Logic: From Greeks to philosophers to circuits.

COS 116, Spring 2010
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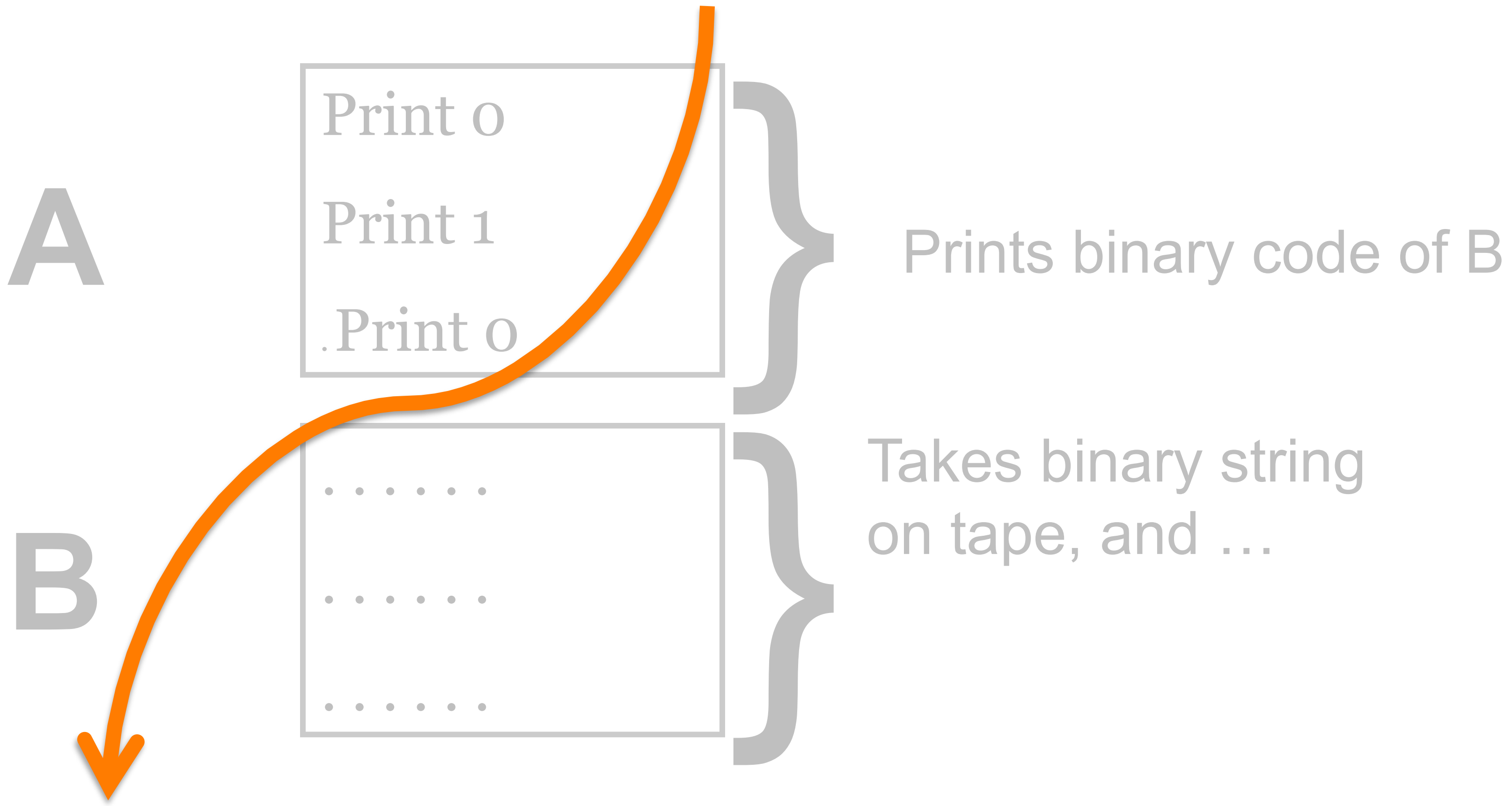
High-level view of self-reproducing program



A

Print 0

Print 1

.Print 0

Prints binary code of B

B

. . . . . .

. . . . . .

. . . . . .

