Parametric & Implicit Surfaces

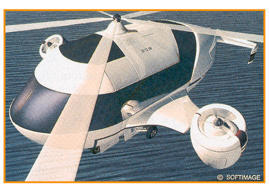
Adam Finkelstein & Tim Weyrich
Princeton University
COS 426, Spring 2008

3D Object Representations

- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - Subdivision
  - Parametric
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

Surfaces

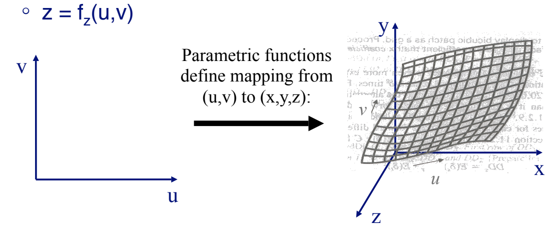
- What makes a good surface representation?
  - Accurate
  - Concise
  - Intuitive specification
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed continuity
  - Natural parameterization
  - Efficient display
  - Efficient intersections



H&B Figure 10.46

Parametric Surfaces

- Boundary defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

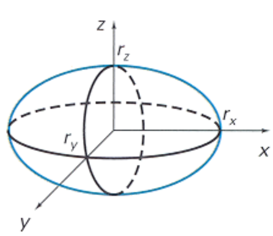


FvDFH Figure 11.42

Parametric Surfaces

- Boundary defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$
- Example: ellipsoid

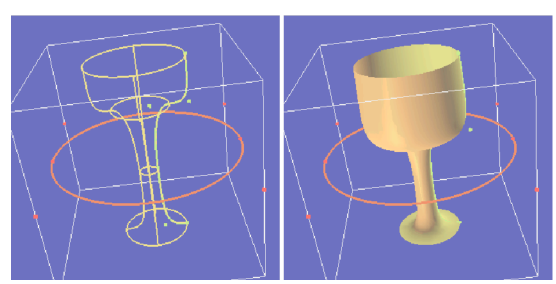
$$x = r_x \cos\phi \cos\theta$$
$$y = r_y \cos\phi \sin\theta$$
$$z = r_z \sin\phi$$



H&B Figure 10.10

Parametric Surfaces

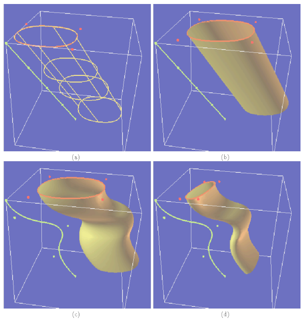
- Example: surface of revolution
  - Take a curve and rotate it about an axis



Demetri Terzopoulos

Parametric Surfaces

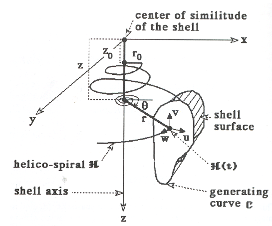
- Example: swept surface
  - Sweep one curve along path of another curve

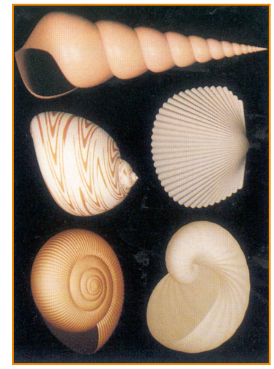


Demetri Terzopoulos

Parametric Surfaces

- Example: swept surface
  - Making sea shells

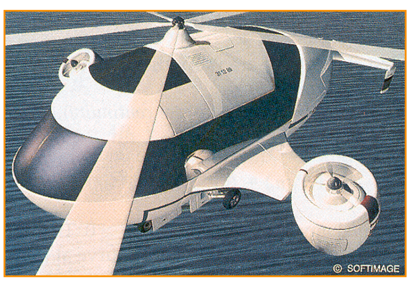




Fowler

Parametric Surfaces

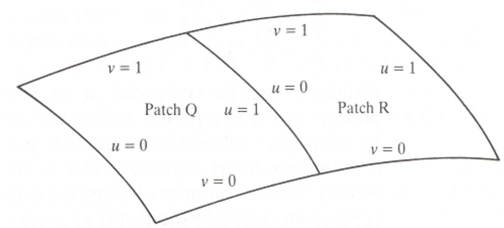
- How do we describe arbitrary smooth surfaces with parametric functions?



