Clustering and the k-means Algorithm

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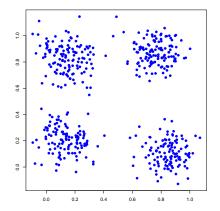
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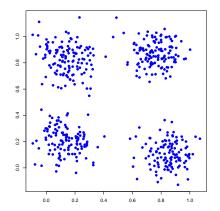
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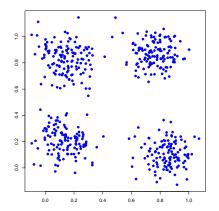
$$\{z_1,\ldots,z_N\}$$
 where $z_i \in \{1,\ldots,K\}$.



500 2-dimensional data points: $\mathbf{x}_n = \langle x_{n,1}, x_{n,2} \rangle$

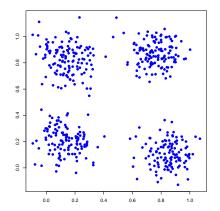


• What is a good distance function here?

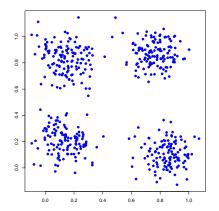


- What is a good distance function here?
- Squared Euclidean distance is reasonable

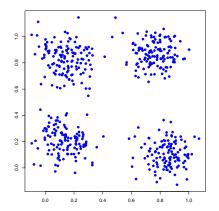
$$d(\mathbf{x}_n, \mathbf{x}_m) = \sum_{i=1}^{p} (x_{n,i} - x_{m,i})^2 = ||x_n - x_m||^2$$



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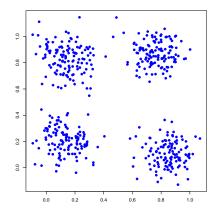


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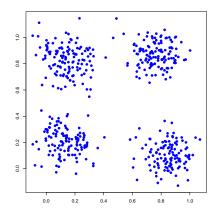
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- What should k be?
- Automatically choosing k is complicated; for now, 4.

k-means



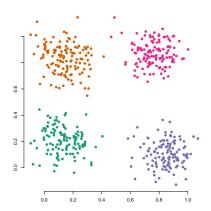
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- The goal of *k*-means is to assign data to clusters and deine these clusters with their means.

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Output the each cluster mean to be the coordinate-wise average over data points assigned to that cluster,

$$\mathbf{m}_k = \frac{1}{N_k} \sum_{\{n: z_n = k\}} \mathbf{x}_n$$

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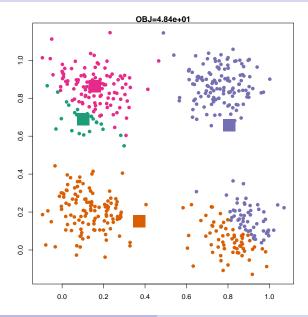
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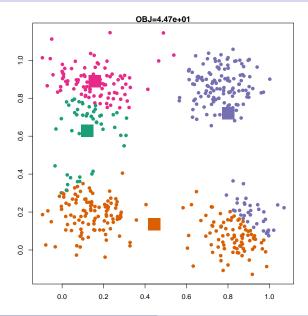
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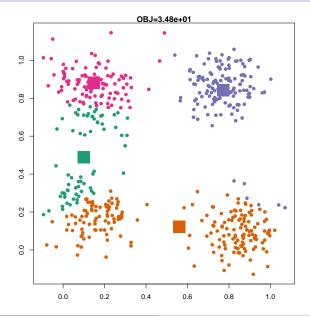
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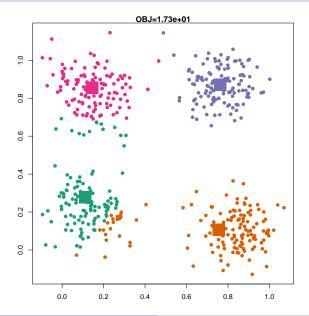
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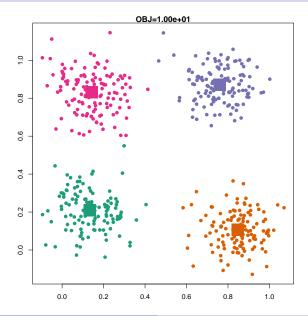


