Radix Sorts

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- application: LRS

References: Algorithms in Java, Chapter 10
Intro to Algs and Data Structs, Section 6.1

Review: summary of the performance of sorting algorithms

Frequency of execution of instructions in the inner loop:

algorithm	guarantee	average	extra space	assumptions on keys
insertion sort	$N^2/2$	$N^2/4$	no	Comparable
selection sort	$N^2/2$	$N^2/2$	no	Comparable
mergesort	N lg N	N lg N	N	Comparable
quicksort	1.39 N lg N	1.39 N lg N	c lg N	Comparable

lower bound: N lg N -1.44 N compares are required by any algorithm

Q: Can we do better (despite the lower bound)?

Digital keys

Many commonly-use key types are inherently digital
(sequences of fixed-length characters)

Examples
- Strings
- 64-bit integers

interface

```
interface Digital
{
   public int charAt(int k);
   public int length(int);
   static int R();
}
```

This lecture:
- refer to fixed-length vs. variable-length strings
- `R` different characters
- key type implements `charAt()` and `length()` methods
- code works for `String` and key types that implement `Digital`.

Widely used in practice
- low-level bit-based sorts
- string sorts

key-indexed counting
LSD radix sort
MSD radix sort
3-way radix quicksort
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Key-indexed counting: assumptions about keys

Assume that keys are integers between 0 and R-1

Examples:

- char (R = 256)
- short with fixed R, enforced by client
- int with fixed R, enforced by client

Reminder: equal keys are allowed in sorts

Applications:

- sort phone numbers by area code
- sort classlist by precept
- Requirement: sort must be stable
- Ex: Full sort on primary key, then stable radix sort on secondary key

Implication: Can use key as an array index

Key-indexed counting

Task: sort an array `a[]` of N integers between 0 and R-1

Plan: produce sorted result in array `temp[]`

1. Count frequencies of each letter using key as index

```
int N = a.length;
int[] count = new int[R];
```

count frequencies

```
for (int i = 0; i < N; i++)
   count[a[i]+1]++;
```

	a[]	temp[]
0	d	
1	a	
2	c	
3	f	
4	f	
5	b	
6	d	
7	b	
8	f	
9	b	
10	e	
11	a	

offset by 1 [stay tuned]

count[]	
a	0
b	2
c	3
d	1
e	2
f	1

2 d's

Key-indexed counting

Task: sort an array `a[]` of N integers between 0 and R-1

Plan: produce sorted result in array `temp[]`

1. Count frequencies of each letter using key as index
2. Compute frequency cumulates which specify destinations

```
int N = a.length;
int[] count = new int[R];
```

count frequencies

```
for (int i = 0; i < N; i++)
   count[a[i]+1]++;
```

compute cumulates

```
for (int i = 1; i < 256; i++)
   count[i] += count[i-1];
```

	a[]	temp[]
0	d	
1	a	
2	c	
3	f	
4	f	
5	b	
6	d	d
7	b	d
8	f	
9	b	
10	e	
11	a	

count[]	
a	0
b	2
c	5
d	6
e	8
f	9

6 keys < d, 8 keys < e
so d's go in a[6] and a[7]

Key-indexed counting

Task: sort an array `a[]` of N integers between 0 and R-1

Plan: produce sorted result in array `temp[]`

1. Count frequencies of each letter using key as index
2. Compute frequency cumulates which specify destinations
3. Access cumulates using key as index to move records.

```
int N = a.length;
int[] count = new int[R];
```

count frequencies

```
for (int i = 0; i < N; i++)
   count[a[i]+1]++;
```

compute cumulates

```
for (int i = 1; i < 256; i++)
   count[i] += count[i-1];
```

move records

```
for (int i = 0; i < N; i++)
   temp[count[a[i]]++] = a[i];
```

	a[]	temp[]
0	d	a
1	a	a
2	c	b
3	f	b
4	f	b
5	b	c
6	d	d
7	b	d
8	f	e
9	b	f
10	e	f
11	a	f

count[]	
a	2
b	5
c	6
d	8
e	9
f	12

Key-indexed counting

Task: sort an array `a[]` of N integers between 0 and R-1

Plan: produce sorted result in array `temp[]`

1. Count frequencies of each letter using key as index
2. Compute frequency cumulates
3. Access cumulates using key as index to find record positions.
4. Copy back into original array

```
int N = a.length;
int[] count = new int[R];

for (int i = 0; i < N; i++)        // count frequencies
   count[a[i]+1]++;

for (int k = 1; k < 256; k++)      // compute cumulates
   count[k] += count[k-1];

for (int i = 0; i < N; i++)        // move records
   temp[count[a[i]++]] = a[i]

for (int i = 0; i < N; i++)        // copy back
   a[i] = temp[i];
```

