Quantum Property Testing[‡]

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Abstract

A language L has a property tester if there exists a probabilistic algorithm that given an input x only asks a small number of bits of x and distinguishes the cases as to whether x is in L and x has large Hamming distance from all y in L. We define a similar notion of quantum property testing and show that there exist languages with quantum property testers but no good classical testers. We also show there exist languages which require a large number of queries even for quantumly testing.

1 Introduction

Suppose we have a large data set, for example, a large chunk of the world-wide web or a genomic sequence. We would like to test whether the data has a certain property, but we may not have the time to look at the entire data set or even a large portion of it.

To handle these types of problems, Rubinfeld and Sudan [23] and Goldreich, Goldwasser and Ron [17] have developed the notion of property testing. Testable properties come in many varieties including graph properties, e.g., [17, 3, 13, 14, 1, 18], algebraic properties of functions [8, 23, 11], and regular languages [4]. Nice surveys of this area can be found in [22] [12].

In this model, the property tester has random access to the n input bits similar to the black-box oracle model. The tester can query only a small number of input bits; the set of indices is usually of constant size and chosen probabilistically. Clearly we cannot determine from these small number of bits whether the input sits in some language L. However, for many languages we can distinguish the case that the input is in L from the case that the input differs from all inputs in L of the same length by some constant fraction of input bits.

Since there are many examples where quantum computation gives us an advantage over classical computation [7, 25, 24, 19] one may naturally ask whether using quantum computation may lead to better property testers. By using the quantum oracle-query model developed by Beals, Buhrman, Cleve, Mosca and de Wolf [5] we can easily extend the definitions of property testing to the quantum setting.

Beals et al. [5] have shown that for all total functions we have a polynomial relationship between the number of queries required by quantum machine and that needed by a deterministic machine.

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For greater separations one needs to impose a promise on the input. The known examples, such as those due to Simon [25] and Bernstein and Vazirani [7], require considerable structure in the promise. Property testing amounts to the natural promise of either being in the language or far from each input in the language. This promise would seem to have too little structure to give a separation but in fact we can prove that quantum property testing can greatly improve on classical testing.

We show that every subset of Hadamard codes has a quantum property tester with $O(1)$ queries and that most subsets would require $\Theta(\log n)$ queries to test with a probabilistic tester. This shows that indeed quantum property testers are more powerful than classical testers. Moreover, we also give an example of a language where the quantum tester is exponentially more efficient.

Beals et al. observed that every k-query quantum algorithm gives rise to a degree-$2k$ polynomial in the input bits, which gives the acceptance probability of the algorithm; thus, a quantum property tester for P gives rise to a polynomial that is on all binary inputs between 0 and 1, that is at least $2/3$ on inputs with the property P and at most $1/3$ on inputs far from having the property P. Szegedy [27] suggested to algebraically characterize the complexity of classical testing by the minimum degree of such polynomials; however, our separation results imply that there are for example properties, for which such polynomials have constant degree, but for which the best classical tester needs $\Omega(\log n)$ queries. Hence, the minimum degree is only a lower bound, which sometimes is not tight.

A priori it is conceivable that every language has a quantum property tester with a small number of queries. We show that this is not the case. We prove that for most properties of a certain size, every quantum algorithm requires $\Omega(n)$ queries. We then show that a natural property, namely, the range of a d-wise independent pseudorandom generator cannot be quantumly tested with less than $(d+1)/2$ queries for every odd $d \leq n/\log n - 1$.

2 Preliminaries

We will use the following formal definition of property testing from Goldreich [16]:

Definition 1 *Let S be a finite set, and P a set of functions mapping S to $\{0,1\}$. A property tester for P is a probabilistic oracle machine M, which given a distance parameter $\varepsilon > 0$ and oracle access to a function $f : S \to \{0,1\}$, satisfies the following conditions:*

1. *the tester accepts f if it is in P: if $f \in P$ then $\Pr(M^f(\varepsilon) = 1) \geq 2/3$*
2. *the tester rejects f if it is far from P: if $|\{x \in S : f(x) \neq g(x)\}| > \varepsilon \cdot |S|$, for every $g \in P$, then $\Pr(M^f(\varepsilon) = 1) \leq 1/3$.*

Here M^f denotes that the machine M is provided with the oracle for f.

Definition 2 *The complexity of the tester is the number of oracle queries it makes: A property P has an (ε, q)-*tester *if there is a tester for P that makes at most q oracle queries for distance parameter ε.*

We often consider a language $L \subseteq \{0,1\}^*$ as the family of properties $\{P_n\}$ with P_n the characteristic functions of the length-n strings from L, and analyze the query complexity $q = q(\varepsilon, n)$ asymptotically for large n.

To define quantum property testing we simply modify Definition 1 by allowing M to be a quantum oracle machine. We need to be careful to make sure our oracle queries are unitary operations. If $|f(x)| = |g(y)|$ for all $x, y \in S$ and $f, g \in P$, we use the oracle-query model by Beals, Buhrman, Cleve, Mosca and de Wolf [5]: we define the unitary transformation U_f that maps the basis state $|x, y, z\rangle$ to $|x, y \oplus f(x), z\rangle$ where $|x| = \lceil \log |S| \rceil$, $|y| = |f(x)|$ and $\oplus$ denotes bitwise exclusive or. In case there are x, y, f, g so that $|f(x)| \neq |g(y)|$, we define U_f as mapping $|x, l, y, z\rangle$ to $|x, l + |f(x)| \mod k, y \oplus 0^{k-|f(x)|} f(x), z\rangle$ where $k = \max\{|f(x)| : f \in P \text{ and } x \in S\}$, $|x| = \lceil \log |S| \rceil$, $|l| = \lceil \log k \rceil$, and $|y| = k$.

