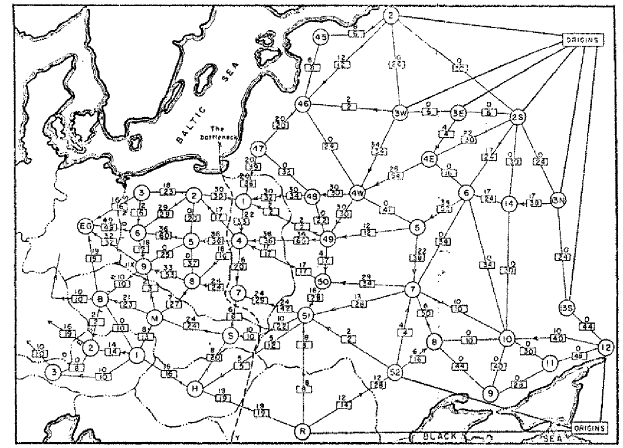
7. Network Flow

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in Math Programming, 91: 3, 2002.

Maximum Flow and Minimum Cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

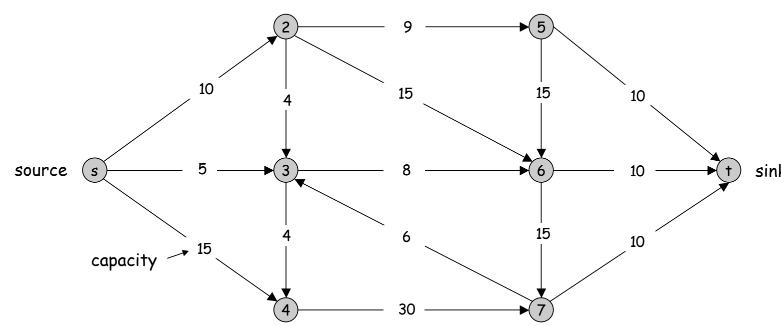
Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more . . .

Minimum Cut Problem

Flow network.

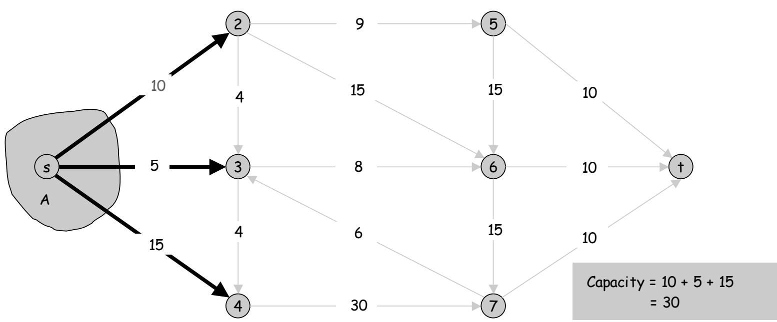
- Abstraction for material flowing through the edges.
- $G = (V, E)$ = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- $c(e)$ = capacity of edge e.



Cuts

Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



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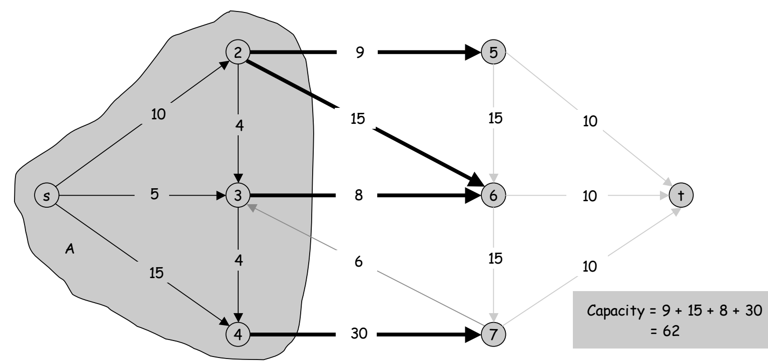
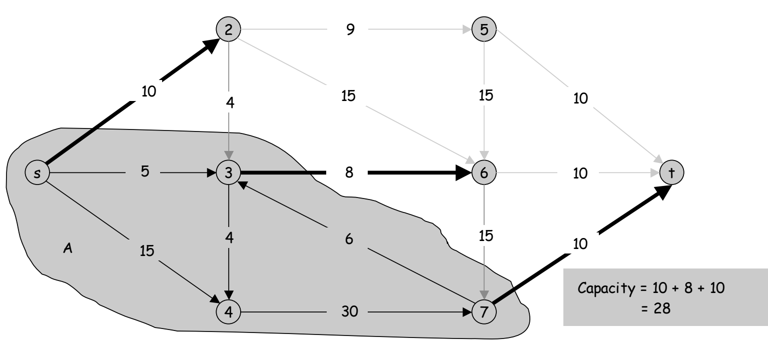


