9.5 n-Dimensional Euclidean Space

n-Dimensional space merely means that we have n independent variables $x_1, x_2, \ldots, x_n$. Euclidean space means that we use the Pythagorean distance

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2 \tag{9.5-1}$$

and we can define a *sphere of radius* r by this expression.

A little reflection on classical Euclidean geometry will suffice to see why the volume of an n-dimensional sphere depends on the radius r as r^n. Therefore, we have the formula for the volume

$$V_n(r) = C_n r^n \tag{9.5-2}$$

where C_n is some constant depending on n. For example,

$$C_2 = \pi \qquad \text{and} \qquad C_3 = \frac{4\pi}{3} \qquad (\text{note that } C_1 = 2)$$

To find the values of C_n, we use the same trick as in Section 9.4 of multiplying the gamma integral by itself and then going to polar coordinates (which worked so well). Let us consider the product of n of the integrals. We have [using (9.5-1)]

$$[\Gamma(\tfrac{1}{2})]^n = \pi^{n/2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-r^2}\,dx_1\,dx_2\ldots dx_n$$

$$= \int_0^{\infty} e^{-r^2}\frac{dV_n(r)}{dr}\,dr$$

This last step takes a moment's thinking about the shell of thickness Δr and comparing this with our result in two dimensions,

$$\int_0^{\infty} e^{-r^2}\frac{d(\pi r^2)}{dr}\,dr = 2\pi\int_0^{\infty} e^{-r^2} r\,dr$$

as we really did in Section 9.4 (although it was disguised somewhat by the conventional method).

We have, therefore, using (9.5-2),

$$\pi^{n/2} = C_n\int_0^{\infty} e^{-r^2} n r^{n-1}\,dr$$

Set $r^2 = t$. Then $dr = \frac{1}{2}t^{-1/2}\,dt$, and

$$\pi^{n/2} = \frac{nC_n}{2}\int_0^{\infty} e^{-t}\frac{t^{(n-1)/2}\,dt}{t^{1/2}}$$

$$= \frac{nC_n}{2}\int_0^{\infty} e^{-t}t^{(n/2-1)}\,dt$$

$$= \frac{nC_n}{2}\Gamma\left(\frac{n}{2}\right) = C_n\Gamma\left(\frac{n}{2}+1\right)$$

Therefore,

$$C_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

It is easy to show that

$$C_n = \frac{2\pi}{n}C_{n-2}$$

and we can compute the table (of the volume of a unit sphere in n dimensions)

n	C_n	
1	2	2.00000 . . .
2	π	3.14159 . . .
3	$4\pi/3$	4.18879 . . .
4	$\pi^2/2$	4.93480 . . .
5	$8\pi^2/15$	5.26379 . . .
6	$\pi^3/6$	5.16771 . . .
7	$16\pi^3/105$	4.72477 . . .
8	$\pi^4/24$	4.05871 . . .
⋮	⋮	⋮
$2k$	$\pi^k/k!$	$\longrightarrow 0$

From the table we see that the volume of the unit sphere, or equivalently the coefficient of r^n, comes to a maximum at $n = 5$ and falls off rather rapidly toward zero as n approaches infinity.

For $n = 2k$, the volume of an n-dimensional sphere of radius r is

$$\left(\frac{\pi^k}{k!}\right)r^{2k} = \frac{(\pi r^2)^k}{k!}$$

From this we see that once

$$k > \pi r^2$$

then increasing k (or equivalently n) will *decrease* the volume. Indeed, given any radius, no matter how large, the dimension of the space n can be increased until the volume of the sphere is arbitrarily small. [For odd-dimensional spaces, we have, from the original definition of $\Gamma(n)$, equation (9.4-1), that

$$V_n(r) = C_n r^n = \frac{\pi^{n/2} r^n}{\Gamma(n/2 + 1)}$$

will change smoothly with increasing n.]

We now consider the *fraction* of the volume of an n-dimensional sphere that is within a distance ε of the surface (ε is used here as any small positive number). We have

$$\frac{\text{shell}}{\text{volume}} = \frac{C_n r^n - C_n(r - \varepsilon)^n}{C_n r^n} = 1 - \left(1 - \frac{\varepsilon}{r}\right)^n$$

which approaches 1 as n gets large. Thus for a high-dimensional space, almost all the volume of a sphere is arbitrarily close to the surface. Indeed, no matter how thin a shell you wish to use, and how close you wish to get to 1 (say 99.44%), it is possible to find an n_0 such that for all greater n, the two conditions are met. There simply is not much volume inside a high-dimensional sphere; almost all of it is "on the surface."

Let us now examine the angle between the vector from the origin $(0, 0, \ldots, 0)$ to the point $(1, 1, \ldots, 1)$ and any coordinate axis. The projection on each axis is, obviously, exactly equal to 1 (its coordinate value on that axis). The length of the vector is $\sqrt{n}$; hence the angle we want, θ, is given by

$$\cos\theta = \frac{1}{\sqrt{n}}$$

As $n \longrightarrow \infty$, this approaches 0; hence $\theta \longrightarrow \pi/2$. Thus for sufficiently large n, the diagonal line is "almost perpendicular" to each coordinate axis!

Exercises

9.5-1 Show for any family of convex similar figures that "almost all of the volume is on the surface."

9.5-2 Show that the volume of a $2k$-dimensional sphere divided by the corresponding volume of an enclosing cube is $(\pi/4)^k(1/k!)$.

9.6 A Paradox

The results of this section are not needed but are included to show how unreliable your intuition is about spheres in n-dimensional Euclidean space.

