More Curves and Surfaces

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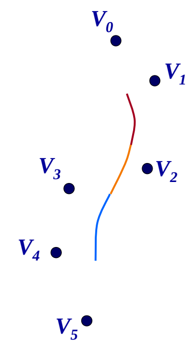
Uniform Cubic B-Splines

- Properties:
  - Local control
  - C^2 continuity
  - Approximating



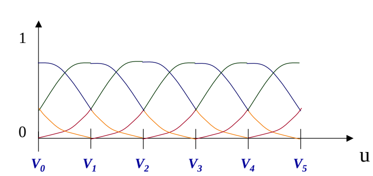
Types of Spline Curves

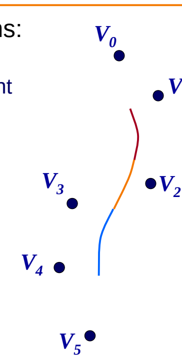
- Splines covered in last lecture
  - Hermite
  - Bezier
  - Catmull-Rom
  - ➢ B-Spline

Each has different blending functions resulting in different properties

B-Spline Blending Functions

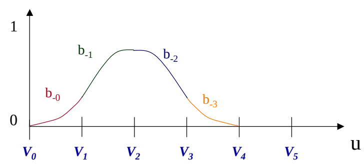
- Properties imply blending functions:
  - Cubic polynomials
  - Four control vertices affect each point
  - C^2 continuity

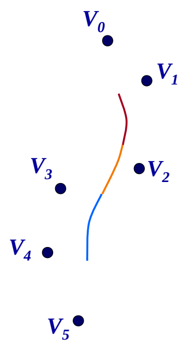




B-Spline Blending Functions

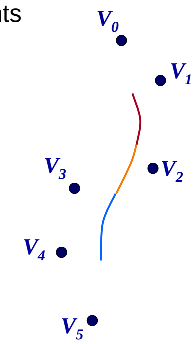
- How derive blending functions?
  - Cubic polynomials
  - Local control
  - C^2 continuity





B-Spline Blending Functions

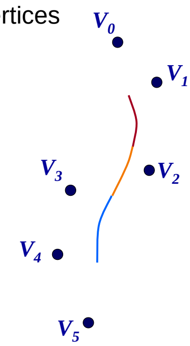
- C2 continuity implies 15 constraints
  - Position of two curves same
  - Derivative of two curves same
  - Second derivatives same



B-Spline Blending Functions

- Four cubic polynomials for four vertices
  - 16 variables (degrees of freedom)
  - Variables are a_i, b_i, c_i, d_i for four blending functions

$$b_{-0}(u) = a_0u^3 + b_0u^2 + c_0u^1 + d_0$$
$$b_{-1}(u) = a_1u^3 + b_1u^2 + c_1u^1 + d_1$$
$$b_{-2}(u) = a_2u^3 + b_2u^2 + c_2u^1 + d_2$$
$$b_{-3}(u) = a_3u^3 + b_3u^2 + c_3u^1 + d_3$$



B-Spline Blending Functions

Fifteen continuity constraints:

$$0 = b_{-0}(0) \qquad 0 = b_{-0}'(0) \qquad 0 = b_{-0}''(0)$$
$$b_{-0}(1) = b_{-1}(0) \qquad b_{-0}'(1) = b_{-1}'(0) \qquad b_{-0}''(1) = b_{-1}''(0)$$
$$b_{-1}(1) = b_{-2}(0) \qquad b_{-1}'(1) = b_{-2}'(0) \qquad b_{-1}''(1) = b_{-2}''(0)$$
$$b_{-2}(1) = b_{-3}(0) \qquad b_{-2}'(1) = b_{-3}'(0) \qquad b_{-2}''(1) = b_{-3}''(0)$$
$$b_{-3}(1) = 0 \qquad b_{-3}'(1) = 0 \qquad b_{-3}''(1) = 0$$

One more convenient constraint:

$$b_{-0}(0) + b_{-1}(0) + b_{-2}(0) + b_{-3}(0) = 1$$

B-Spline Blending Functions

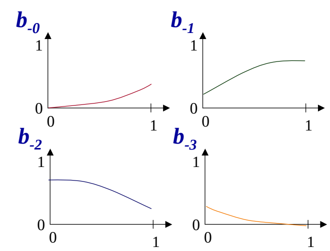
- Solving the system of equations yields:

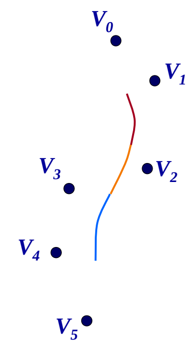
$$b_{-3}(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$b_{-2}(u) = \tfrac{1}{2}u^3 - u^2 + \tfrac{2}{3}$$
$$b_{-1}(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$b_{-0}(u) = \tfrac{1}{6}u^3$$