H&B Figure 10.46

Piecewise Polynomial Parametric Surfaces

- Surface is partitioned into parametric patches:

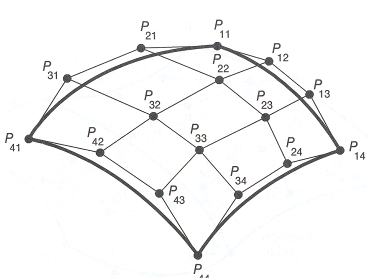


Same ideas as parametric splines!

Watt Figure 6.25

Parametric Patches

- Each patch is defined by blending control points

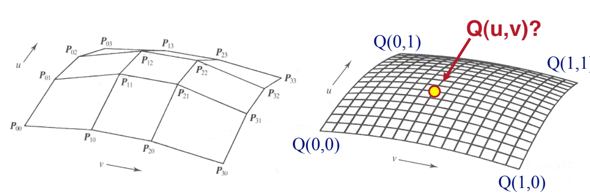


Same ideas as parametric curves!

FvDFH Figure 11.44

Parametric Patches

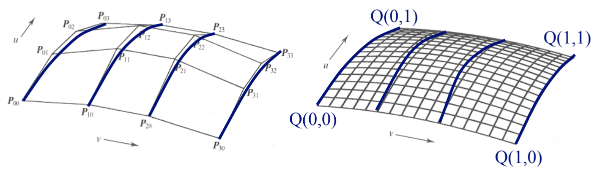
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

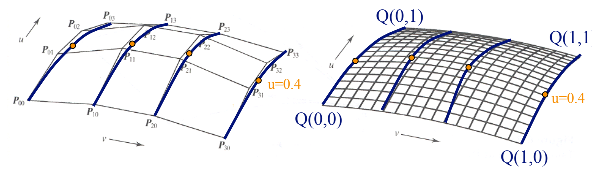
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

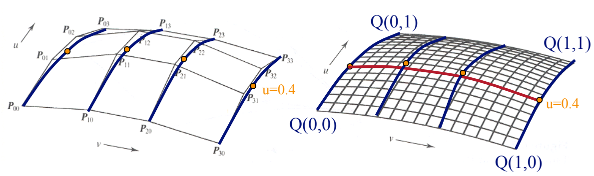
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

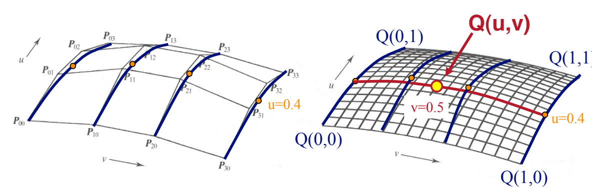
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{U}\mathbf{M}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}$$

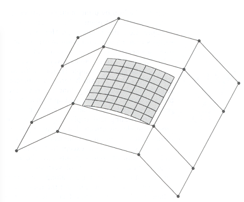
$$\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bezier, B-spline, etc.)

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{B-Spline}}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}_{\mathbf{B-Spline}}\mathbf{V}$$

$$\mathbf{M}_{\mathbf{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$

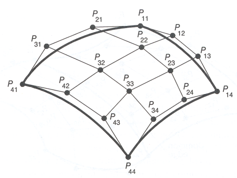


Watt Figure 6.28

Bezier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\text{T}}_{\text{Bezier}} \mathbf{V}$$

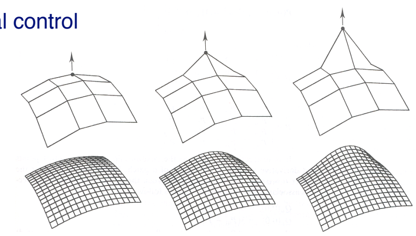
$$\mathbf{M}_{\text{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



FvDFH Figure 11.42

Bezier Patches

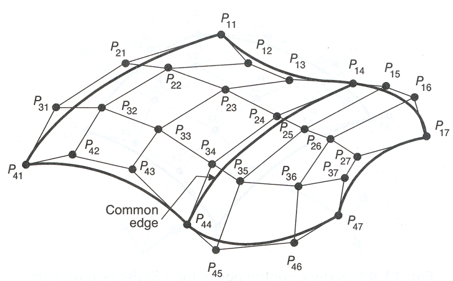
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