	a[]	temp[]
0	a	a
1	a	a
2	b	b
3	b	b
4	b	b
5	c	c
6	d	d
7	d	d
8	e	e
9	f	f
10	f	f
11	f	f

	count[]
a	2
b	5
c	6
d	8
e	9
f	12

key-indexed counting
LSD radix sort
MSD radix sort
3-way radix quicksort
application: LRS

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quicksort	1.39 N lg N	1.39 N lg N	c lg N	Comparable
key-indexed counting	N	N	N + R	integers

inplace version is possible and practical

Q: Can we do better (despite the lower bound)?
A: Yes, if we do not depend on comparisons

Least-significant-digit-first radix sort

LSD radix sort.

- Consider characters d from right to left
- Stably sort using dth character as the key via key-indexed counting.

	input	sort key (3rd char)	sort key (2nd char)	sort key (1st char)
0	dab	dab	dab	ace
1	add	cab	cab	add
2	cab	ebb	fad	bad
3	fad	add	bad	bed
4	fee	fad	dad	bee
5	bad	bad	ebb	cab
6	dad	dad	ace	dab
7	bee	fed	add	dad
8	fed	bed	fed	ebb
9	bed	fee	bed	fad
10	ebb	bee	fee	fed
11	ace	ace	bee	fee

sort must be stable
arrows do not cross

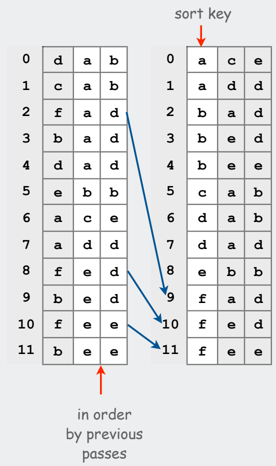
LSD radix sort: Why does it work?

Pf 1. [thinking about the past]

- If two strings differ on first character, key-indexed sort puts them in proper relative order.
- If two strings agree on first character, stability keeps them in proper relative order.

Pf 2. [thinking about the future]

- If the characters not yet examined differ, it doesn't matter what we do now.
- If the characters not yet examined agree, stability ensures later pass won't affect order.



LSD radix sort implementation

Use k-indexed counting on characters, moving right to left

```
public static void lsd(String[] a)
{
   int N = a.length;
   int W = a[0].length;
   for (int d = W-1; d >= 0; d--)
   {
      int[] count = new int[R];
      for (int i = 0; i < N; i++)
         count[a[i].charAt(d) + 1]++;
      for (int k = 1; k < 256; k++)
         count[k] += count[k-1];
      for (int i = 0; i < N; i++)
         temp[count[a[i].charAt(d)]++] = a[i];
      for (int i = 0; i < N; i++)
         a[i] = temp[i];
   }
}
```

key-indexed counting

count frequencies

compute cumulates

move records

copy back

Assumes fixed-length keys (length = W)

Review: summary of the performance of sorting algorithms

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mergesort	N lg N	N lg N	N	Comparable
quicksort	1.39 N lg N	1.39 N lg N	c lg N	Comparable
LSD radix sort	WN	WN	N + R	Digital

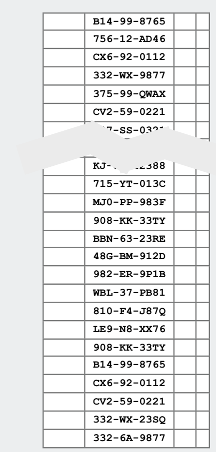
Sorting Challenge

Problem: sort a huge commercial database on a fixed-length key field

Ex: account number, date, SS number

Which sorting method to use?