We recommend the book of Nielsen and Chuang [21] for background information on quantum computing.

3 Separating Quantum and Classical Property Testing

We show that there exist languages with $(\varepsilon, \mathrm{O}(1))$ quantum property testers that do not have $(\varepsilon, \mathrm{O}(1))$ classical testers.

Theorem 1 *There is a language L that is ε-testable by a quantum test with $\mathrm{O}(1/\varepsilon)$ number of queries but for which every probabilistic $1/3$-test requires $\Omega(\log n)$ queries.*

We use Hadamard codes to provide examples for Theorem 1:

Definition 3 *The* Hadamard code *of $y \in \{0,1\}^{\log n}$ is $x = h(y) \in \{0,1\}^n$ such that $x_i = y \cdot i$ where $y \cdot i$ denotes the inner product of two vectors $y, i \in \mathbb{F}_2^{\log n}$.*

Note: the Hadamard mapping $h : \{0,1\}^{\log n} \to \{0,1\}^n$ is one-to-one. Bernstein and Vazirani [7] showed that a quantum computer can extract y with one query to an oracle for the bits of x, whereas a classical probabilistic procedure needs $\Omega(\log n)$ queries. Based on this separation for a decision problem we construct for $A \subseteq \{0,1\}^{\log n}$ the property $P_A \subseteq \{0,1\}^n$,

$$P_A := \{x : \exists y \in A \text{ s.t. } x = h(y)\}.$$

Theorem 1 follows from the following two lemmas.

Lemma 2 *For every A, P_A has an $(\varepsilon, \mathrm{O}(1/\varepsilon))$ quantum tester. Furthermore, the test has one-sided error.*

Lemma 3 *For most A of size $|A| = n/2$, P_A requires $\Omega(\log n)$ queries for a probabilistic $1/3$-test, even for testers with two-sided error.*

Before we prove Lemma 2 we note that for every A, P_A can be tested by a one-sided algorithm with $\mathrm{O}(1/\varepsilon + \log n)$ queries even nonadaptively; hence, the result of Lemma 3 is tight. An $\mathrm{O}(1/\varepsilon \log n)$-test follows from Theorem 4 below. The slightly more efficient test is the following: First we query x_{2^i}, $i = 0, \ldots, \log n$. Note that if $x = h(y)$ then $y_i = x_{2^i}$ for $i = 0, \ldots, \log n$. Thus a candidate y for $x = h(y)$ is found. If $y \notin A$ then x is rejected. Then $k = \mathrm{O}(1/\varepsilon)$ times the following check is performed: a random index $i \in \{1, \ldots, n\}$ is chosen independently at random and if $x_i \neq y \cdot i$, then x is rejected. Otherwise, x is accepted. Clearly if x is rejected then $x \notin P_A$. It is easily verified that if x has Hamming distance more than εn from every z in P_A then with constant probability x is rejected.

Proof of Lemma 2. P_A can be checked with $\mathrm{O}(1/\varepsilon)$ queries on a quantum computer: The test is similar to the test above except that y can be found in $\mathrm{O}(1)$ queries: k times query for random i, j values x_i, x_j, and $x_{i\oplus j}$. If $x_i \oplus x_j \neq x_{i\oplus j}$ reject. $k = \mathrm{O}(1/\varepsilon)$ is sufficient to detect an input x that is εn-far from being a Hadamard codeword with high probability. Now run the Bernstein-Vazirani algorithm to obtain y. Accept if and only if $y \in A$. Obviously, if $x \in P_A$, the given procedure accepts, and if x is far from each $x' \in P_A$, then it is either far from being a Hadamard codeword or it is close to a Hadamard codeword $h(y')$ for a $y' \notin A$; note that in this case x is far from every $h(y)$, $y \in A$ as two distinct Hadamard codewords are of Hamming distance $n/2$. Thus, in this case the second part of the tester succeeds with high probability in finding y' and rejects because $y' \notin A$. We note also that this algorithm has one-sided error. □

Proof of Lemma 3. The lower bound makes use of the Yao principle [28]: let D be an arbitrary probability distribution on positive and negative inputs, i.e., on inputs that either belong to P_A or are εn-far from P_A. Then if every deterministic algorithm that makes at most q queries, errs with probability at least 1/8 with respect to input chosen according to D, then q is a lower bound on the number of queries of any randomized algorithm for testing P_A with error probability bounded by 1/8.

D will be the uniform distribution over Hadamard codewords of length n, namely, generated by choosing $y \in \{0,1\}^{\log n}$ uniformly at random and setting $x = h(y)$. Note that for any $A \subset \{0,1\}^{\log n}$, D is concentrated on positive and negative inputs as required, as two Hadamard codewords are of Hamming distance $n/2$ apart.

The lower bound will be established by a counting argument. We show that for a fixed tester that makes $q \leq (\log n)/2$ queries, the probability over random choices of A that the algorithm errs on at most 1/8 of the inputs is bounded from above by $1/(10T)$ where T is the number of such algorithms. By the union bound it follows that for most properties there is no such algorithm.