Bezier Surfaces

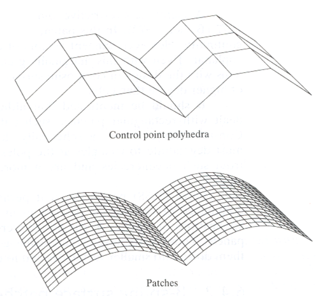
- Continuity constraints are similar to the ones for Bezier splines



FvDFH Figure 11.43

Bezier Surfaces

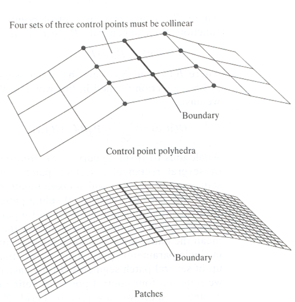
- C^0 continuity requires aligning boundary curves



Watt Figure 6.26a

Bezier Surfaces

- C^1 continuity requires aligning boundary curves and derivatives

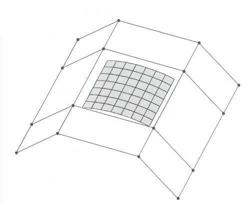


Watt Figure 6.26b

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\text{T}}_{\text{B-Spline}} \mathbf{V}$$

$$\mathbf{M}_{\text{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$



Watt Figure 6.28

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes
- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain
  - Hard to find intersections

25

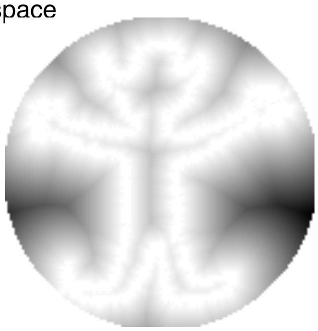
3D Object Representations

- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - Subdivision
  - Parametric
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

26

Implicit Surfaces

- Represent surface with function over all space

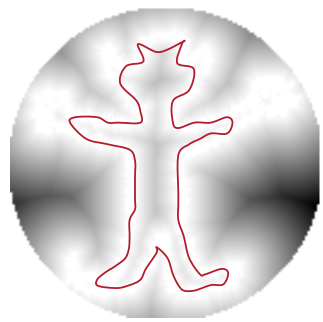


Kazhdan

27

Implicit Surfaces

- Surface defined implicitly by function

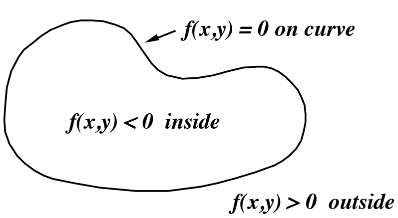


Kazhdan

28

Implicit Surfaces

- Surface defined implicitly by function:
  - f (x, y, z) = 0 (on surface)
  - f (x, y, z) < 0 (inside)
  - f (x, y, z) > 0 (outside)

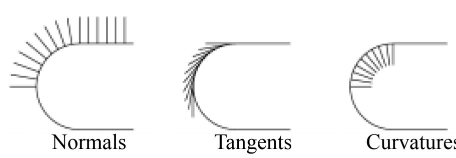


Turk

29

Implicit Surfaces

- Normals defined by partial derivatives
  - normal(x, y, z) = (df/dx, df/dy, df/dz)



Bloomenthal

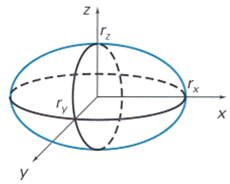
30

Implicit Surface Properties

(1) Efficient check for whether point is inside

- Evaluate f(x,y,z) to see if point is inside/outside/on

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 - 1 = 0$$



H&B Figure 10.10

Implicit Surface Properties

(2) Efficient surface intersections

- Substitute to find intersections

Ray: $P = P_0 + tV$

Sphere: $|P - O|^2 - r^2 = 0$

Substituting for P, we get:

$|P_0 + tV - O|^2 - r^2 = 0$

Solve quadratic equation:

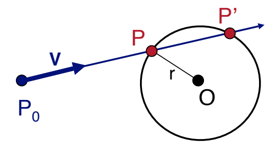
$at^2 + bt + c = 0$

where:

$a = 1$

$b = 2\, V \cdot (P_0 - O)$

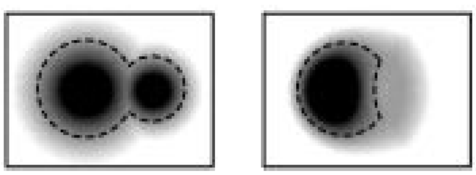
$c = |P_0 - C|^2 - r^2 = 0$



Implicit Surface Properties

(3) Efficient boolean operations (CSG)

- Union, difference, intersect



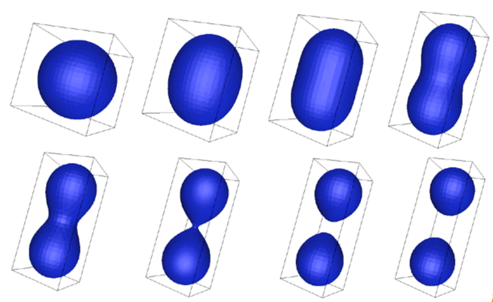
Union Difference

Bloomenthal

Implicit Surface Properties

(4) Efficient topology changes

- Surface is not represented explicitly!

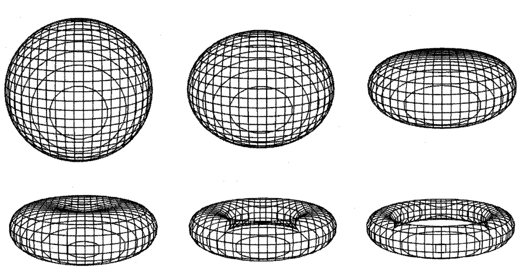


Bourke

Implicit Surface Properties

(4) Efficient topology changes

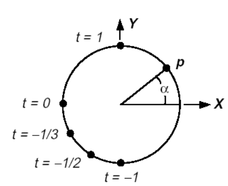
- Surface is not represented explicitly!



Bloomenthal

Comparison to Parametric Surfaces

- Implicit
  - Efficient intersections & topology changes
- Parametric
  - Efficient "marching" along surface & rendering



equiangular parametric
(transcendental trigonometric)
$p = (\cos(\alpha), \sin(\alpha)),\ \alpha \in [0, 2\pi]$

non-equiangular parametric (rational)
$p = (\pm(1-t^2)/(1+t^2),\ 2t/(1+t^2)),\ t \in [-1, 1]$

implicit
$p_x^2 + p_y^2 - 1 = 0$

Bloomenthal

Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Blobby models
  - Skeletons
  - Procedural
  - Samples
  - Variational

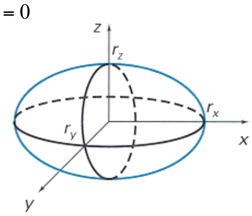
Implicit Surface Representations

- How do we define implicit function?
  - ➤ Algebraics
  - Blobby models
  - Skeletons
  - Procedural
  - Samples
  - Variational

Algebraic Surfaces

- Implicit function is polynomial
  - $f(x,y,z)=ax^d+by^d+cz^d+dx^{d-1}y+dx^{d-1}z+dy^{d-1}x+\ldots$

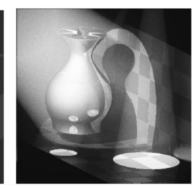
$$\left(\frac{x}{r_x}\right)^2+\left(\frac{y}{r_y}\right)^2+\left(\frac{z}{r_z}\right)^2-1=0$$



H&B Figure 10.10

Algebraic Surfaces

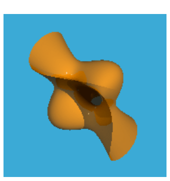
- Most common form: quadrics
  - $f(x,y,z)=ax^2+by^2+cz^2+2dxy+2eyz+2fxz+2gx+2hy+2jz+k$
- Examples
  - Sphere
  - Ellipsoid
  - Torus
  - Paraboloid
  - Hyperboloid

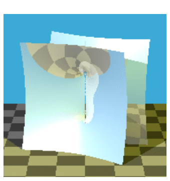



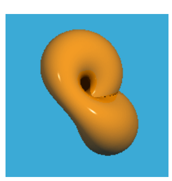
Menon

Algebraic Surfaces

- Higher degree algebraics

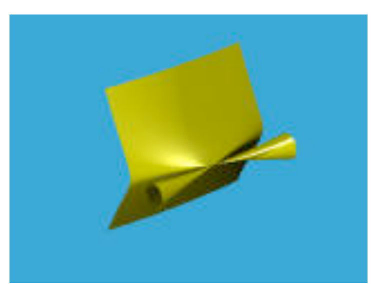

Cubic


Quartic


Degree six

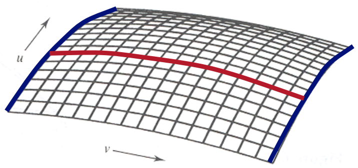
Algebraic Surfaces

- Function extends to infinity
  - Must trim to get desired patch (this is difficult!)