Minimum Cut Problem

Min s-t cut: find an s-t cut of minimum capacity.

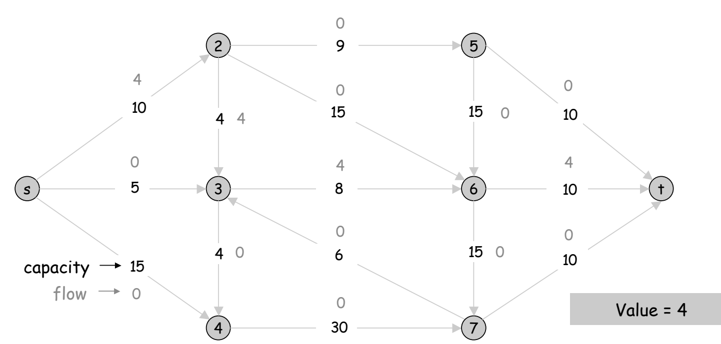


Flows

Def. An s-t flow is a function that satisfies:

- For each e ∈ E: $0 \leq f(e) \leq c(e)$ (capacity)
- For each v ∈ V – {s, t}: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



Flows

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- For each $e \in E$: $0 \le f(e) \le c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

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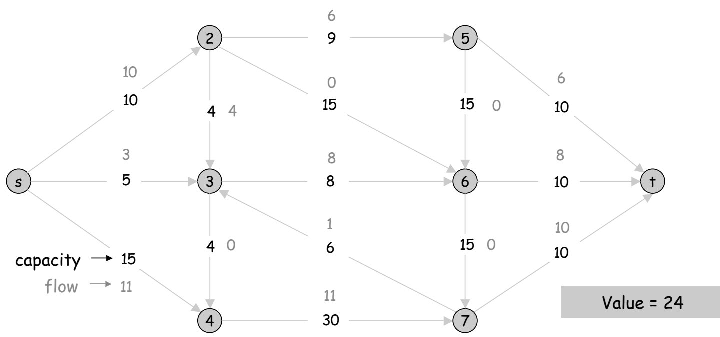
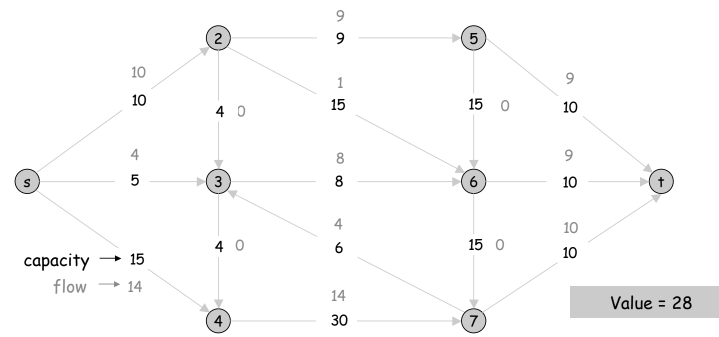


Maximum Flow Problem

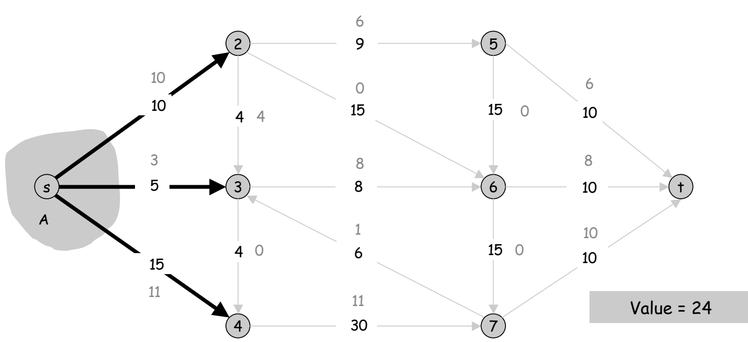
Max flow problem: find s-t flow of maximum value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