Indeed, let $A \subseteq \{0,1\}^{\log n}$ be chosen by picking independently each $i \in \{0,1\}^{\log n}$ to be in A with probability 1/2; this will not necessarily result in a set A of size $n/2$ but we can condition on the event that $|A| = n/2$ and will not lose much. Let $\mathcal{T}$ be any fixed deterministic decision tree performing at most q queries in every branch. Then let $c(\mathcal{T}) := \{y | \mathcal{T}(h(y)) = \text{accept}\}$ and let $\mu(\mathcal{T}) := |c(\mathcal{T})|/n$, i.e., $\mu(\mathcal{T})$ is the fraction of inputs that $\mathcal{T}$ accepts. Assume first that $\mu(\mathcal{T}) \leq 1/2$. Since for a random y we have $\Pr_y[\mathcal{T}(h(y)) = \text{accept}] = \mu(\mathcal{T}) \leq 1/2$, it follows by a Chernoff-type bound that $\Pr_A[|A \cap c(\mathcal{T})| \geq 3/4|A|] \leq 2^{-n/8}$. However, if $|A \cap c(\mathcal{T})| < 3/4|A|$ then $\mathcal{T}$ will be wrong on at least 1/4 of the positive inputs which is at least $n/8$ of all inputs. Hence, with probability at most $2^{-n/8}$, $\mathcal{T}$ will be correct on at least 7/8 of the inputs. If $\mu(\mathcal{T}) > 1/2$ the same reasoning shows that with probability of at most $1 - 2^{-n/8}$ it will err on at least a 1/4-fraction of the negative inputs. Hence, in total, for every fixed $\mathcal{T}$, $\Pr_A[\mathcal{T} \text{ is correct on at least 7/8 of the inputs}] \leq 2^{-n/8}$.

Now, let us bound from above the number of algorithms that make at most q queries. As an algorithm may be adaptive, it can be defined by $2^q - 1$ query positions for all queries on all branches and a Boolean function $f : \{0,1\}^q \to \{\text{accept}, \text{reject}\}$ of the decision made by the algorithm for the possible answers. Hence, there are at most $T \leq (2n)^{2^q}$ such algorithms. However, for $q < (\log n)/2$, we have $T \cdot 2^{-n/8} = \mathrm{o}(1)$, which shows that for most A as above, every ε-test that queries at most $(\log n)/2$ many queries has error probability of at least 1/8. Standard amplification techniques then imply that for some constant c every algorithm that performs $c \log n$ many queries has error at least 1/3. □

Theorem 4 *Let $P \subseteq \{0,1\}^n$ be a property with $|P| = s > 0$. For any $\varepsilon > 0$, P can be ε-tested by a one-sided classical algorithm using $\mathrm{O}((\log s)/\varepsilon)$ many queries.*

Proof. Denote the input by $y \in \{0,1\}^n$. Consider the following algorithm: query the input y in $k := \ln(3s^2)/\varepsilon$ random places; accept if there is at least one $x \in P$ consistent with the bits from the input and reject otherwise. Clearly, if $y \in P$, this algorithm works correctly.

If y is ε-far from each $x \in P$, then for every specific $x \in P$, $\Pr[x_i = y_i] \leq 1 - \varepsilon$ when choosing an $i \in [n]$ uniformly at random. With k indices chosen independently and uniformly at random, the probability for no disagreement with x becomes $(1-\varepsilon)^k \leq 1/(3s^2)$. Therefore, the probability that there is no disagreement for at least one of the s members of P is at most $1/(3s)$, so with probability $2/3$ for a y that is far from P, we will rule out every $x \in P$ as being consistent with y. $\square$

4 An Exponential Separation

In this section, we show that a quantum computer can be exponentially more efficient in testing certain properties than a classical computer.

Theorem 5 *There exists a language L that for every $\varepsilon = \Omega(1)$ is $(\varepsilon, \log n \log\log n)$ quantumly testable but every probabilistic $1/8$-test for L requires $n^{\Omega(1)}$ queries.*

The language that we provide is inspired by Simon's problem [25] and our quantum testing algorithm makes use of Brassard and Høyer's algorithm for Simon's problem [9]. Simon's problem is to find $s \in \{0,1\}^n \setminus \{0^n\}$ from a function-query oracle for some $f : \{0,1\}^n \to \{0,1\}^n$, such that $f(x) = f(y) \Leftrightarrow x = y \oplus s$. Simon proved that classically, $\Omega(2^{n/2})$ queries are required on average to find s, and gave a quantum algorithm for determining s with an expected number of queries that is polynomial in n; Brassard and Høyer improved the algorithm to worst-case polynomial time. Their algorithm produces in each run a z with $z \cdot s = 0$ that is linearly independent to all previously computed such zs. Essentially, our quantum tester uses this subroutine to try to extract information about s until it fails repeatedly. Høyer [20] and also Friedl et al. [15] analyzed this approach in group-theoretic terms, obtaining an alternative proof to Theorem 7.

In the following, let $N = 2^n$ denote the length of the binary string encoding a function $f : \{0,1\}^n \to \{0,1\}$. For $x \in \{0,1\}^n$ let $x[j]$ be the jth bit of x, i.e., $x = x[1] \ldots x[n]$; the inner product of $x, y \in \{0,1\}^n$ as vectors in $\mathbb{F}_2^n$ is $x \cdot y = \sum_{j=1}^n x[j]y[j] \mod 2$. We define

$$L := \{f \in \{0,1\}^N : \exists s \in \{0,1\}^n \setminus \{0^n\}\ \forall x \in \{0,1\}^n\ f(x) = f(x \oplus s)\}$$

Theorem 5 follows from the following two theorems.