Algebraic Surfaces

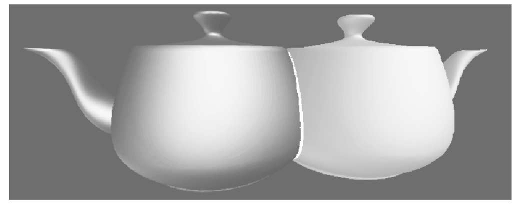
- Equivalent parametric surface
  - Tensor product patch of degree m and n curves yields algebraic function with degree 2mn



Bicubic patch has degree 18!

Algebraic Surfaces

- Intersection
  - Intersection of degree m and n algebraic surfaces yields curve with degree mn



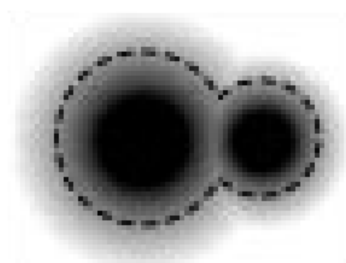
Intersection of bicubic patches has degree 324!

Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Blobby models
  - Skeletons
  - Procedural
  - Samples
  - Variational

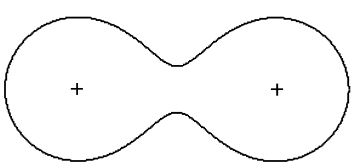
Blobby Models

- Implicit function is sum of spherical basis functions



Blobby Models

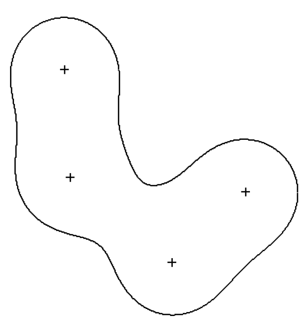
- Sum of two blobs



Turk

Blobby Models

- Sum of four blobs

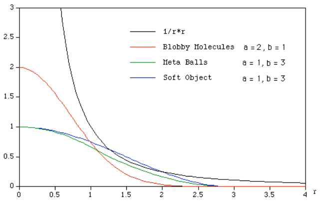


Turk

Blobby Models: radial basis funcs

- Blobby molecules

$$D(r) = ae^{-br^2}$$



- Meta balls

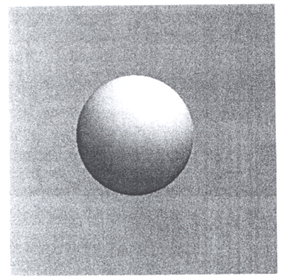
$$D(r) = \begin{cases} a(1 - \frac{3r^2}{b^2}) & 0 \le r \le b/3 \\ \frac{3a}{2}(1 - \frac{r}{b})^2 & b/3 \le r \le b \\ 0 & b \le r \end{cases}$$

- Soft objects

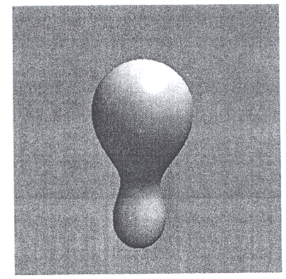
$$D(r) = \begin{cases} a(1 - \frac{4r^6}{9b^6} + \frac{17r^4}{9b^4} - \frac{22r^2}{9b^2} & r \le b \\ 0 & r \ge b \end{cases}$$