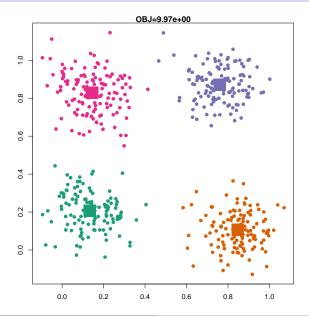


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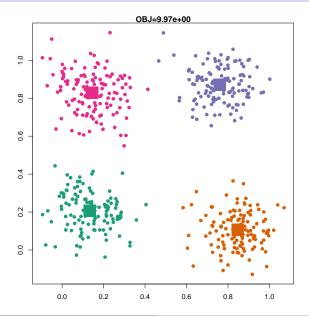


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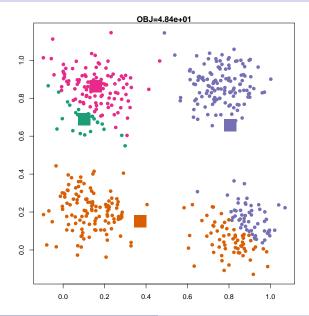


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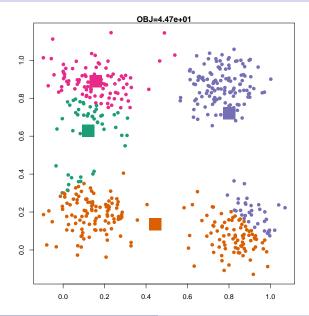
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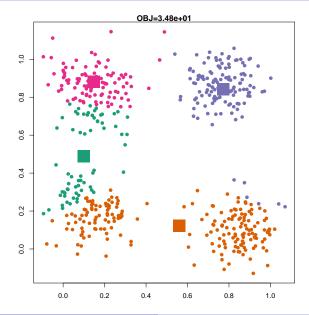
- How can we measure how well our algorithm is doing?
- The *k*-means objective function is the sum of the squared distances of each point to each assigned mean

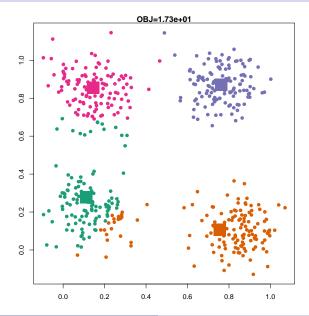
$$F(z_{1:N},\mathbf{m}_{1:k}) = \frac{1}{2}\sum_{n=1}^{N}||\mathbf{x}_n - \mathbf{m}_{z_n}||^2$$

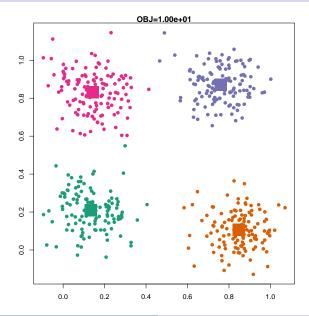


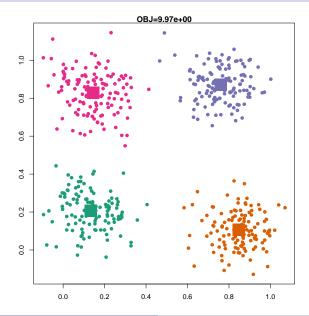
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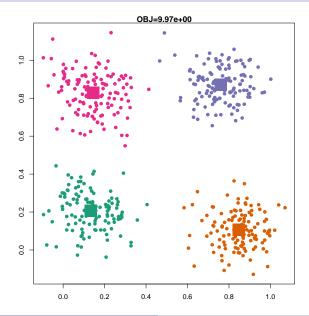








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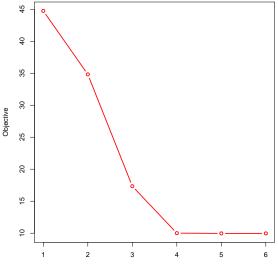
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- It finds a local minimum. (Multiple restarts are often necessary.)

Objective for the example data



Round of k-means

Compressing images



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- How can we use *k*-means to compress this image?

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- With k = 100, we need 7 bits per pixel plus 100×3 bits ≈ 897 K.