Theorem 6 *Every classical $1/8$-tester for L must make $\Omega(\sqrt{N})$ queries, even when allowing two-sided error.*

Theorem 7 *There is a quantum property tester for L making $\mathrm{O}(\log N \log\log N)$ queries. Moreover, this quantum property tester makes all its queries nonadaptively.*

Proof of Theorem 6. We again apply the Yao principle [28] as in the proof of Lemma 3: we construct two distributions, P and U, on positive and at least $N/8$-far negative inputs, respectively, such that every deterministic adaptive decision tree $\mathcal{T}$ has error $1/2-\mathrm{o}(1)$ when trying to distinguish whether an input is chosen from U or P. Indeed, we will show a stronger statement: Let $\mathcal{T}$ be any deterministic decision tree. Let v be a vertex of $\mathcal{T}$. Let $\Pr_P(v)$ and $\Pr_U(v)$ be the probability the an input chosen according to P and U, respectively, is consistent with v. We will show that for every vertex v of $\mathcal{T}$ we have $|\Pr_P(v) - \Pr_U(v)| = \mathrm{o}(1)$; hence, $\mathcal{T}$ has error $1/2 - \mathrm{o}(1)$.

The distribution P is defined as follows: We first choose $s \in \{0,1\}^n$ at random. This defines a matching M_s of $\{0,1\}^n$ by matching x with $x \oplus s$. Now a function f_s is defined by choosing for each matched pair independently $f_s(x) = f_s(x \oplus s) = 1$ with probability $1/2$ and $f_s(x) = f_s(x \oplus s) = 0$ with probability $1/2$. Clearly, this defines a distribution that is concentrated on positive inputs. Note that it might be that by choosing different s's we end up choosing the same function, however, this will be considered different events in the probability space. Namely, the atomic events in P really are the pairs (s, f_s) as described above.

Now let U be the uniform distribution over all functions, namely, we select the function by choosing for each x independently $f(x) = 1$ with probability $1/2$ and 0 with probability $1/2$. Since every function has a nonzero probability, U is not supported exclusively on the negative instances. However, as we proceed to show, a function chosen according to U is $N/8$-far from having the property with very high probability, and hence U will be a good approximation to the desired distribution:

Definition 4 *For $f : \{0,1\}^n \to \{0,1\}$ and $s \in \{0,1\}^n$ we define $n_s := |\{x : f(x) = f(x \oplus s)\}|$.*

Lemma 8 *Let f be chosen according to U. Then $\Pr_U[\exists s \in \{0,1\}^n : n_s \geq N/8] \leq e^{-\Omega(N)}$.*

Proof. Let f be chosen according to U and $s \in \{0,1\}^n$. By a Chernoff bound we obtain $\Pr_U[n_s \geq N/8] \leq e^{-\Omega(N)}$. Together with the union bound over all s's this yields $\Pr_U[\exists s \in \{0,1\}^n : n_s \geq N/8] \leq 2^n \cdot e^{-\Omega(N)} \leq e^{-\Omega(N)}$. □

In particular, a direct consequence of Lemma 8 is that with probability $1 - e^{-\Omega(N)}$ an input chosen according to U will be $N/8$-far from having the property.

From the definition of U, we immediately obtain the following:

Lemma 9 *Let $\mathcal{T}$ be any fixed deterministic decision tree and let v be a vertex of depth d in $\mathcal{T}$. Then $\Pr_U[f$ is consistent with the path to $v] = 2^{-d}$.*

We now want to derive a similar bound as in the lemma for functions chosen according to P. For this we need the following definition for the event that after d queries, nothing has been learned about the hidden s:

Definition 5 *Let $\mathcal{T}$ be a deterministic decision tree and u a vertex in $\mathcal{T}$ at depth d. We denote the path from the root of $\mathcal{T}$ to u by $\mathrm{path}(u)$. Every vertex v in $\mathcal{T}$ defines a query position $x_v \in \{0,1\}^n$. For $f = f_s$ chosen according to P, we denote by B_u the event $B_u := \{(s, f_s) : s \neq x_v \oplus x_w$ for all $v, w \in \mathrm{path}(u)\}$.*

Lemma 10 *Let v be a vertex of depth d in a decision tree $\mathcal{T}$. Then $\Pr_P[B_v] \geq 1 - \binom{d-1}{2}/N$*

Proof. B_v does not occur if for some v, w on the path to v we have $s = x_v \oplus x_w$. As there are $d-1$ such vertices, there are at most $\binom{d-1}{2}$ pairs. Each of these pairs excludes exactly one s and there are N possible s's. □

Lemma 11 *Let v be a vertex of depth d in a decision tree $\mathcal{T}$ and let f be chosen according to P. Then* $\Pr_P[f$ *is consistent with* $v|B_v] = 2^{-d}$.

Proof. By the definition of P, f gets independently random values on vertices that are not matched. But if B_v occurs, then no two vertices along the path to v are matched and hence the claim follows. □

Now we can complete the proof of the theorem: assume that $\mathcal{T}$ is a deterministic decision tree of depth $d = \mathrm{o}(\sqrt{N})$ and let v be any leaf of $\mathcal{T}$. Then by Lemmas 10 and 11, we get that $\Pr_P[f \text{ is consistent with } v] = (1 - \mathrm{o}(1))2^{-d}$. On the other hand, let U' be the distribution on negative inputs defined by U conditioned on the event that the input is at least $N/8$-far from the property. Then by Lemmas 8 and 9 we get that $\Pr_{U'}[f \text{ is consistent with } v] = (1 - \mathrm{o}(1))2^{-d}$ and hence $\mathcal{T}$ has only o(1) bias of being right on every leaf. This implies that its error probability is $1/2 - \mathrm{o}(1)$. □

Proof of Theorem 7. We give a quantum algorithm making $\mathrm{O}(\log N \, \log\log N)$ queries to the quantum oracle for input $f \in \{0,1\}^N$. We will show that it accepts with probability 1 if $f \in L$ and rejects with high probability if the Hamming distance between f and every $g \in L$ is at least εN. Pseudo code for our algorithm is given on page 8; it consists of a classical main program SimonTester and a quantum subroutine SimonSampler adapted from Brassard and Høyer's algorithm for Simon's problem [9, Section 4]. The quantum gates used are the 2^n-dimensional Hadamard transform H_{2^n}, which applies

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

individually to each of n qubits, the quantum oracle query U_f, and classical reversible operations run in quantum superposition.