1. insertion sort
2. mergesort
3. quicksort
4. LSD radix sort



Sorting Challenge

Problem: sort a huge commercial database on a fixed-length key field
Ex: account number, date, SS number

Which sorting method to use?
1. insertion sort
2. mergesort
3. quicksort
4. LSD radix sort ✓

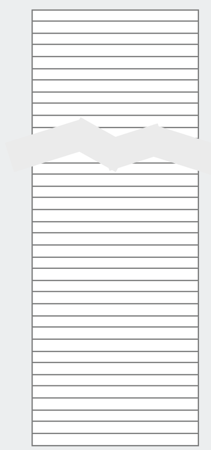
B14-99-8765
756-12-AD46
CX6-92-0112
332-WX-9877
375-99-QWAX
CV2-59-0221
7-SS-032
KJ-...388
715-YT-013C
MJ0-PP-983F
908-KK-33TY
BBN-63-23RE
48G-BM-912D
982-ER-9P1B
WBL-37-PB81
810-F4-J87Q
LE9-N8-XX76
908-KK-33TY
B14-99-8765
CX6-92-0112
CV2-59-0221
332-WX-23SQ
332-6A-9877

256 (or 65536) counters
Fixed-length strings
sort in W passes

Sorting Challenge

Problem: sort huge files of random 128-bit numbers
Ex: supercomputer sort, internet router

Which sorting method to use?
1. insertion sort
2. mergesort
3. quicksort
4. LSD radix sort



Sorting Challenge

Problem: sort huge files of random 128-bit numbers
Ex: supercomputer sort, internet router

Which sorting method to use?
1. insertion sort
2. mergesort
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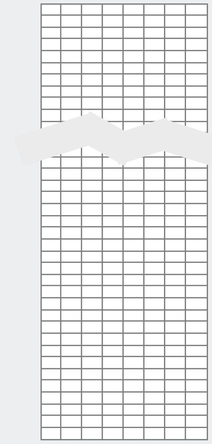
4. LSD radix sort ✓



2^{16} = 65536 counters
divide each word into 16-bit "chars"
sort in 8 passes

LSD radix sort: a moment in history (1960s)


card punch

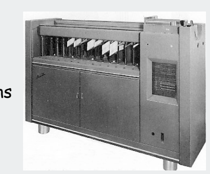

punched cards


card reader

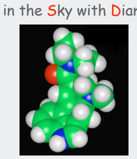

mainframe


line printer

To sort a card deck
1. start on right column
2. put cards into hopper
3. machine distributes into bins
4. pick up cards (stable)
5. move left one column
6. continue until sorted


card sorter

LSD not related to sorting
↓
"Lucy in the Sky with Diamonds"


Lysergic Acid Diethylamide

LSD radix sort actually predates computers

key-indexed counting
LSD radix sort
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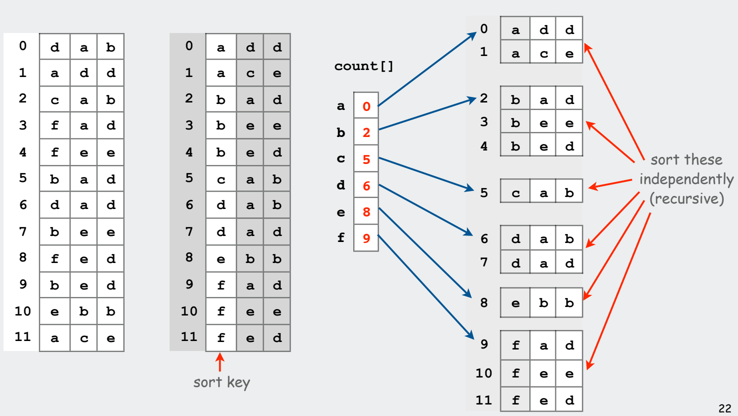
application: LRS

MSD Radix Sort

Most-significant-digit-first radix sort.

- Partition file into R pieces according to first character (use key-indexed counting)
- Recursively sort all strings that start with each character (key-indexed counts delineate files to sort)



MSD radix sort implementation

Use key-indexed counting on first character, recursively sort subfiles

```
public static void msd(String[] a)
{  msd(a. 0. a.length, 0);

private static void msd(String[] a, int l, int r, int d)
{
   if (r <= l + 1) return;
   int[] count = new int[R];
   for (int i = 0; i < N; i++)
      count[a[i].charAt(d) + 1]++;
   for (int k = 1; k < 256; k++)
      count[k] += count[k-1];
   for (int i = 0; i < N; i++)
      temp[count[a[i].charAt(d)]++] = a[i];
   for (int i = 0; i < N; i++)
      a[i] = temp[i];
   for (int i = 0; i < 255; i++)
      msd(a, l + count[i], l + count[i+1], d+1);
}
```

key-indexed counting

count frequencies

compute cumulates

move records

copy back

MSD radix sort: potential for disastrous performance

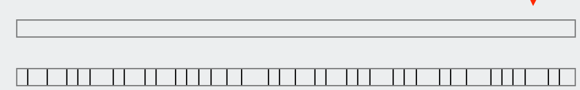
Observation 1: Much too slow for small files