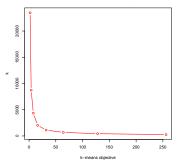








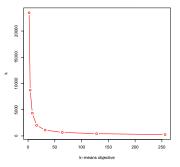
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- For more clusters, the picture is less distorted.

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- Each of the clusters is associated with its most typical example

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③ Until assignments $\mathbf{z}_{1:N}$ do not change

• Choosing k is a nagging problem in cluster analysis

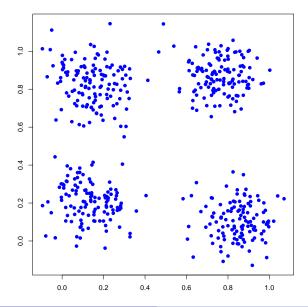
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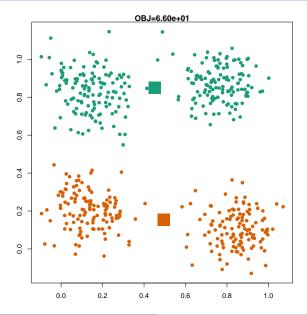
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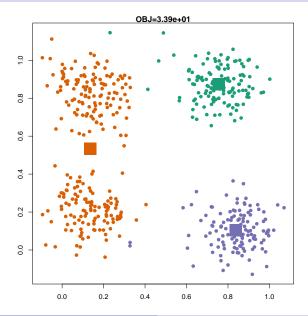
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 - Clustering customers for k salespeople in a business
- Usually, we seek the "natural" clustering, but what does this mean?
- It is not well-defined.



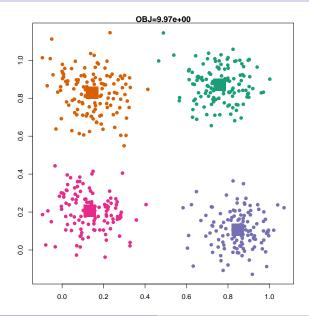
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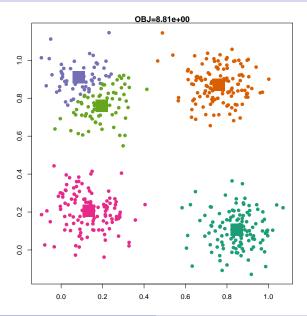
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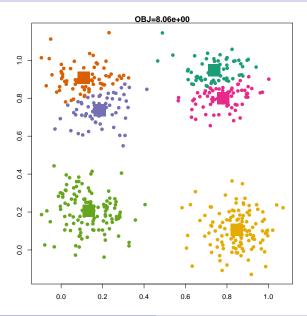
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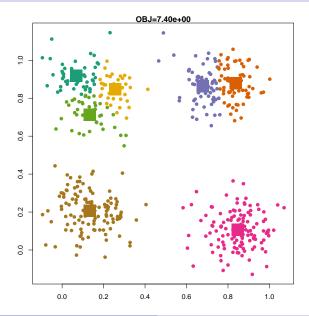
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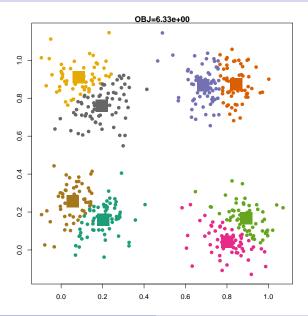
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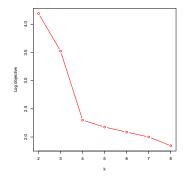


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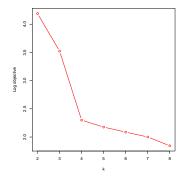
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Heuristic: A kink in the objective



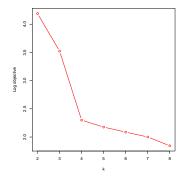
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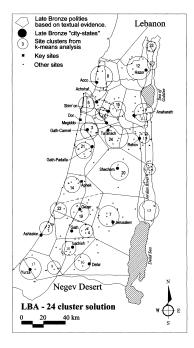
- Notice the "kink" in the objective between 3 and 5.
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- Tibshirani (2001) presents a method for finding this kink.

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- Make inferences about political history based on the clusters
- Choose k very carefully, with a complicated computational technique.