Takes binary string on tape, and …

SEE HANDOUT ON COURSE WEB

Recap: Boolean Logic Example

Ed goes to the party if
Dan does not and Stella does.

Choose “Boolean variables” for 3 events:

$$\left\{ \begin{array}{l} \textbf{E}\text{: Ed goes to party} \\ \textbf{D}\text{: Dan goes to party} \\ \textbf{S}\text{: Stella goes to party} \end{array} \right\}$$

Each is either
TRUE or FALSE

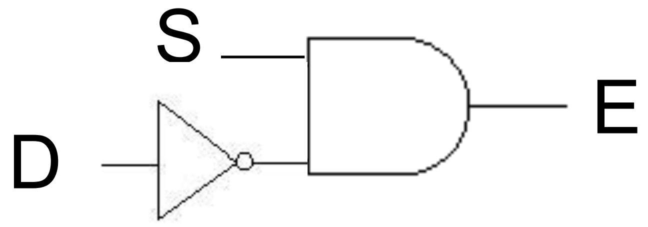
$$E = S \ \textbf{AND} \ (\textbf{NOT}\ D)$$

Alternately: $E = S \ \textbf{AND} \ \overline{D}$

Three Equivalent Representations

Boolean Expression $E = S \text{ AND } \overline{D}$

Boolean Circuit



Truth table:

Value of E for every possible D, S.

TRUE=1; FALSE= 0.

D	S	E
0	0	0
0	1	1
1	0	0
1	1	0

Boolean “algebra”

A **AND** B written as $A \bullet B$

$0 \bullet 0 = 0$

$0 \bullet 1 = 0$

$1 \bullet 1 = 1$

A **OR** B written as $A + B$

$0 + 0 = 0$

$1 + 0 = 1$

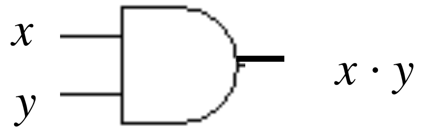
$1 + 1 = 1$

Funny arithmetic

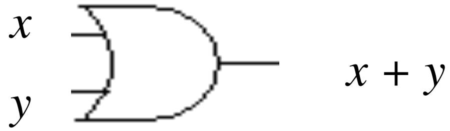
Boolean gates

Shannon (1939)

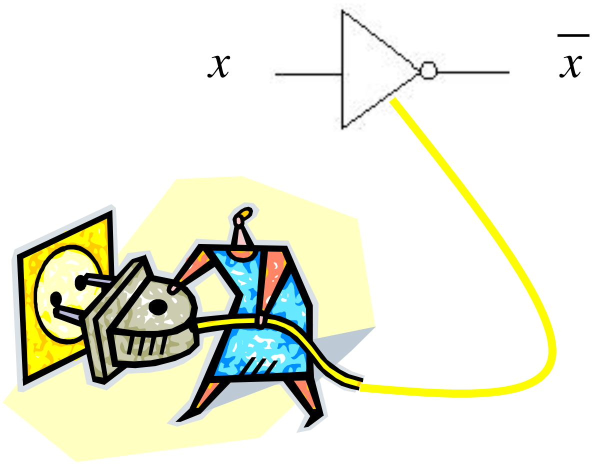
High voltage = 1
Low voltage = 0



Output voltage is high
if **both** of the input voltages are high;
otherwise output voltage low.



Output voltage is high
if **either** of the input voltages are high;
otherwise output voltage low.



Output voltage is high
if the input voltage is low;
otherwise output voltage high.

(implicit extra wires for power)

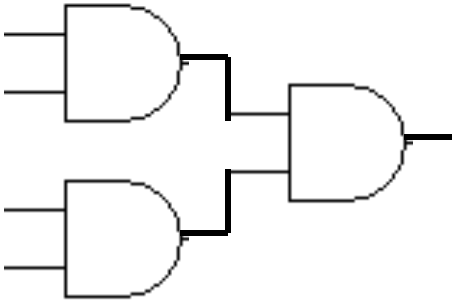
Claude Shannon (1916-2001)

Founder of many fields
(circuits, information theory, artificial intelligence…)