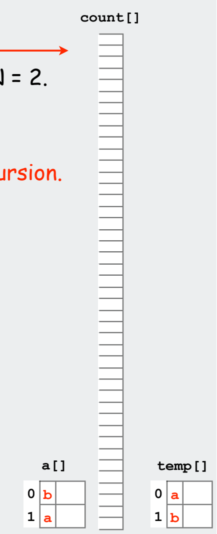
- all counts must be initialized to zero
- ASCII (256 counts): 100x slower than copy pass for N = 2.
- Unicode (65536 counts): 30,000x slower for N = 2

Observation 2: Huge number of small files because of recursion.

- keys all different: up to N/2 files of size 2
- ASCII: 100x slower than copy pass for all N.
- Unicode: 30,000x slower for all N

switch to Unicode might be a big surprise!





Solution. Switch to insertion sort for small N.

MSD radix sort bonuses

Bonus 1: May not have to examine all of the keys.

0	a	c	e
1	a	d	d
2	b	a	d
3	b	e	d
4	b	e	e
5	c	a	b
6	d	a	b
7	d	a	d

← 19/24 ≈ 80% of the characters examined

Bonus 2: Works for variable-length keys (Strings)

0	a	c	e	t	o	n	e	\0		
1	a	d	d	i	t	i	o	n	\0	
2	b	a	d	g	e	\0				
3	b	e	d	a	z	z	l	e	d	\0
4	b	e	e	h	i	v	e	\0		
5	c	a	b	i	n	e	t	r	y	\0
6	d	a	b	b	l	e	\0			
7	d	a	d	\0						

← 19/64 ≈ 30% of the characters examined

Implication: sublinear sorts (!)

MSD string sort implementation

Use key-indexed counting on first character, recursively sort subfiles

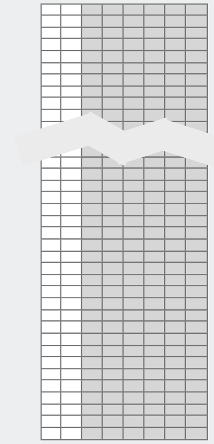
```
public static void msd(String[] a)
{  msd(a. 0. a.length, 0);

private static void msd(String[] a, int l, int r, int d)
{
   if (r <= l + 1) return;
   int[] count = new int[256];
   for (int i = 0; i < N; i++)
      count[a[i].charAt(d) + 1]++;
   for (int k = 1; k < 256; k++)
      count[k] += count[k-1];
   for (int i =0; i < N; i++)
      temp[count[a[i].charAt(d)]++] = a[i];
   for (int i = 0; i < N; i++)
      a[i] = temp[i];
   for (int i = 1; i < 255; i++)
      msd(a, l + count[i], l + count[i+1], d+1);
}
```

don't sort strings that start with '\0' (end of string char)

Sorting Challenge (revisited)

Problem: sort huge files of random 128-bit numbers
Ex: supercomputer sort, internet router

Which sorting method to use?
1. insertion sort
2. mergesort
3. quicksort
✓ 4. LSD radix sort on MSDs



2^{16} = 65536 counters
divide each word into 16-bit "chars"
sort on leading 32 bits in 2 passes
finish with insertion sort
examines only ~25% of the data

MSD radix sort versus quicksort for strings

Disadvantages of MSD radix sort.
- Accesses memory "randomly" (cache inefficient)
- Inner loop has a lot of instructions.
- Extra space for counters.
- Extra space for temp (or complicated inplace key-indexed counting).