Bourke

Blobby Model of Face

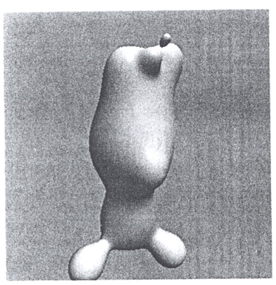


(a) $N = 1$

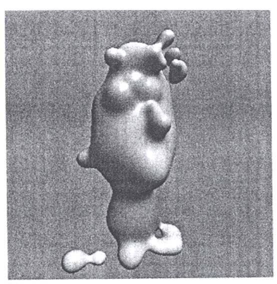


(b) $N = 2$

Blobby Model of Face

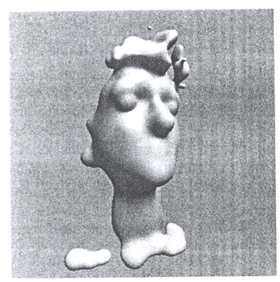


(c) $N = 10$

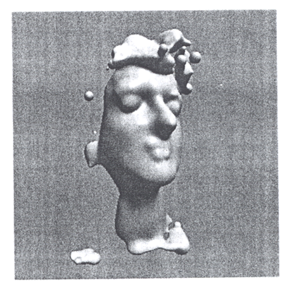


(d) $N = 35$

Blobby Model of Face

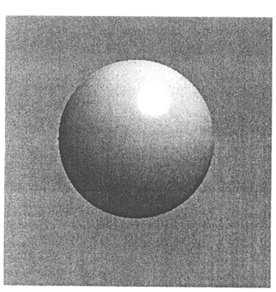


(e) $N = 70$

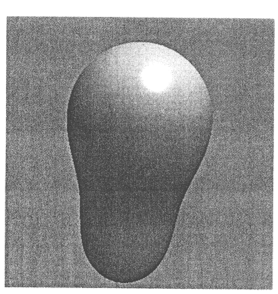


(f) $N = 243$

Blobby Model of Head

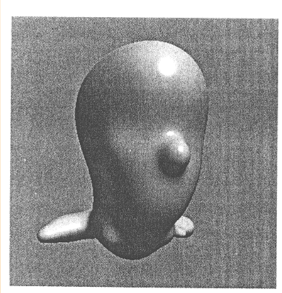


(a) $N = 1$

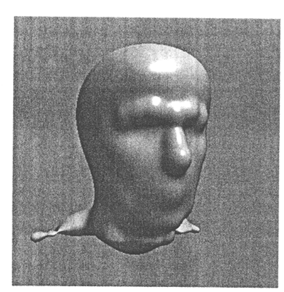


(b) $N = 2$

Blobby Model of Head

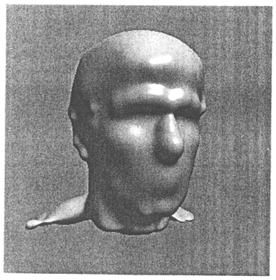


(c) $N = 20$

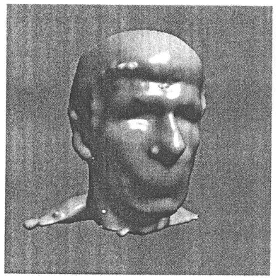


(d) $N = 60$

Blobby Model of Head



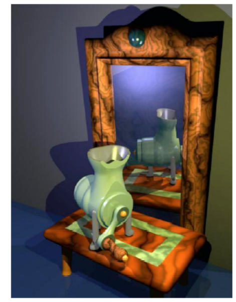
(e) $N = 120$



(f) $N = 451$

Blobby Models

Objects resulting from CSG of implicit soft objects and other primitives



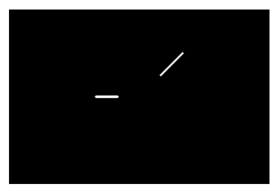
Menon

Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Blobby models
  - ➢ Skeletons
  - Procedural
  - Samples
  - Variational