B-Spline Blending Functionss

In plot form:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$





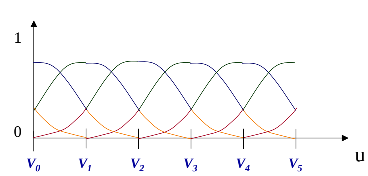
B-Spline Blending Functions

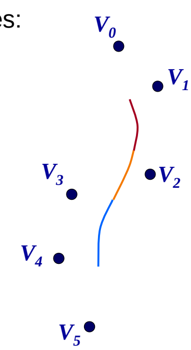
- In matrix form:

$$Q(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

B-Spline Blending Functions

- Blending functions imply properties:
  - Local control
  - Approximating
  - C^2 continuity
  - Convex hull





Curved Surfaces

- Motivation
  - Exact boundary representation for some objects
  - More concise representation than polygonal mesh

H&B Figure 10.46

Curved Surface Representations

- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Curved Surfaces

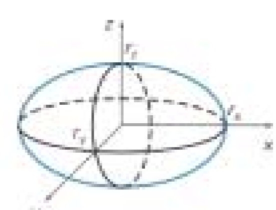
- What makes a good surface representation?
  - Accurate
  - Concise
  - Intuitive specification
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed continuity
  - Natural parameterization
  - Efficient display
  - Efficient intersections

Parametric Surfaces

- Boundary defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

- Example: ellipsoid

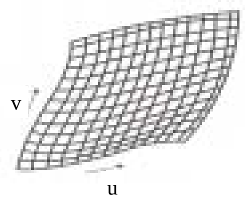
$$x = r_x \cos\phi \cos\theta$$
$$y = r_y \cos\phi \sin\theta$$
$$z = r_z \sin\phi$$



H&B Figure 10.10

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface

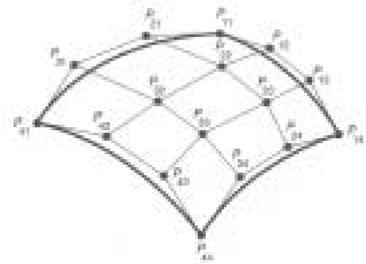


- Problem:
  - Need piecewise-parametrics surfaces to describe complex shapes

FvDFH Figure 11.42

Parametric Patches

- Each patch is defined by blending control points

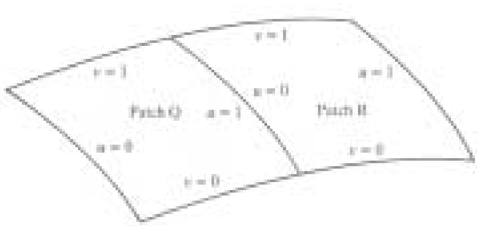


Same ideas as parametric curves!

FvDFH Figure 11.44

Piecewise Parametric Surfaces

- Surface is partitioned into parametric patches:

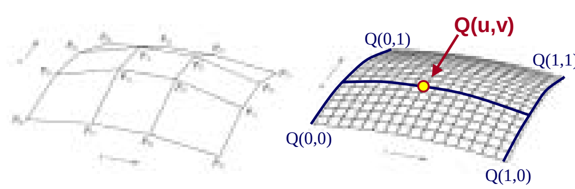


Same ideas as parametric splines!

Watt Figure 6.25

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{U}\mathbf{M}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}$$

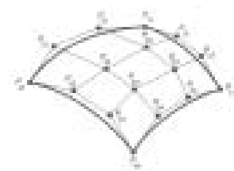
$$\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bezier, B-spline, etc.)

Bezier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathrm{Bezier}}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathrm{T}}_{\mathrm{Bezier}}\mathbf{V}$$

$$\mathbf{M}_{\mathrm{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

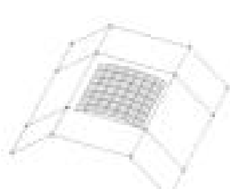


FvDFH Figure 11.42

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathrm{B-Spline}}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathrm{T}}_{\mathrm{B-Spline}}\mathbf{V}$$

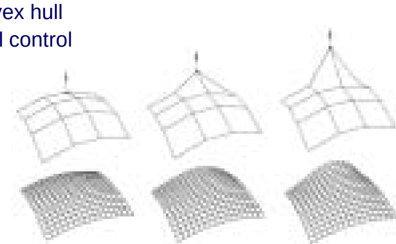
$$\mathbf{M}_{\mathrm{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$



Watt Figure 6.28

Bezier Patches

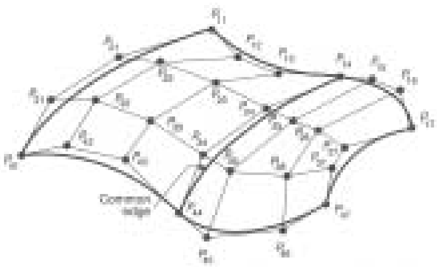
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

Bezier Surfaces

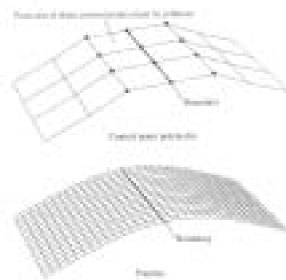
- Continuity constraints are similar to the ones for Bezier splines