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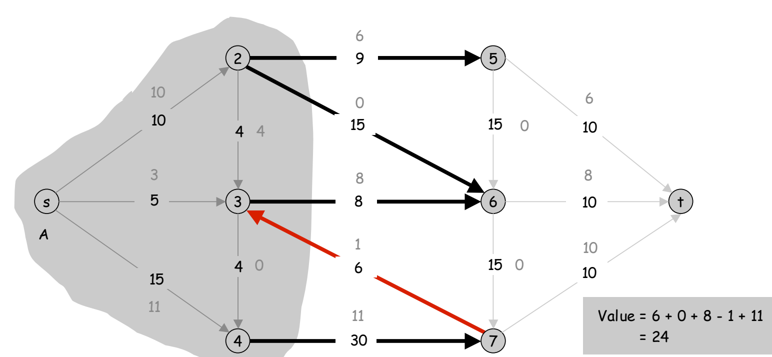

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

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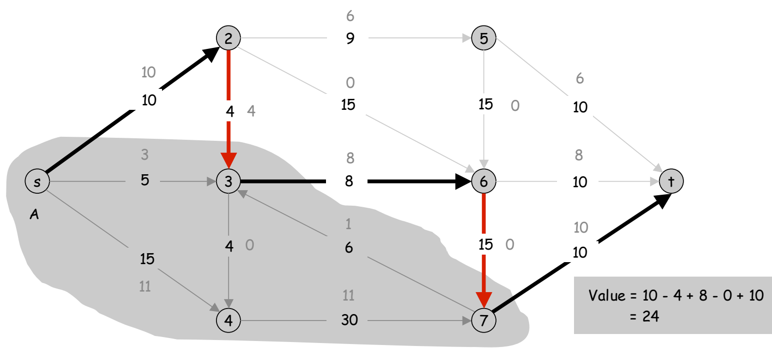


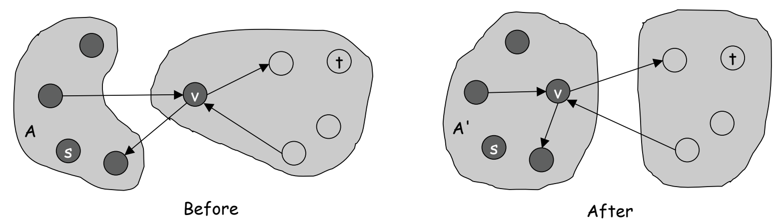
Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf. (by induction on |A|)

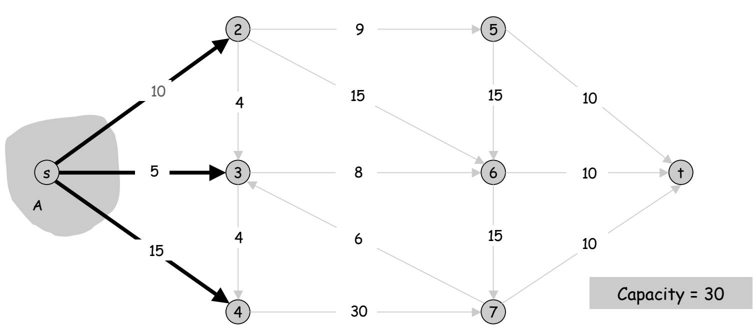
- Base case: A = { s }.
- Inductive hypothesis: assume true for all cuts (A, B) with |A| < k.
  - consider cut (A', B') with |A'| = k
  - A' = A ∪ { v } for some v ≠ {s, t}
  - By induction, cap(A, B) = v(f).
  - adding v to A increase cut capacity by $\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) = 0$ ▪



Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 ⇒ Flow value ≤ 30

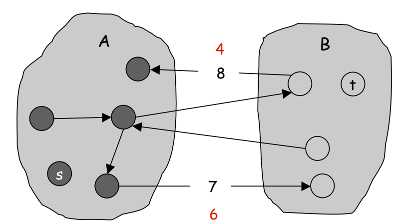


Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have v(f) ≤ cap(A, B).