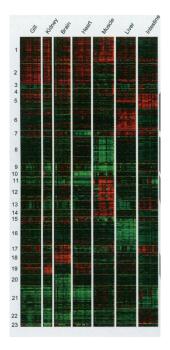


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- Exposed carp to different levels of cold
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- (No mention of how k = 23 was chosen.)



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- Clustered survey results of 206 students
- Used the clusters to identify groups to buttress an analysis of what affects motivation.
- I.e., the levels of encouragement are corrected for

- Teachers as Sources of Middle School Students' Motivational Identity: Variable-Centered and Person-Centered Analytic Approaches (Murdock and Miller, 2003)
- Clustered survey results of 206 students
- Used the clusters to identify groups to buttress an analysis of what affects motivation.
- I.e., the levels of encouragement are corrected for
- Chose the number of clusters to get nice results

	Cluster 1	Cluster 2	Cluster 3	Cluster 4	Cluster 5
Teacher caring Peers' academic support	5 1.0	5 to .5 5	5 to .5	5 5	1.0 5 to .5
Parents' academic support	.5	-1.0	5 to .5	– .5 to .5	5 to .5

TABLE 3. Five-Cluster Solution: Z scores on Each Clustering Variable

TABLE 4. Means and Standard Deviations for Each Cluster on Grade 8 Motivational Variables

	Academic Self-Efficacy		Intrinsic Valuing of Education		Teacher-Rated Effort	
Cluster	М	SD	М	SD	М	SD
1. All positive	3.59	.48ª	2.99	.55ª	3.74	.26ª
2. Peer negative, parents very negative	2.44	.66 ^b	2.16	.51 ^b	3.05	.61 ^b
3. Peer positive	3.01	.73°	2.43	.66 ^b	3.26	.66 ^b
Negative teacher and peer	2.47	.63 ^b	2.24	.51 ^b	3.17	.59 ^b
5. Positive teacher and parents	3.19	.65°	2.89	.62ª	3.54	.47ª

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- Draw the conclusion that patterns exist. What's wrong with this?
- k-means will find patterns everywhere!

Cluster type and behavioral patterns	%
Light substance dabbiers—infrequent or no current use of substances† None have had sex	24.
Abstainers—none have ever used substances† or had sex	22.3
Sex dabblers—all have had sex	14.
Median no. of partners=1 60% used a condom at last sex	
60% used a condom at last sex Infrequent use of substances†	
Drinkers-all consumed alcohol in past 12 mos.	7,
49% report binge drinking	
Infrequent or no illicit drug use None have had sex	
Smokers—all smoke cigarettes daily	7.
infrequent use of alcohol/illicit drugs 62% have had sex	
Alcohol-and-sex dabblers—all drink occasionally; all have had sex Infrequent tobacco/illicit drug use	5.
Binge drinkers—all binge frequently	4
Infrequent cigarette, marijuana and other drug use	
60% binge ≥1 time/wk.	
45% have had sex	
Heavy dabblers—all smoke, drink and binge drink with moderate frequency	3/
45% use marijuana; few use other illicit drugs	
91% have had sex	
Combination sex and drug use—all have had sex; all used alcohol/illicit drug at last sex	3.
Marijuana users—all use marijuana frequently; few have used other illicit drugs	Ľ
94% use alcohol 79% smoke cigarettes	
74% have had sex	
Multiple partners—all report ≥14 sexual partners	1
75% report low or moderate use of substances†	
Sex for drugs or money—all have had sex for drugs or money	1.
50% report low or moderate use of substances†	
Median no. of partners=3	
High marijuana use and sex—all use marijuana frequently; all have had sex	1.
All used alcohol/other drug at last sex	
82% have had >1 partner (median=6)	
Marijuana and other drug users—95% report heavy marijuana use; all use other illicit drugs	0,
58% have had sex	
28% used alcohol/other drug at last sex	
injection-drug users—all have injected drugs	0.0
82% have had sex Median no. of partners=4	
,	
Males who have sex with males—all are males who have had sex with another male	0.3
78% have had multiple partners (median=5) 40% used mariluana in past 30 days	
40% used manjuana in past 30 days 50% use alcohol >1 time/mo	
17% have had sex for drugs or money	

D. Blei

Clustering 01

Summary