Disadvantage of quicksort.
- N lg N, not linear.
- Has to rescan long keys for compares
- [but stay tuned]

key-indexed counting
LSD radix sort
MSD radix sort
3-way radix quicksort
application: LRS

3-Way radix quicksort (Bentley and Sedgewick, 1997)

Idea. Do 3-way partitioning on the dth character.
- cheaper than R-way partitioning of MSD radix sort
- need not examine again chars equal to the partitioning char

qsortX(0, 12, 0)

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	a	c	e
10	e	b	b
11	b	e	d

partition 0th char on b

0	b	e	e
1	b	a	d
2	a	c	e
3	a	d	d
4	f	e	e
5	f	a	d
6	d	a	d
7	c	a	b
8	f	e	d
9	d	a	b
10	e	b	b
11	b	e	d

swap b's to ends as in 3-way quicksort

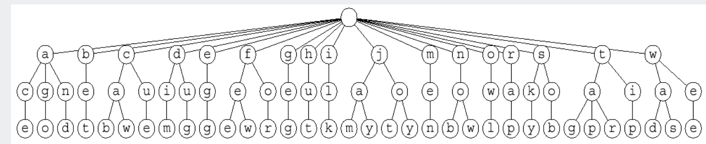
0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	f	a	d
6	d	a	d
7	c	a	b
8	f	e	d
9	d	a	b
10	e	b	b
11	f	e	e

3-way partition on b

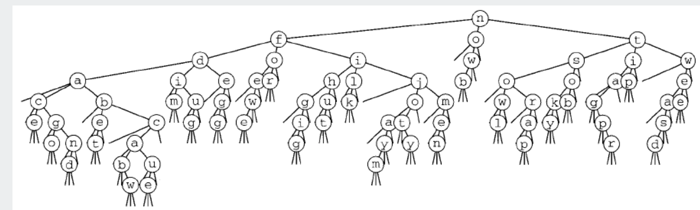
0	a	d	d	qsortX(0, 2, 0)
1	a	c	e	
2	b	a	d	qsortX(2, 5, 1)
3	b	e	e	
4	b	e	d	
5	f	a	d	qsortX(5, 12, 0)
6	d	a	d	
7	c	a	b	
8	f	e	d	
9	d	a	b	
10	e	b	b	
11	f	e	e	

Recursive structure: MSD radix sort vs. 3-Way radix quicksort

3-way radix quicksort collapses empty links in MSD recursion tree.



MSD radix sort recursion tree
(1035 null links, not shown)



3-way radix quicksort recursion tree
(155 null links)

3-Way radix quicksort

```
private static void quicksortX(String a[], int lo, int hi, int d)
{
   if (hi - lo <= 0) return;
   int i = lo-1, j = hi;
   int p = lo-1, q = hi;
   char v = a[hi].charAt(d);
   while (i < j)
   {
      while (a[++i].charAt(d) < v) if (i == hi) break;
      while (v < a[--j].charAt(d)) if (j == lo) break;
      if (i > j) break;
      exch(a, i, j);
      if (a[i].charAt(d) == v) exch(a, ++p, i);
      if (a[j].charAt(d) == v) exch(a, j, --q);
   }

   if (p == q)
   {
      if (v != '\0') quicksortX(a, lo, hi, d+1);
      return;
   }

   if (a[i].charAt(d) < v) i++;
   for (int k = lo; k <= p; k++) exch(a, k, j--);
   for (int k = hi; k >= q; k--) exch(a, k, i++);

   quicksortX(a, lo, j, d);
   if ((i == hi) && (a[i].charAt(d) == v)) i++;
   if (v != '\0') quicksortX(a, j+1, i-1, d+1);
   quicksortX(a, i, hi, d);
}
```

4-way partition with equals at ends

special case for all equals

swap equals back to middle

sort 3 pieces recursively

3-Way Radix quicksort vs. standard quicksort

standard quicksort.

- uses 2N ln N string comparisons on average.
- uses costly compares for long keys that differ only at the end, and this is a common case!

3-way radix quicksort.

- avoids re-comparing initial parts of the string.
- adapts to data: uses just "enough" characters to resolve order.
- uses 2 N ln N character comparisons on average for random strings.
- is sub-linear when strings are long

Theorem. Quicksort with 3-way partitioning is OPTIMAL.
No sorting algorithm can examine fewer chars on any input

asymptotically

Pf. Ties cost to entropy. Beyond scope of 226.

key-indexed counting
LSD radix sort
MSD radix sort
3-way radix quicksort
application: LRS

3-Way Radix quicksort vs. MSD radix sort

MSD radix sort

- has a long inner loop
- is cache-inefficient
- repeatedly initializes counters for long stretches of equal chars, and this is a common case!

Ex. Library call numbers

```
WUS--------10706-----7---10
WUS--------12692-----4---27
WLSOC------2542----30
LTK--6015-P-63-1988
LDS---361-H-4
 ...
```

3-way radix quicksort

- uses one compare for equal chars.
- is cache-friendly
- adapts to data: uses just "enough" characters to resolve order.