The following technical lemma captures the operation of the quantum subroutine SimonSampler.

Lemma 12 *When* SimonSampler *is passed k linearly independent vectors $z_1, \ldots, z_k$ so that all $i_j := \min\{i : z_j[i] = 1\}$ are distinct for $1 \le j \le k$, then the state $|\psi\rangle$ before the measurement is*

$$\frac{\sqrt{2^k}}{N} \sum_{x \in \{0,1\}^n} \sum_{\substack{y \in \{0,1\}^n \\ y[i_j]=0 \; \forall j \le k}} (-1)^{x \cdot y} |y\rangle |f(x)\rangle |x \cdot z_1\rangle \cdots |x \cdot z_k\rangle \ .$$

Proof. We follow the steps of subroutine SimonSampler when passed k linearly independent vectors $z_1, \ldots, z_k$ so that all $i_j := \min\{i : z_j[i] = 1\}$ are distinct for $1 \le j \le k$.

$$\begin{aligned}|0^n\rangle|0\rangle|0^k\rangle &\mapsto \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle|0\rangle|0^k\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle|f(x)\rangle|0^k\rangle \\ &\mapsto \frac{1}{N} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle|f(x)\rangle|0^k\rangle\end{aligned}$$

Procedure SimonTester

```
for $k = 0$ to $n - 1$ do
    $l \leftarrow 0$
    repeat
        $z \leftarrow$ SimonSampler$(z_1, \ldots, z_k)$
        $l \leftarrow l + 1$
    until $z \neq 0$ or $l > 2(\log n)/\varepsilon^2$
    if $z = 0$ then
        accept
    else
        $z_{k+1} \leftarrow z$
reject
```

Procedure SimonSampler$(z_1, \ldots, z_k)$

```
input: $z_1, \ldots, z_k \in \{0,1\}^n$
output: $z \in \{0,1\}^n$
quantum workspace: $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}$ where
$\mathcal{X}$ is $n$ qubits $\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$, $\mathcal{X}_i = \mathbb{C}^2$,
$\mathcal{Y} = \mathbb{C}^2$ is one qubit, and
$\mathcal{Z}$ is $k$ qubits $\mathcal{Z} = \mathcal{Z}_1 \otimes \cdots \otimes \mathcal{Z}_k$, $\mathcal{Z}_j = \mathbb{C}^2$
initialize the workspace to $|0^n\rangle|0\rangle|0^k\rangle$
apply $H_{2^n}$ to $\mathcal{X}$
apply $U_f$ to $\mathcal{X} \otimes \mathcal{Y}$
apply $H_{2^n}$ to $\mathcal{X}$
for $j = 1$ to $k$ do
    $i \leftarrow \min\{i : z_j[i] = 1\}$
    apply CNOT with control $\mathcal{X}_i$ and target $\mathcal{Z}_j$
    apply $|x\rangle \mapsto |x \oplus z_j\rangle$ to $\mathcal{X}$ conditional on $\mathcal{Z}_j$
    apply $H_2$ to $\mathcal{Z}_j$
return measurement of $\mathcal{X}$
```

This is the state before the **for** loop is entered. We claim and proceed to show by induction that after the Jth execution of the loop body, the state is

$$\frac{\sqrt{2^J}}{N}\sum_{x\in\{0,1\}^n}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J}}(-1)^{x\cdot y}|y\rangle|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle|0^{k-J}\rangle\ .$$

Executing the body of the loop for $j=J+1$,

$$\begin{aligned}
&\frac{\sqrt{2^J}}{N}\sum_{x\in\{0,1\}^n}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J}}(-1)^{x\cdot y}|y\rangle|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle|0\rangle|0^{k-J-1}\rangle\\
\mapsto&\frac{\sqrt{2^J}}{N}\sum_{x\in\{0,1\}^n}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J}}(-1)^{x\cdot y}|y\rangle|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle|y[i_{j+1}]\rangle|0^{k-J-1}\rangle\\
=&\frac{\sqrt{2^J}}{N}\sum_{x\in\{0,1\}^n}\sum_{b\in\{0,1\}}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J+1}}(-1)^{x\cdot(y\oplus bz_{J+1})}|y\oplus bz_{J+1}\rangle\\
&\qquad|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle|b\rangle|0^{k-J-1}\rangle\\
\mapsto&\frac{\sqrt{2^J}}{N}\sum_{x\in\{0,1\}^n}\sum_{b\in\{0,1\}}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J+1}}(-1)^{x\cdot(y\oplus bz_{J+1})}|y\rangle\\
&\qquad|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle|b\rangle|0^{k-J-1}\rangle\\
=&\frac{\sqrt{2^{J+1}}}{N}\sum_{x\in\{0,1\}^n}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J+1}}(-1)^{x\cdot y}|y\rangle|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_J\rangle\\
&\qquad\frac{1}{\sqrt2}\sum_{b\in\{0,1\}}(-1)^{x\cdot(bz_{J+1})}|b\rangle|0^{k-J-1}\rangle\\
\mapsto&\frac{\sqrt{2^{J+1}}}{N}\sum_{x\in\{0,1\}^n}\sum_{\substack{y\in\{0,1\}^n\\ y[i_j]=0\,\forall j\le J+1}}(-1)^{x\cdot y}|y\rangle|f(x)\rangle|x\cdot z_1\rangle\cdots|x\cdot z_{J+1}\rangle|0^{k-J-1}\rangle\ .
\end{aligned}$$