Pf.

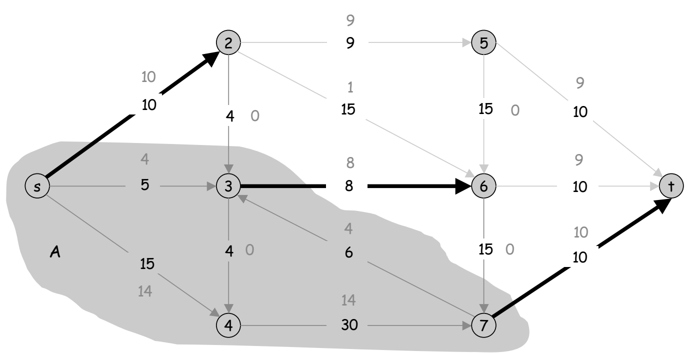
$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\le \sum_{e \text{ out of } A} f(e) \\ &\le \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \quad \blacksquare \end{aligned}$$



Certificate of Optimality

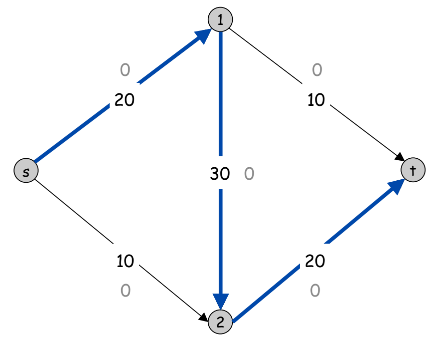
Corollary. Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e $\in$ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e $\in$ E.
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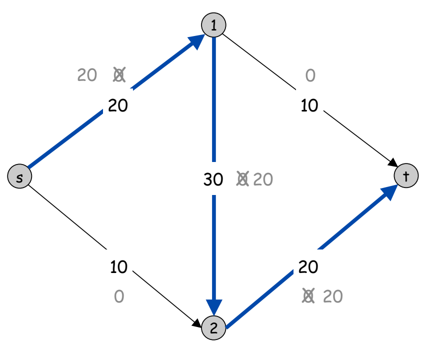
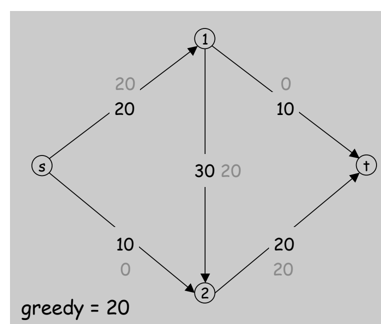


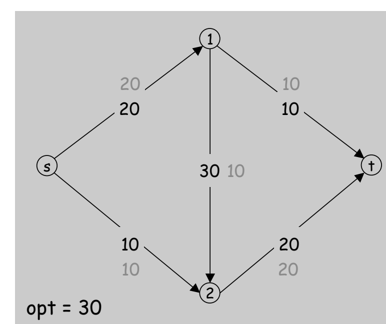
Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e $\in$ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.

locally optimality $\not\Rightarrow$ global optimality

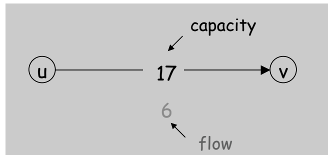




Residual Graph

Original edge: $e = (u, v) \in E$.

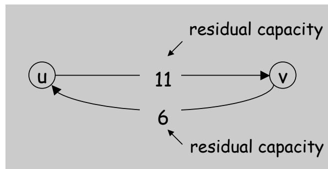
- Flow $f(e)$, capacity $c(e)$.



Residual edge.