With “Theseus” mouse

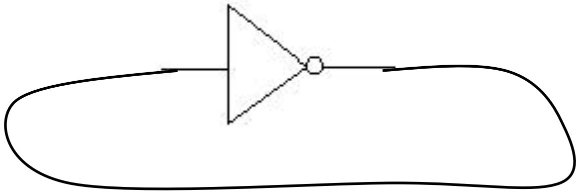
Combinational circuit

- Boolean gates connected by wires



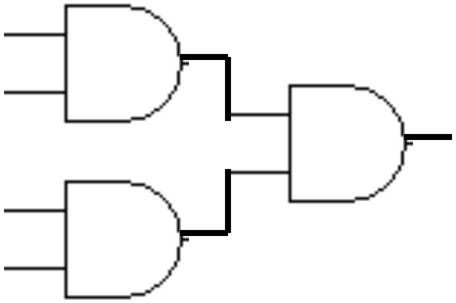
Wires: transmit voltage
(and hence value)

- Important: no cycles allowed

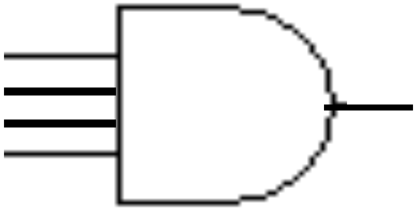


Examples

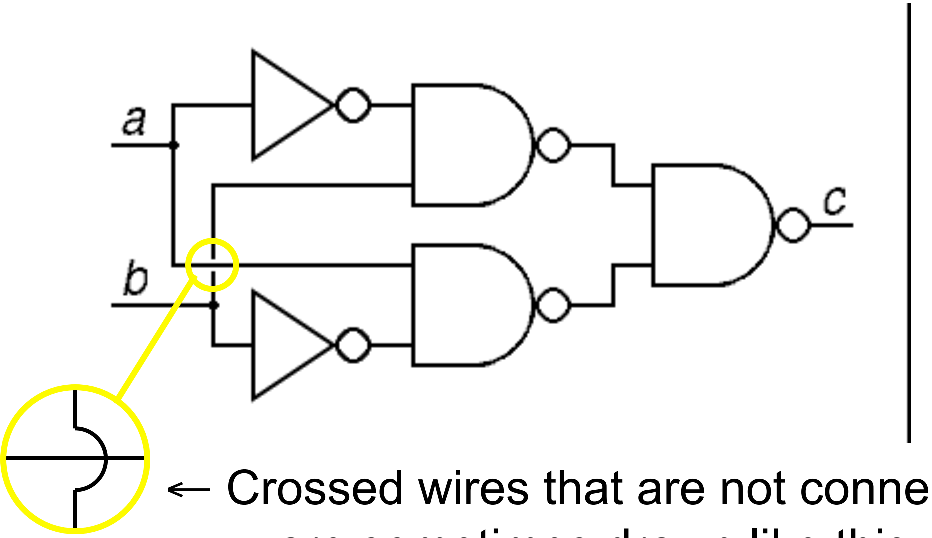
4-way AND



(Sometimes we use this for shorthand)



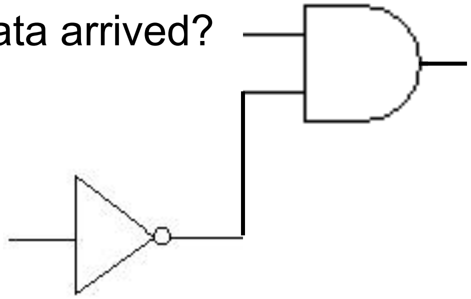
More complicated example



← Crossed wires that are not connected are sometimes drawn like this.

Combinational circuits and control

- “If data has arrived and packet has not been sent, send a signal”



D Data arrived?

