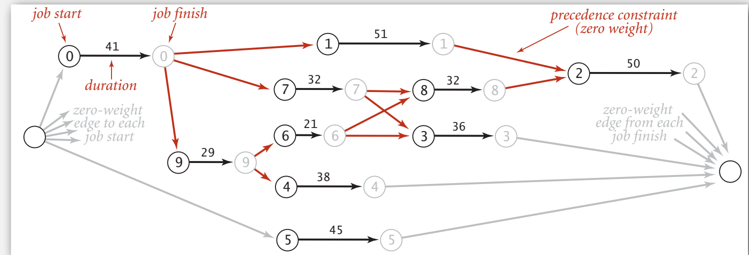


Critical path method

CPM. To solve a job-scheduling problem, create a network

- source, sink
- two vertices (begin and end) for each job
- three edges for each job
  - begin to end (weighted by duration)
  - source to begin
  - end to sink

		precedence constraints	
job	duration	count	successors
0	41.0	3	1 7 9
1	51.0	1	2
2	50.0	0	
3	36.0	0	
4	38.0	0	
5	45.0	0	
6	21.0	2	3 8
7	32.0	2	3 8
8	32.0	1	2
9	29.0	2	4 6



Critical path method: Use longest path from the source to schedule each job

Critical path method

CPM. To solve a job-scheduling problem, create a network

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- two vertices (begin and end) for each job
- three edges for each job
  - begin to end (weighted by duration)
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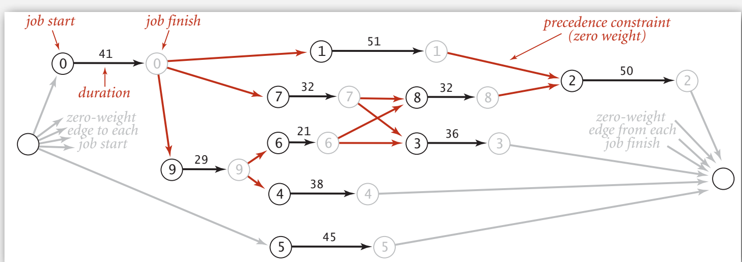
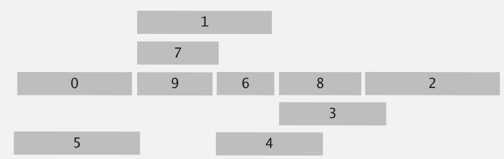
		precedence constraints	
job	duration	count	successors
0	41.0	3	1 7 9
1	51.0	1	2
2	50.0	0	
3	36.0	0	
4	38.0	0	
5	45.0	0	
6	21.0	2	3 8
7	32.0	2	3 8
8	32.0	1	2
9	29.0	2	4 6

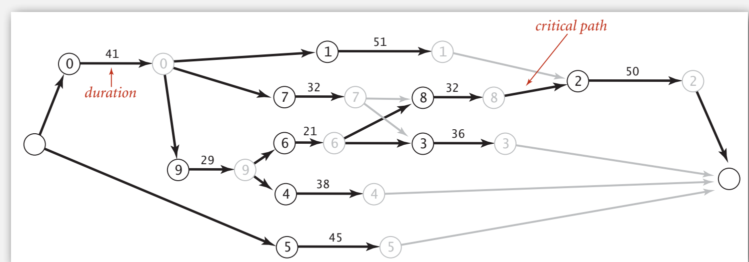


Critical path method: Use longest path from the source to schedule each job

Critical path method

Use longest path from the source to schedule each job.