□

As an immediate consequence, we can establish the invariant that in SimonTester $\{z_1,\ldots,z_k\}$ is always linearly independent with $i_j=\min\{i:z_j[i]=1\}$ distinct for $1\le j\le k$; moreover, if $f\in L$, then just as in Simon's algorithm, a nonzero z is orthogonal to the hidden s:

Lemma 13 *If measuring the first register, $\mathcal{X}$, yields a nonzero value z, then*

1. $\{z_1,\ldots,z_k,z\}$ *is linearly independent,*
2. $\min\{i:z[i]=1\}$ *is distinct from i_j for $1\le j\le k$, and*

3. *if $f \in L$, then $z \cdot s = 0$ for every s such that $f(x) = f(x \oplus s)$ for all x.*

Next, we want to assess the probability of obtaining $z = 0$ in SimonTester Line 4. We let P_0 denote the projection operator mapping $|0\rangle|y\rangle|z\rangle \mapsto |0\rangle|y\rangle|z\rangle$ and $|x\rangle|y\rangle|z\rangle \mapsto 0$ for $x \neq 0$; hence, $\|P_0|\psi\rangle\|^2$ is the probability of obtaining 0 when measuring subspace $\mathcal{X}$ of the quantum register in state $|\psi\rangle$. We can characterize the probability for outcome $z = 0$ in terms of the following definition and lemma:

Definition 6 *For $c \in \{0,1\}^k$ and $z_1, \ldots, z_k \in \{0,1\}^n$ we define $D_c := \{x \in \{0,1\}^n : x \cdot z_1 = c[1], \ldots, x \cdot z_k = c[k]\}$.*

Lemma 14 *Let $|\psi\rangle$ be the state before the measurement in* SimonSampler*, when* SimonSampler *is passed k linearly independent vectors $z_1, \ldots, z_k$ so that all $i_j := \min\{i : z_j[i] = 1\}$ are distinct for $1 \leq j \leq k$.*

1. *$\|P_0|\psi\rangle\|^2 = 1$ if and only if for every $c \in \{0,1\}^k$, f is constant when restricted to D_c.*
2. *If $\|P_0|\psi\rangle\|^2 \geq 1 - \varepsilon^2/2$, then f differs in at most εN points from some function g that is constant when restricted to D_c for every $c \in \{0,1\}^k$.*

Proof. For $b \in \{0,1\}$ let $D_{b,c} := D_c \cap f^{-1}\{b\} = \{x : f(x) = b \text{ and } x \cdot z_1 = c[1], \ldots, x \cdot z_k = c[k]\}$. Note that the $D_{b,c}$ and D_c also depend on $z_1, \ldots, z_k$ and the $D_{b,c}$ depend on f. Let

$$\begin{aligned} |\psi_0\rangle &:= \frac{\sqrt{2^k}}{N} \sum_{x \in \{0,1\}^n} |0\rangle|f(x)\rangle|x \cdot z_1\rangle \cdots |x \cdot z_k\rangle \\ &= \frac{\sqrt{2^k}}{N} \sum_{b \in \{0,1\}} \sum_{c \in \{0,1\}^k} |D_{b,c}||0\rangle|b\rangle|c[1]\rangle \cdots |c[k]\rangle \ . \end{aligned}$$

By Lemma 12, at the end of SimonSampler the system is in state

$$|\psi\rangle = |\psi_0\rangle + \frac{\sqrt{2^k}}{N} \sum_{x \in \{0,1\}^n} \sum_{\substack{y \in \{0,1\}^n \setminus \{0\} \\ y[i_j]=0 \, \forall j \leq k}} (-1)^{x \cdot y} |y\rangle|f(x)\rangle|x \cdot z_1\rangle \cdots |x \cdot z_k\rangle \ .$$

We consider the case $\|P_0|\psi\rangle\|^2 = 1$. Then the register $\mathcal{X}$ must be in state $|0\rangle$ and thus $|\psi\rangle = |\psi_0\rangle$. Since the state has norm 1, we know that

$$\sum_{b \in \{0,1\}} \sum_{c \in \{0,1\}^k} |D_{b,c}|^2 = \frac{N^2}{2^k} \ . \tag{1}$$

The $D_{b,c}$ partition $\{0,1\}^n$ and the $D_c = D_{0,c} \cup D_{1,c}$ have the same size for all $c \in \{0,1\}^k$ because they are cosets of D_0. Therefore,

$$\sum_{b \in \{0,1\}} \sum_{c \in \{0,1\}^k} |D_{b,c}| = N \text{ and } |D_{0,c}| + |D_{1,c}| = \frac{N}{2^k} \text{ for all } c \in \{0,1\}^k \ . \tag{2}$$

$|D_{0,c}|^2 + |D_{1,c}|^2 \leq N^2/2^{2k}$, but in order for equation (1) to hold, $|D_{0,c}|^2 + |D_{1,c}|^2$ must be exactly $N^2/2^{2k}$. This can only be achieved if either $D_{0,c}$ or $D_{1,c}$ is empty. So f must be constant when restricted to D_c for any $c \in \{0,1\}^k$. Conversely, if f is constant when restricted to D_c for any $c \in \{0,1\}^k$, then equation (1) holds, therefore $\| |\psi_0\rangle \| = 1$ and $|\psi\rangle = |\psi_0\rangle$. This concludes the proof of case 1 of the lemma.