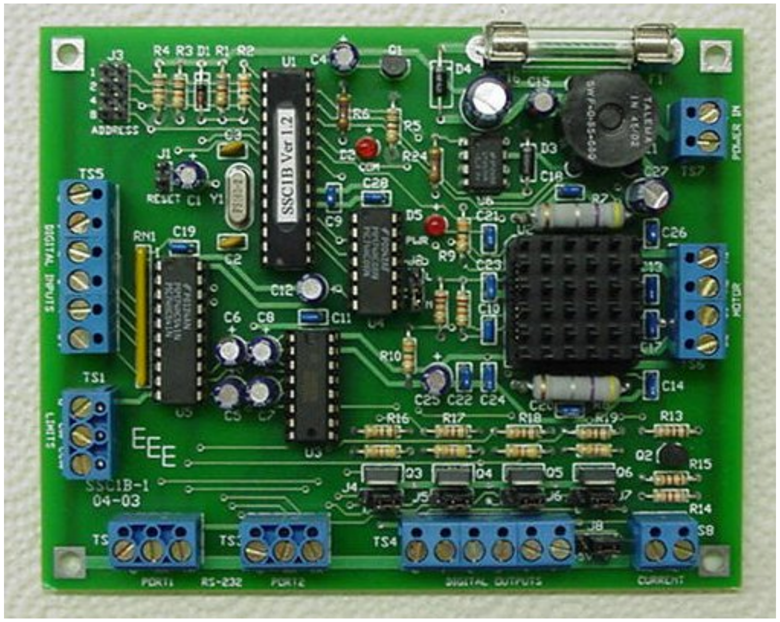
P Packet sent?

S Send signal

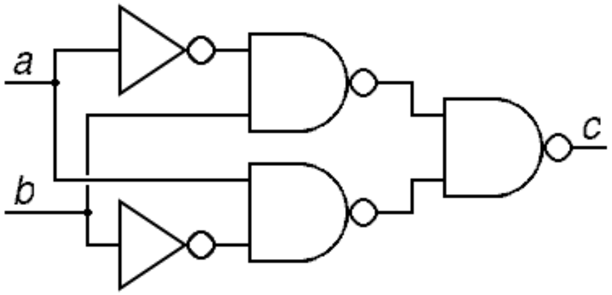
S = D AND (NOT P)

Circuits compute functions

- Every combinational circuit computes a Boolean function of its inputs



Inputs



Outputs

Ben Revisited

Ben only rides to class if he overslept, but even then if it is raining he’ll walk and show up late (he hates to bike in the rain). But if there’s an exam that day he’ll bike if he overslept, even in the rain.

B: Ben Bikes
R: It is raining
E: There is an exam today
O: Ben overslept

How to write a boolean expression for B in terms of R, E, O?

Ben’s truth table

O	R	E	B
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Truth table → Boolean expression

Use **OR** of all
input combinations
that lead to TRUE (1)

$$B = O\cdot\overline{R}\cdot\overline{E} + O\cdot\overline{R}\cdot E + O\cdot R\cdot E$$

O	R	E	B
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Note:
AND, OR, and NOT gates suffice to implement every Boolean function!

Sizes of representations

- For *k* variables:

k	10	20	30
2^k	1024	1048576	1073741824

2^k

A	B	…	X
0	0	…	0
0	0	…	0
0	1	…	0
0	1	…	1
…	…	…	…
…	…	…	…
1	1	…	1

$k + 1$

For an arbitrary function,
expect roughly half of X’s to be 1
(for 30 inputs roughly 1/2 billion!)

Tools for reducing size:
(a) circuit optimization (b) modular design

Expression simplification

- Some simple rules:

  $x + \overline{x} = 1$

  $x \cdot 1 = x$

  $x \cdot 0 = 0$

  $x + 0 = x$

  $x + 1 = 1$

  $x + x = x \cdot x = x$

  $x \cdot (y + z) = x \cdot y + x \cdot z$

  $x + (y \cdot z) = (x+y) \cdot (x+z)$

$$
\begin{aligned}
& x \cdot y + x \cdot \overline{y} \\
& \quad = x \cdot (y + \overline{y}) \\
& \quad = x \cdot 1 \\
& \quad = x
\end{aligned}
$$

De Morgan's Laws:

$\overline{x \cdot y} = \overline{x} + \overline{y}$

$\overline{x + y} = \overline{x} \cdot \overline{y}$

Simplifying Ben’s circuit

- $B = O\cdot\overline{R}\cdot\overline{E} + O\cdot\overline{R}\cdot E + O\cdot R\cdot E$

$$= O\cdot(\overline{R}\cdot\overline{E} + \overline{R}\cdot E + R\cdot E)$$

$$= O\cdot(\overline{R}\cdot(\overline{E} + E) + R\cdot E)$$

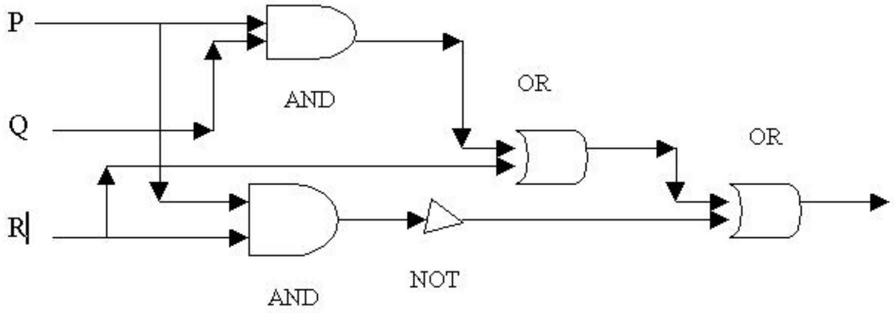
$$= O\cdot(\overline{R} + R\cdot E)$$

…

$$= O\cdot(\overline{R} + E)$$

Something to think about:
How hard is Circuit Verification?

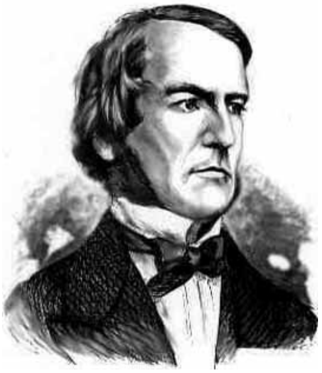
- Given a circuit, decide if it is “trivial” (no matter the input, it either always outputs 1 or always outputs 0)



- Alternative statement: Decide if there is any setting of the inputs that makes the circuit evaluate to 1.

Time required?

Boole’s reworking of Clarke’s “proof” of existence of God (see handout)



- General idea: Try to prove that Boolean expressions E_1, E_2, …, E_k cannot simultaneously be true

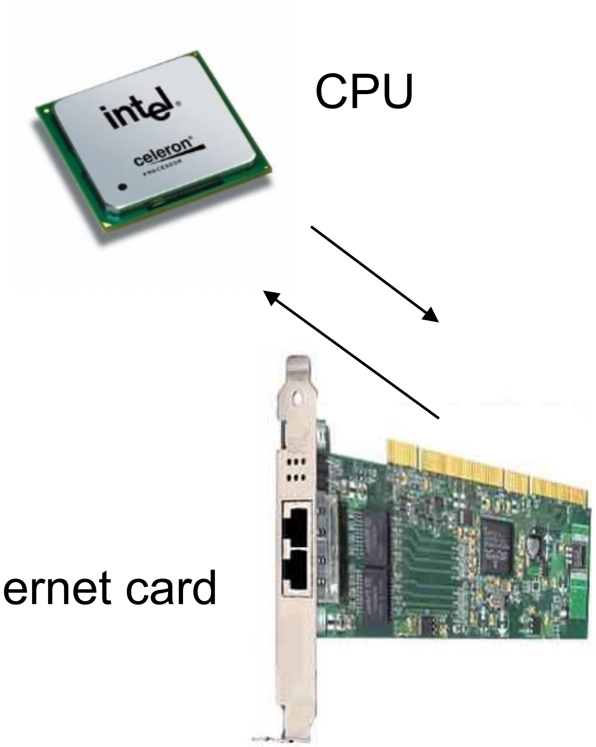
- Method: Show $E_1 \cdot E_2 \cdot \ldots \cdot E_k = 0$

- Discussion for after Break: What exactly does Clarke’s “proof” prove? How convincing is such a proof to you?

Also: Do Google search for “Proof of God’s Existence.”

Beyond combinational circuits …

- Need 2-way communication (must allow cycles!)



CPU

Ethernet card

- Need memory (scratchpad)

Circuit for binary addition?

$$\begin{array}{rrr} & 25 & 11001 \\ + & 29 & 11101 \\ \hline & 54 & 110110 \end{array}$$

- Want to design a circuit to add any two *N*-bit integers.

Is the truth table method useful for N=64?

After Break: Modular Design

Design an N-bit adder using N 1-bit adders

Read:
(a) handout on boolean logic.
(b) handout on Boole’s “proof” of existence of God.