- "Undo" flow sent.
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity:

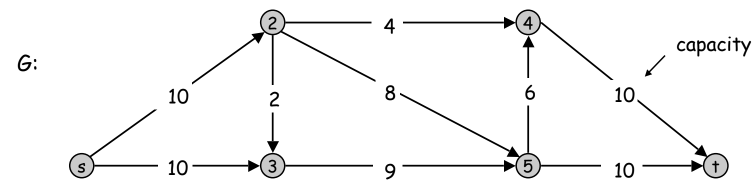
$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$



Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

Ford-Fulkerson Algorithm



Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson, 1956] The value of the max flow is equal to the value of the min cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

(i) There exists a cut (A, B) such that v(f) = cap(A, B).
(ii) Flow f is a max flow.
(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

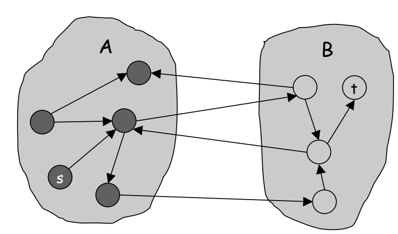
- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

(iii) $\Rightarrow$ (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, $s \in A$.
- By definition of f, $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= cap(A, B) \quad \blacksquare \end{aligned}$$



original network

Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b        forward edge
      else       f(eR) ← f(e) - b       reverse edge
   }
   return f
}
```

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Gf ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update Gf
   }
   return f
}
```


Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value f(e) and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \le nC$ iterations.
Pf. Each augmentation increase value by at least 1.

Corollary. If $C = 1$, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.
Pf. Since algorithm terminates, theorem follows from invariant.

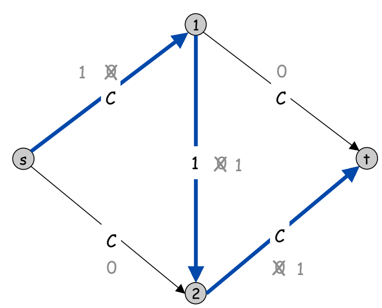
7.3 Choosing Good Augmenting Paths

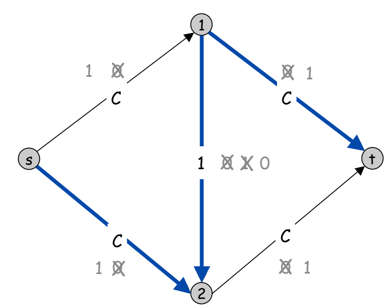
Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.





Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

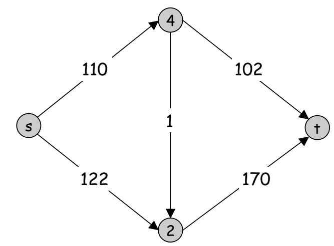
Choose augmenting paths with: [Edmonds-Karp, 1972]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

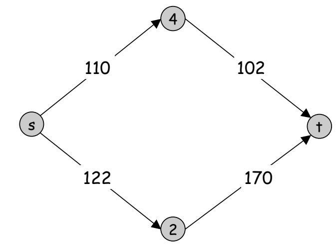
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ.



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Δ ← smallest power of 2 greater than or equal to C
   Gf ← residual graph

   while (Δ ≥ 1) {
      Gf(Δ) ← Δ-residual graph
      while (there exists augmenting path P in Gf(Δ)) {
         f ← augment(f, c, P)
         update Gf(Δ)
      }
      Δ ← Δ / 2
   }
   return f
}
```


Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.
Pf. Initially $C \le \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ▪

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most $v(f) + m\,\Delta$. ← proof on next slide

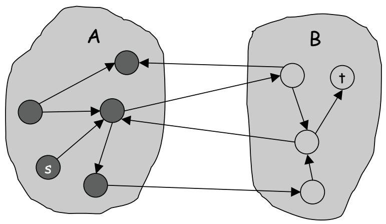
Lemma 3. There are at most 2m augmentations per scaling phase.
- Let f be the flow at the end of the previous scaling phase.
- L2 $\Rightarrow$ $v(f^*) \le v(f) + m\,(2\Delta)$.
- Each augmentation in a Δ-phase increases v(f) by at least Δ. ▪

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▪

Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then value of the maximum flow is at most $v(f) + m\,\Delta$.
Pf. (almost identical to proof of max-flow min-cut theorem)
- We show that at the end of a Δ-phase, there exists a cut (A, B) such that $cap(A, B) \le v(f) + m\,\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A, $s \in A$.
- By definition of f, $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\ge \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\ &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\ &\ge cap(A, B) - m\Delta \qquad ▪ \end{aligned}$$



original network