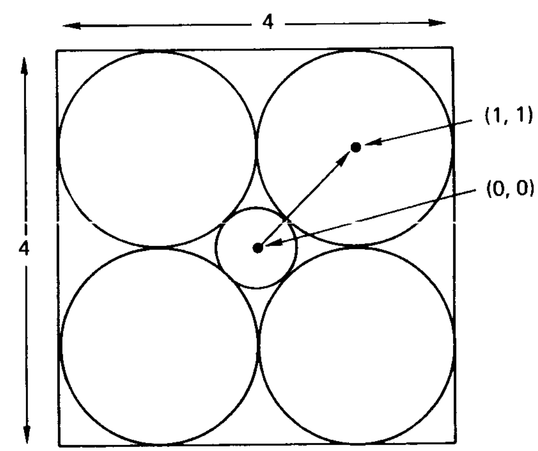

Figure 9.6-1 A paradox

Suppose, as shown in Figure 9.6-1, that we have a 4×4 square centered about the origin (0, 0) and have four unit circles in each of the four corners. Now consider the radius of the circle about the origin that is tangent on the inside to the four circles. It has a radius

$$r_2 = \sqrt{(1-0)^2 + (1-0)^2} - 1 = \sqrt{2} - 1 = 0.414\ldots$$

Next consider the same situation in three dimensions. We have a $4 \times 4 \times 4$ cube with eight unit spheres in the corners. The inner sphere has the radius

$$r_3 = \sqrt{3} - 1 = 0.732\ldots$$

Finally, consider the similar situation in n dimensions. We have a $4 \times 4 \times \ldots \times 4$ cube with 2^n unit spheres in the corners, each one touching all its n neighboring spheres. The spheres are packed in properly. The distance from the origin to the center of a sphere is given by the expression

$$\sqrt{(1-0)^2 + (1-0)^2 + \ldots + (1-0)^2} = \sqrt{n}$$

Again subtracting the radius of the corner unit sphere, we get the radius of the inner sphere in n dimensions as

$$r_n = \sqrt{n} - 1 \tag{9.6-1}$$

When $n = 10$, this is

$$r_{10} = \sqrt{10} - 1 = 3.16\ldots - 1 = 2.16\ldots > 2$$

and the inner sphere reaches outside the cube! Does this seem impossible? The spheres are convex surfaces to be sure, the distance is surely the correct Euclidean distance, and the radius of the corner sphere is surely 1—so there is no escape from the conclusion. Contrary to any normal intuition, the inner sphere for $n \geq 10$ reaches outside the cube.

To compound the paradox further, consider the volume of this inner sphere *relative* to the whole cube as a function of n. We have, for the special case of n an even number $n = 2k$, and using (9.6-1),

$$\begin{aligned}
\text{ratio} &= \frac{\text{volume of sphere}}{\text{volume of whole cube}} = C_n \frac{(\sqrt{n} - 1)^n}{4^n} \\
&= C_{2k} \frac{(\sqrt{2k} - 1)^{2k}}{4^{2k}} \\
&= \frac{\pi^k (\sqrt{2})^{2k} (\sqrt{k})^{2k}}{k! 4^{2k}} \left[\left(1 - \frac{1}{\sqrt{2k}}\right)^{\sqrt{2k}} \right]^{\sqrt{2k}} \\
&= \frac{\pi^k 2^k k^k}{4^{2k} k!} \left[\left(1 - \frac{1}{\sqrt{2k}}\right)^{\sqrt{2k}} \right]^{\sqrt{2k}}
\end{aligned} \tag{9.6-2}$$

Using Stirling's approximation (9.3-1) for $k!$,

$$k! \simeq k^k e^{-k} \sqrt{2\pi k}$$

and the result from the calculus,

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$$

We get for (9.6-2),

$$\begin{aligned}
\text{ratio} &\simeq \frac{\pi^k 2^k k^k}{4^{2k} k^k e^{-k} \sqrt{2\pi k}} e^{-\sqrt{2k}} \\
&\simeq \left(\frac{\pi e}{8}\right)^k \frac{e^{-\sqrt{2k}}}{\sqrt{2\pi k}}
\end{aligned} \tag{9.6-3}$$

But

$$\frac{\pi e}{8} = 1.06747\ldots$$

Taking $\log_e$ of both sides, we get, as k approaches infinity, $\log_e$ (ratio) $= (0.065288\ldots)k - \sqrt{2k} - \frac{1}{2}\log_e k - \frac{1}{2}\log_e 2 \longrightarrow \infty$. Therefore, we see that the factor

$$\left(\frac{\pi e}{8}\right)^k$$

will go to infinity faster than the other two factors in the product (9.6-3),

$$e^{-\sqrt{2k}} \quad \text{and} \quad \frac{1}{\sqrt{2\pi k}}$$

can decrease it.

We conclude that the volume of the *inner* sphere becomes arbitrarily larger than the volume of the cube which contains all the $2^{2k} = 2^n$ unit spheres in the corners.

The case of the odd dimension $n = 2k - 1$ does not change matters; it only makes the details more messy.

9.7 Chebyshev's Inequality and the Variance

If a random variable X is discrete, then the mean square is given by the expectation

$$E\{X^2\} = \sum_{i=-\infty}^{\infty} x_i^2 p(x_i)$$

and if the variable is continuous, it is given by

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 p(x)\, dx$$

For any $\varepsilon > 0$, since the integrand is positive, we have

$$\int_{-\infty}^{\infty} x^2 p(x)\, dx \geq \int_{|x| \geq \varepsilon} x^2 p(x)\, dx$$