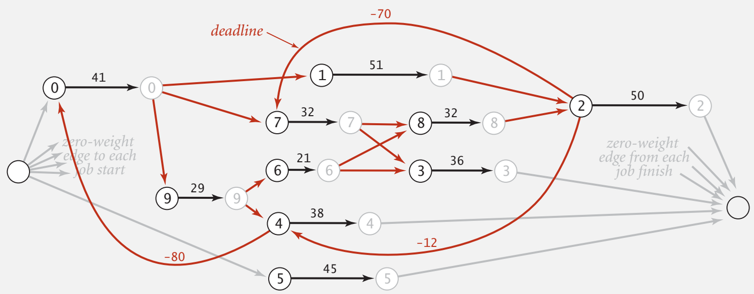




Deep water

Add deadlines to the job-scheduling problem.

Ex. "Job 2 must start no later than 70 time units after job 7."
Or, "Job 7 must start no earlier than 70 times units before job 2"

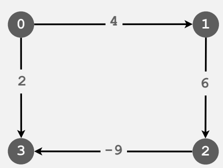


Need to solve longest paths problem in general networks (cycles, neg weights).
Possibility of infeasible problem (negative cycles)

- Dijkstra's algorithm
- implementation
- **negative weights**

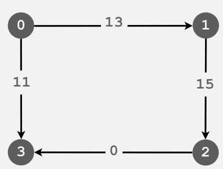
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight also doesn't work.

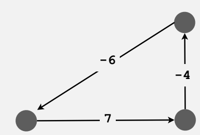


Adding 9 to each edge changes the shortest path
because it adds 9 to each edge;
wrong thing to do for paths with many edges.

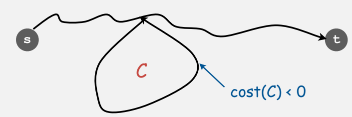
Bad news. Need a different algorithm.

Negative cycles

Def. A negative cycle is a directed cycle whose sum of edge weights is negative.



Observations. If negative cycle C is on a path from s to t, then shortest path can be made arbitrarily negative by spinning around cycle.

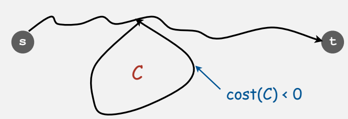


Worse news. Need a different problem.

Shortest paths with negative weights

Problem 1. Does a given digraph contain a negative cycle?

Problem 2. Find the shortest simple path from `s` to `t`.



Bad news. Problem 2 is intractable.

Good news. Can solve problem 1 in O(VE) steps;
if no negative cycles, can solve problem 2 with same algorithm!

Shortest paths with negative weights: dynamic programming algorithm

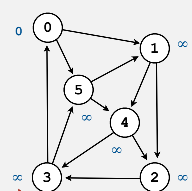
A simple solution that works!

- Initialize `distTo[v] = ∞`, `distTo[s]= 0`.
- Repeat `V` times: relax each edge `e`.

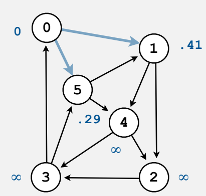
```
for (int i = 1; i <= G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v)) relax(e);
```

phase i

Dynamic programming algorithm trace

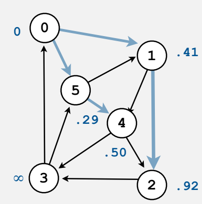


distTo[v]



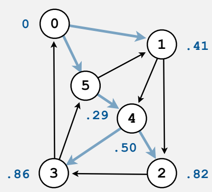
0→1 (.41 = 0 + .41)
0→5 (.50 = 0 + .50)

relaxed edges that update `distTo[]`

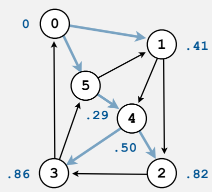


1→2 (.92 = .41 + .51)
1→4 (.73 = .41 + .32)
5→4 (.50 = .29 + .21)

0→1	.41
0→5	.29
1→2	.51
1→4	.32
2→3	.50
3→0	.45
3→5	.38
4→2	.32
4→3	.36
5→1	.29
5→4	.21



2→3 (1.33 = .83 + .50)
4→3 (.86 = .50 + .36)
4→2 (.82 = .50 + .32)



can stop early since
no entries in `distTo[]` updated

Dynamic programming algorithm: analysis

Running time. Proportional to E V.