3-way radix quicksort is the method of choice for sorting strings

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Ex:

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Ex:

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```


String processing

String. Sequence of characters.

Important fundamental abstraction

Natural languages, Java programs, genomic sequences, …

> The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. -M. V. Olson

Using Strings in Java

String concatenation: append one string to end of another string.

Substring: extract a contiguous list of characters from a string.

s	t	r	i	n	g	s
0	1	2	3	4	5	6

```
String s = "strings";          // s = "strings"
char   c = s.charAt(2);        // c = 'r'
String t = s.substring(2, 6);  // t = "ring"
String u = s + t;              // u = "stringsring"
```


Implementing Strings In Java

Memory. 40 + 2N bytes for a virgin string!

could use byte array instead of String to save space

```
public final class String implements Comparable<String>
{
   private char[] value;  // characters
   private int offset;    // index of first char into array
   private int count;     // length of string
   private int hash;      // cache of hashCode()

   private String(int offset, int count, char[] value)
   {
       this.offset = offset;
       this.count  = count;
       this.value  = value;
   }
   public String substring(int from, int to)
   {
      return new String(offset + from, to - from, value);  }
   …
}
```

java.lang.String

String vs. StringBuilder

`String`. [immutable] Fast substring, slow concatenation.
`StringBuilder`. [mutable] Slow substring, fast (amortized) append.

Ex. Reverse a string

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
         rev += s.charAt(i);
   return rev;
}
```

quadratic time

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

linear time

Warmup: longest common prefix

Given two strings, find the longest substring that is a prefix of both

p	r	e	f	i	x		
0	1	2	3	4	5	6	7
p	r	e	f	e	t	c	h

```
public static String lcp(String s, String t)
{
   int n = Math.min(s.length(), t.length());
   for (int i = 0; i < n; i++)
   {
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   }
   return s.substring(0, n);
}
```

linear time

Would be quadratic with `StringBuilder`
Lesson: cost depends on implementation

This lecture: need constant-time `substring()`, use `String`

Longest repeated substring

Given a string of N characters, find the longest repeated substring.

Classic string-processing problem.

Ex: `a a c a a g t t t a c a a g c`
(1 … 9)

Applications
- bioinformatics.
- cryptanalysis.

Brute force.
- Try all indices `i` and `j` for start of possible match, and check.
- Time proportional to $M N^2$, where M is length of longest match.

k k

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c

i j

Longest repeated substring

Suffix sort solution.
- form N suffixes of original string.
- sort to bring longest repeated substrings together.
- check LCP of adjacent substrings to find longest match

suffixes

0	a a c a a g t t t a c a a g c
1	a c a a g t t t a c a a g c
2	c a a g t t t a c a a g c
3	a a g t t t a c a a g c
4	a g t t t a c a a g c
5	g t t t a c a a g c
6	t t t a c a a g c
7	t t a c a a g c
8	t a c a a g c
9	a c a a g c
10	c a a g c
11	a a g c
12	a g c
13	g c
14	c

sorted suffixes

0	a a c a a g t t t a c a a g c
11	a a g c
3	a a g t t t a c a a g c
9	a c a a g c
1	a c a a g t t t a c a a g c
12	a g c
4	a g t t t a c a a g c
14	c
10	c a a g c
2	c a a g t t t a c a a g c
13	g c
5	g t t t a c a a g c
8	t a c a a g c
7	t t a c a a g c
6	t t t a c a a g c

Suffix Sorting: Java Implementation

```
public class LRS {
   public static void main(String[] args) {
      String s = StdIn.readAll();
      int N = s.length();

      String[] suffixes = new String[N];
      for (int i = 0; i < N; i++)
         suffixes[i] = s.substring(i, N);

      Arrays.sort(suffixes);

      String lrs = "";
      for (int i = 0; i < N - 1; i++) {
         String x = lcp(suffixes[i], suffixes[i+1]);
         if (x.length() > lrs.length()) lrs = x;
      }
      System.out.println(lrs);