If $\|P_0|\psi\rangle\|^2 = \||\psi_0\rangle\|^2 \geq 1-\delta$, then

$$\sum_{b\in\{0,1\}} \sum_{c\in\{0,1\}^k} |D_{b,c}|^2 \geq (1-\delta)\,\frac{N^2}{2^k} \quad . \tag{3}$$

Still, the constraints (2) hold; let $r2^k$ be the number of $c \in \{0,1\}^k$ so that $\min\{|D_{0,c}|, |D_{1,c}|\} \geq \gamma N/2^k$. Then

$$\sum_{b\in\{0,1\}} \sum_{c\in\{0,1\}^k} |D_{b,c}|^2 \leq r2^k(\gamma^2 + (1-\gamma)^2)\frac{N^2}{2^{2k}} + (1-r)2^k\frac{N^2}{2^{2k}} \quad ,$$

and using (3), we obtain $r \leq \delta/(1-\gamma^2-(1-\gamma)^2)$. With $\delta = \varepsilon^2/2$ and $\gamma = \varepsilon/2$, this implies $r \leq \varepsilon$. But then

$$\sum_{c\in\{0,1\}^k} \min\{|D_{0,c}|, |D_{1,c}|\} \leq r2^k\frac{N}{2^{k+1}} + (1-r)2^k\gamma\frac{N}{2^k} \leq \varepsilon N \quad .$$

□

We need to relate these two cases to membership in L and bound the number of repetitions needed to distinguish between the two cases. This is achieved by the following two lemmas.

Lemma 15 *Let k be the minimum number of linearly independent vectors z_1, …, z_k so that for each $c \in \{0,1\}^k$, f is constant when restricted to D_c. Then $f \in L$ if and only if $k < n$.*

Proof. If $k < n$, then there exists an s with $s \cdot z_1 = 0, \ldots, s \cdot z_k = 0$. For each such s and all x, we have $x \cdot z_1 = (x \oplus s)\cdot z_1, \ldots, x\cdot z_k = (x\oplus s)\cdot z_k$ and $x \in D_{f(x),x\cdot z_1,\ldots,x\cdot z_k}$ and $x \oplus s \in D_{f(x\oplus s),x\cdot z_1,\ldots,x\cdot z_k}$, therefore $f(x) = f(x\oplus s)$. Conversely, for $f \in L$, $S := \{s : \forall x f(x) = f(x\oplus s)\}$ is a nontrivial subspace of $\{0,1\}^n$, therefore $S^\perp = \{z : z\cdot s = 0 \forall s \in S\}$ is a proper subspace of $\{0,1\}^n$. Let z_1, …, z_k be an arbitrary basis of $S^\perp$. □

Lemma 16 *Let $0 < q < 1$, and $|\varphi_1\rangle$, …, $|\varphi_m\rangle$ be quantum states satisfying $\|P_0|\varphi_j\rangle\|^2 < 1-\delta$ for $1 \leq j \leq m$. If $m = \log q/\log(1-\delta) = \Theta(-\log q/\delta)$, then with probability at most q measuring the $\mathcal{X}$ register of $|\varphi_1\rangle$, …, $|\varphi_m\rangle$ will yield m times outcome 0.*

Proof.

$$\Pr\left[m \text{ times } 0 \,\middle|\, \forall j : \|P_0|\varphi_j\rangle\|^2 < 1-\delta\right] < (1-\delta)^m = (1-\delta)^{\log q/\log(1-\delta)} = q \quad .$$

□

Now all the ingredients for wrapping up the argument are at hand; first consider $f \in L$. Let $S := \{s : f(x) = f(x\oplus s)\ \forall x\}$ be the set of all "Simon promises" of f, and $S^\perp := \{z : z\cdot s = 0\ \forall s \in S\}$ the vectors that are orthogonal to all such promises. By Lemma 13 the nonzero z computed by the algorithm lie in $S^\perp$ and are linearly independent, therefore after $\dim S^\perp$ rounds of **for** loop in SimonTester, we measure $z = 0$ with certainty. Since $f \in L$, $\dim S > 0$ and thus $\dim S^\perp < n$.

If f is εn-far from being in L, then by Lemma 15 f is εn-far from being close to a function for which a $k < n$ and $z_1, \ldots, z_k$ exist so that f is constant when restricted to D_c for any of the $c \in \{0,1\}^k$. Therefore, by Lemma 14 case 2, for all $k < n$, $\|P_0|\psi\rangle\|^2 < 1 - \varepsilon^2/2$. Thus, Lemma 16 guarantees that we accept with probability at most 1/3 if we let $q = 1/(3n)$ and thus $m \leq 2(\log n)/\varepsilon^2$.

This concludes the proof of Theorem 7. □

5 Quantum Lower Bounds

In this section we prove that not every language has a fast quantum property tester.