Invariant. At end of phase `i`, `distTo[v]` ≤ length of any path from `s` to `v` using at most `i` edges.

Proposition. If there are no negative cycles, upon termination `distTo[v]` is the length of the shortest path from from `s` to `v`.

and `edgeTo[]` gives the shortest paths

Bellman-Ford-Moore algorithm

Observation. If `distTo[v]` doesn't change during phase `i`, no need to relax any edge leaving `v` in phase `i+1`.

FIFO implementation. Maintain queue of vertices whose distance changed.

be careful to keep at most one copy of each vertex on queue

Running time.

- Proportional to EV in worst case.
- Much faster than that in practice.

Bellman-Ford-Moore algorithm

```
public class BellmanFordSPT
{
   private double[] distTo;
   private DirectedEdge[] edgeTo;
   private int phase;
   private int[] beenTo;
   private Queue<Integer> q = new Queue<Integer>();
   private Queue<Integer> relaxed;
   public BellmanFordSPT(Network G, int s)
   {
      distTo = new double[V];
      edgeTo = new DirectedEdge[V];
      beenTo = new int[V];

      for (int v = 0; v < V; v++)
         distTo[v] = Double.POSITIVE_INFINITY;

      q.enqueue(s);
      distanceTo[s] = 0.0;
      for (phase = 1; phase <= V; phase++)
      {
         relaxed = new Queue<Integer>();
         for (int v : q)
            for (DirectedEdge e : G.adj(v))
               relax(e);
         q = relaxed;
         if (q.isEmpty()) break;
      }
   }
}
```

Maintain queue of vertices whose distance changes.