   }
}
```

read input

create suffixes (linear time)

sort suffixes

find LCP

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```


Sorting Challenge

Problem: suffix sort a long string
Ex. Moby Dick ~1.2 million chars

Which sorting method to use?

1. insertion sort
2. mergesort
3. quicksort
4. LSD radix sort
5. MSD radix sort
6. ✓ 3-way radix quicksort

only if LRS is not long (!)

Suffix sort experimental results

algorithm	time to suffix-sort Moby Dick (seconds)
brute-force	36.000 (est.)
quicksort	9.5
LSD	not fixed-length
MSD	395
MSD with cutoff	6.8
3-way radix quicksort	2.8

Suffix Sorting: Worst Case Input

Longest match not long.
- hard to beat 3-way radix quicksort.

Longest match very long.
- radix sorts are quadratic in the length of the longest match
- Ex: two copies of Moby Dick.

Can we do better? $\Theta(N \log N)$? $\Theta(N)$?

Observation. Must find longest repeated substring while suffix sorting to beat N^2.

```
abcdefghi
abcdefghiabcdefghi
bcdefghi
bcdefghiabcdefghi
cdefghi
cdefghiabcdefgh
defghi
efghiabcdefghi
efghi
fghiabcdefghi
fghi
ghiabcdefghi
fhi
hiabcdefghi
hi
iabcdefghi
i
```

Input: "abcdeghiabcdefghi"

Fast suffix sorting

Manber's MSD algorithm.

- phase 0: sort on first character using key-indexed sort.
- phase i: given list of suffixes sorted on first 2^{i-1} characters, create list of suffixes sorted on first 2^i characters in linear time
- finishes after lg N phases.
- total time proportional to N lg N.

Best in theory. O(N) but more complicated to implement.

Linearithmic suffix sort: key idea

		index sort		inverse	
0	babaaaabcbabaaaaa0	17	0babaaaabcbabaaaaa	0	14
1	abaaaabcbabaaaaa0b	16	a0babaaaabcbabaaaa	1	9
2	baaaabcbabaaaaa0ba	15	aa0babaaaabcbabaaa	2	12
3	aaaabcbabaaaaa0bab	14	aaa0babaaaabcbabaa	3	4
4	aaabcbabaaaaa0baba	3	aaaabcbabaaaaa0bab	4	7
5	aabcbabaaaaa0babaa	12	aaaaa0babaaaabcbab	5	8
6	abcbabaaaaa0babaaa	13	aaaa0babaaaabcbaba	6	11
7	bcbabaaaaa0babaaaa	4	aaabcbabaaaaa0baba	7	16
8	cbabaaaaa0babaaaab	5	aabcbabaaaaa0babaa	8	17
9	babaaaaa0babaaaabc	1	abaaaabcbabaaaaa0b	9	15
10	abaaaaa0babaaaabcb	10	abaaaaa0babaaaabcb	10	10
11	baaaaa0babaaaabcba	6	abcbabaaaaa0babaaa	11	13
12	aaaaa0babaaaabcbab	2	baaaabcbabaaaaa0ba	12	5
13	aaaa0babaaaabcbaba	11	baaaaa0babaaaabcba	13	6
14	aaa0babaaaabcbabaa	0	babaaaabcbabaaaaa0	14	3
15	aa0babaaaabcbabaaa	9	babaaaaa0babaaaabc	15	2
16	a0babaaaabcbabaaaa	7	bcbabaaaaa0babaaaa	16	1
17	0babaaaabcbabaaaaa	8	cbabaaaaa0babaaaab	17	0

0 + 4 = 4

9 + 4 = 13

sorted

can compare strings by indexing into inverse

Suffix sort experimental results

algorithm	time to suffix-sort Moby Dick (seconds)	time to suffix-sort AesopAesop (seconds)
brute-force	36.000 (est.)	4000 (est.)
quicksort	9.5	167
MSD	395	out of memory
MSD with cutoff	6.8	162
3-way radix quicksort	2.8	400
Manber MSD	17	8.5

counters in deep recursion

only 2 keys in subfiles with long matches

Radix sort summary

We can develop linear-time sorts.

- comparisons not necessary for some types of keys
- use keys to index an array

We can develop sub-linear-time sorts.

- should measure amount of data in keys, not number of keys
- not all of the data has to be examined

No algorithm can examine fewer bits than 3-way radix quicksort

- 1.39 N lg N bits for random data

Long strings are rarely random in practice.

- goal is often to learn the structure!
- may need specialized algorithms

lecture acronym cheatsheet	
LSD	least significant digit
MSD	most significant digit
LCP	longest common prefix
LRS	longest repeated substring