FvDFH Figure 11.43

Bezier Surfaces

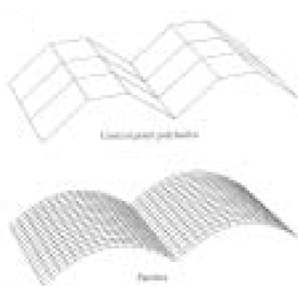
- C^1 continuity requires aligning boundary curves and derivatives



Watt Figure 6.26b

Bezier Surfaces

- C^0 continuity requires aligning boundary curves

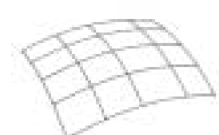


Watt Figure 6.26a

Drawing Bezier Surfaces

- Simple approach is to loop through uniformly spaced increments of u and v

```
DrawSurface(void)
{
    for (int i = 0; i < imax; i++) {
        float u = umin + i * ustep;
        for (int j = 0; j < jmax; j++) {
            float v = vmin + j * vstep;
            DrawQuadrilateral(...);
        }
    }
}
```

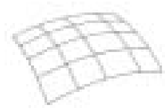


Watt Figure 6.32

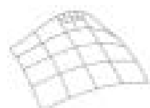
Drawing Bezier Surfaces

- Better approach is to use adaptive subdivision:

```
DrawSurface(surface)
{
    if  Flat (surface, epsilon) {
        DrawQuadrilateral(surface);
    }
    else {
        SubdivideSurface(surface, ...);
        DrawSurface(surfaceLL);
        DrawSurface(surfaceLR);
        DrawSurface(surfaceRL);
        DrawSurface(surfaceRR);
    }
}
```



Uniform subdivision



Adaptive subdivision

Watt Figure 6.32

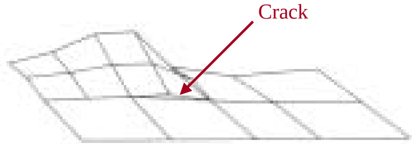
Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes
- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain
  - Hard to find intersections

Drawing Bezier Surfaces

- One problem with adaptive subdivision is avoiding cracks at boundaries between patches at different subdivision levels

Crack



Avoid these cracks by adding extra vertices and triangulating quadrilaterals whose neighbors are subdivided to a finer level

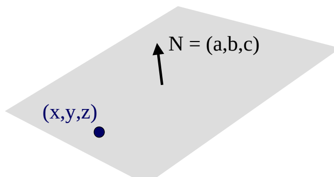
Watt Figure 6.33

Curved Surface Representations

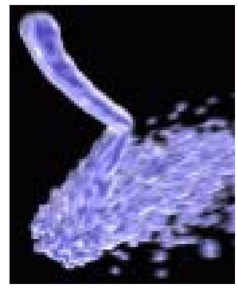
- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Implicit Surfaces

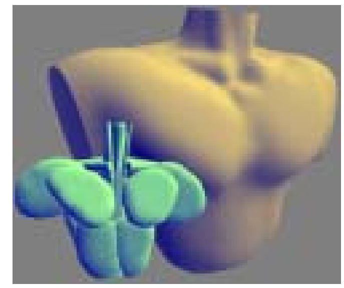
- Boundary defined by implicit function:
  - f (x, y, z) = 0
- Example: linear (plane)
  - ax + by +cz + d = 0



Implicit surface examples



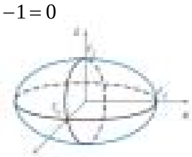
MaxMan Blobby Object



Skin [Markosian99]

Implicit Surfaces

- Example: quadric
  $f(x,y,z)=ax^2+by^2+cz^2+2dxy+2eyz+2fxz+2gx+2hy+2jz+k$
- Common quadric surfaces:
  - Sphere
  - Ellipsoid ⟶ $\left(\frac{x}{r_x}\right)^2+\left(\frac{y}{r_y}\right)^2+\left(\frac{z}{r_z}\right)^2-1=0$
  - Torus
  - Paraboloid
  - Hyperboloid



H&B Figure 10.10

Implicit Surfaces

- Advantages:
  - Easy to test if point is on surface
  - Easy to intersect two surfaces
  - Easy to compute z given x and y
- Disadvantages:
  - Hard to describe specific complex shapes
  - Hard to enumerate points on surface

Summary

Feature	Polygonal Mesh	Implicit Surface	Parametric Surface	Subdivision Surface
Accurate	No	Yes	Yes	Yes
Concise	No	Yes	Yes	Yes
Intuitive specification	No	No	Yes	No
Local support	Yes	No	Yes	Yes
Affine invariant	Yes	Yes	Yes	Yes
Arbitrary topology	Yes	No	No	Yes
Guaranteed continuity	No	Yes	Yes	Yes
Natural parameterization	No	No	Yes	No
Efficient display	Yes	No	Yes	Yes
Efficient intersections	No	Yes	No	No