Relax all edges incident on all vertices in the queue.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
      distTo[w] = distTo[v] + e.weight();
      edgeTo[w] = e;
      if (beenTo[w] < phase)
         relaxed.enqueue(w);
      beenTo[w] = phase;
   }
}
```


Single source shortest paths implementation: cost summary

	algorithm	worst case	typical case
no cycles	topological sort + relax	E	E
nonnegative costs	Dijkstra (binary heap)	$E \log E$	E
no negative cycles	dynamic programming	$E V$	$E V$
	Bellman-Ford	$E V$	E

Remark 1. Cycles make the problem harder.
Remark 2. Negative weights make the problem harder.
Remark 3. Negative cycles makes the problem intractable.

Currency conversion

Problem. Given currencies and exchange rates, what is best way to convert one ounce of gold to US dollars?

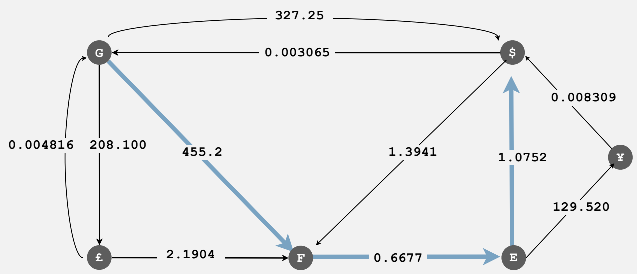
- 1 oz. gold ⇒ \$327.25.
- 1 oz. gold ⇒ £208.10 ⇒ \$327.00. [208.10 × 1.5714]
- 1 oz. gold ⇒ 455.2 Francs ⇒ 304.39 Euros ⇒ \$327.28. [455.2 × .6677 × 1.0752]

currency	£	Euro	¥	Franc	$	Gold
UK pound	1.0000	0.6853	0.005290	0.4569	0.6368	208.100
Euro	1.45999	1.0000	0.007721	0.6677	0.9303	304.028
Japanese Yen	189.50	129.520	1.0000	85.4694	120.400	39346.7
Swiss Franc	2.1904	1.4978	0.01574	1.0000	1.3941	455.200
US dollar	1.5714	1.0752	0.008309	0.7182	1.0000	327.250
Gold (oz.)	0.004816	0.003295	0.0000255	0.002201	0.003065	1.0000

Currency conversion

Graph formulation.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find path that maximizes product of weights.

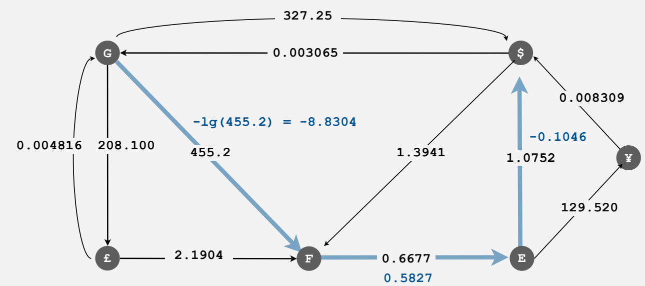


Challenge. Express as a shortest path problem.

Currency conversion

Reduce to shortest path problem by taking logs.

- Let weight of edge `v→w` be - lg (exchange rate from currency `v` to `w`).
- Multiplication turns to addition.
- Shortest path with given weights corresponds to best exchange sequence.

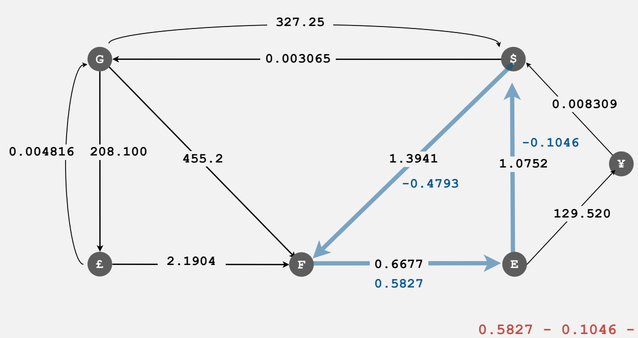


Challenge. Solve shortest path problem with negative weights.

Shortest paths application: arbitrage

Is there an arbitrage opportunity in currency graph?

- Ex: \$1 ⇒ 1.3941 Francs ⇒ 0.9308 Euros ⇒ \$1.00084.
- Is there a negative cost cycle?

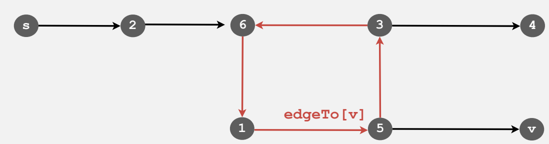


Remark. Fastest algorithm is valuable!

Negative cycle detection

If there is a negative cycle reachable from s.

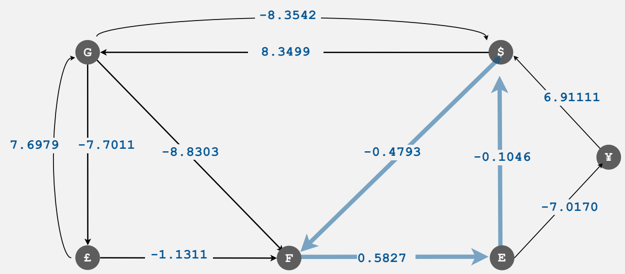
Bellman-Ford-Moore gets stuck in loop, updating vertices in cycle.



Proposition. If any vertex `v` is updated in phase `V`, there exists a negative cycle, and we can trace back `edgeTo[v]` to find it.

Negative cycle detection

Goal. Identify a negative cycle (reachable from any vertex).



Solution. Initialize Bellman-Ford by setting `distTo[v] = 0` for all vertices `v` and putting all vertices on the queue.

Shortest paths summary

Dijkstra's algorithm.

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

Acyclic networks.

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

Negative weights.

- Arise in applications.
- If negative cycles, shortest simple-paths problem is intractable (!)
- If no negative cycles, solvable via classic algorithms.

Shortest-paths is a broadly useful problem-solving model.