Skeletons

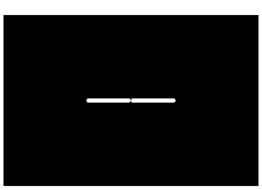
- Bulge problem



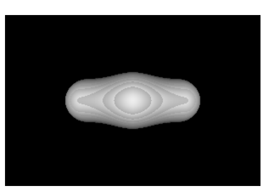
®

Skeletons

- Bulge problem

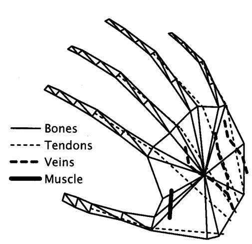


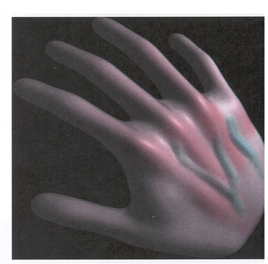
®



Skeletons

- Convolution surfaces





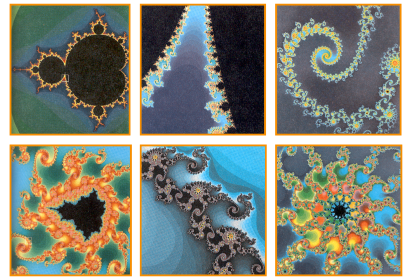
Bloomenthal

Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Blobby models
  - Skeletons
  - ➢ Procedural
  - Samples
  - Variational

Procedural Implicits

- f(x,y,z) is result of procedure
  - Example: Mandelbrot set



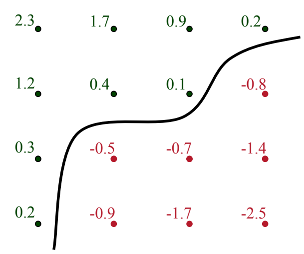
H&B Figure 10.100

Implicit Surface Representations

- How do we define implicit function?
  - Algebraics
  - Blobby models
  - Skeletons
  - Procedural
  - ➢ Samples
  - Variational

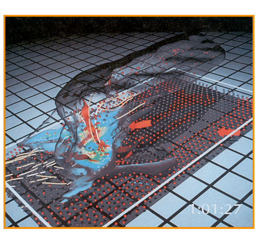
Sampled Functions

- Most common example: voxels
  - Interpolate samples stored on regular grid
  - Isosurface at f(x,y,z) =0 defines surface



Sampled Functions

- Acquired from simulations or scans



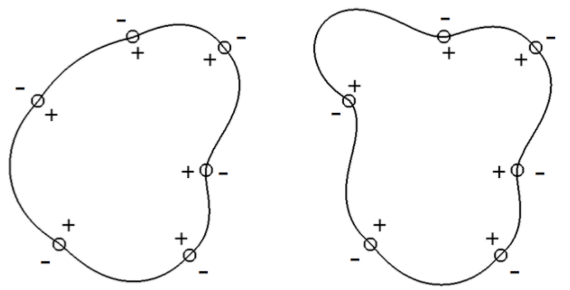
Airflow Inside a Thunderstorm
(Bob Wilhelmson, University of Illinois at Urbana-Champaign)

Visible Human
(National Library of Medicine)

Implicit Surface Representations

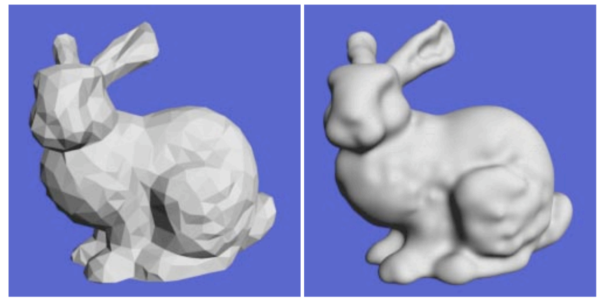
- How do we define implicit function?
  - Algebraics
  - Blobby models
  - Skeletons
  - Procedural
  - Samples
  - ➢ Variational

Variational Implicit Surfaces



Turk

Example Implicit Surface



Turk

Implicit Surface Summary

- Advantages:
  - Easy to test if point is on surface
  - Easy to compute intersections/unions/differences
  - Easy to handle topological changes
- Disadvantages:
  - Indirect specification of surface
  - Hard to describe sharp features
  - Hard to enumerate points on surface
    - » Slow rendering

Summary

Feature	Polygonal Mesh	Implicit Surface	Parametric Surface	Subdivision Surface
Accurate	No	Yes	Yes	Yes
Concise	No	Yes	Yes	Yes
Intuitive specification	No	No	Yes	No
Local support	Yes	No	Yes	Yes
Affine invariant	Yes	Yes	Yes	Yes
Arbitrary topology	Yes	No	No	Yes
Guaranteed continuity	No	Yes	Yes	Yes
Natural parameterization	No	No	Yes	No
Efficient display	Yes	No	Yes	Yes
Efficient intersections	No	Yes	No	No