Theorem 17 *Most properties containing $2^{n/20}$ elements of $\{0,1\}^n$ require quantum property testers using $\Omega(n)$ queries.*

Proof. Fix n, a small ε, and a quantum algorithm A making $q := n/400$ queries. Pick a property P as a random subset of $\{0,1\}^n$ of size $2^{n/20}$. Let

$$P_\varepsilon := \{y : d(x,y) < \varepsilon n \text{ for some } x \in P\} \ ;$$

using $\sum_{k=0}^{\varepsilon n} \binom{n}{k} \leq 2^{H(\varepsilon)n}$, where

$$H(\varepsilon) := -\varepsilon \log \varepsilon - (1-\varepsilon)\log(1-\varepsilon) \ ,$$

we obtain $|P_\varepsilon| \leq 2^{(1/20+H(\varepsilon))n}$. In order for A to test properties of size $2^{n/20}$, it needs to reject with high probability on at least $2^n - 2^{(1/20+H(\varepsilon))n}$ inputs; but then, the probability that A accepts with high probability on a random $x \in \{0,1\}^n$ is bounded by $2^{(1/20+H(\varepsilon))n}/2^n$ and therefore the probability that A accepts with high probability on $|P|$ random inputs is bounded by

$$2^{-(1-1/20-H(\varepsilon))n|P|} = 2^{-2^{n/20+\Theta(\log n)}} \ .$$

We would like to sum this success probability over all algorithms using the union bound to argue that for most properties no algorithm can succeed. However, there is an uncountable number of possible quantum algorithms with arbitrary quantum transitions. But by Beals et al. [5], the acceptance probability of A can be written as a multilinear polynomial of degree at most $2q$ where the n variables are the bits of the input; using results of Bennett, Bernstein, Brassard, and Vazirani [6] and Solovay and Yao [26], every quantum algorithm can be approximated by another algorithm such that the coefficients of the polynomials describing the accepting probability are integers of absolute value less than $2^{n^{\mathrm{O}(1)}}$ over some fixed denominator. There are less than $2^{nH(2q/n)}$ degree-$2q$ polynomials in n variables, thus we can limit ourselves to $2^{n^{\mathrm{O}(1)}2^{nH(2q/n)}} \leq 2^{2^{n/20 \cdot 91/100+\Theta(\log n)}}$ algorithms.

Thus, by the union bound, for most properties of size $2^{n/20}$, no quantum algorithm with q queries will be a tester for it. □

We also give an explicit natural property that requires a large number of quantum queries to test.

Theorem 18 *The range of a d-wise independent pseudorandom generator requires $(d+1)/2$ quantum queries to test for any odd $d \leq n/\log n - 1$.*

We will make use of the following lemma:

Lemma 19 (see [2]) *Suppose $n = 2^k - 1$ and $d = 2t + 1 \leq n$. Then there exists a uniform probability space Ω of size $2(n+1)^t$ and d-wise independent random variables $\xi_1, \ldots, \xi_n$ over Ω each of which takes the values 0 and 1 with probability* $1/2$.

The proof of Lemma 19 is constructive and the construction uniform in n; for given n and d, consider the language P of bit strings $\xi(z) := \xi_1(z) \ldots \xi_n(z)$ for all events $z \in \Omega = \{1, \ldots, 2(n+1)^t\}$. Classically, deciding membership in P takes more than d queries: for all d positions $i_1, \ldots, i_d$ and all strings $v_1 \ldots v_d \in \{0,1\}^d$ there is a z such that $\xi_{i_1}(z) \ldots \xi_{i_d}(z) = v_1 \ldots v_d$. On the other hand, $\lfloor \log |\Omega| \rfloor + 1 = \mathrm{O}(d \log n)$ queries are always sufficient.

Proof of Theorem 18. A quantum computer deciding membership for $x \in \{0,1\}^n$ in $P := \{\xi(z) : z \in \Omega\}$ with T queries gives rise to a degree $2T$ multilinear n-variable approximating polynomial $p(x) = p(x_1, \ldots, x_n)$ [5]. We show that there must be high-degree monomials in p by comparing the expectation of $p(x)$ for randomly chosen $x \in \{0,1\}^n$ with the expectation of $p(x)$ for randomly chosen $x \in P$.

For uniformly distributed $x \in \{0,1\}^n$, we have $\mathrm{E}[p(x)|x \in P] \geq 2/3$ and $\mathrm{E}[p(x)|x \notin P] \leq 1/3$. Since $|P| = \mathrm{o}(2^n)$, $\mathrm{E}[p(x)] \leq 1/3 + \mathrm{o}(1)$ and thus $\Delta := \mathrm{E}[p(x)|x \in P] - \mathrm{E}[p(x)] \geq 1/3 - \mathrm{o}(1)$. Considering $p(x) = \sum_i \alpha_i m_i(x)$ as a linear combination of n-variable multilinear monomials m_i, we have by the linearity of expectation $\mathrm{E}[p(x_1, \ldots, x_n)] = \sum_i \alpha_i \mathrm{E}[m_i(x_1, \ldots, x_n)]$. Because of the d-wise independence of the bits of each $x \in P$, for every m_i of degree at most d holds $\mathrm{E}[m_i(x)] = \mathrm{E}[m_i(x)|x \in P]$. Since $\Delta > 0$, p must comprise monomials of degree greater than d. Hence, the number of queries T is greater than $d/2$.

This proof extends in a straightforward manner to the case of testing the property P: let again $P_\varepsilon := \{y : d(x,y) < \varepsilon n \text{ for some } x \in P\}$. Then

$$|P_\varepsilon| \leq 2^{H(\varepsilon)n} |P| = \mathrm{O}(2^{H(\varepsilon)n + d \log n}) \ ,$$

so

$$\mathrm{E}[p(x)] = \frac{|P_\varepsilon|}{2^n} \mathrm{E}[p(x)|x \in P_\varepsilon] + \left(1 - \frac{|P_\varepsilon|}{2^n}\right) \mathrm{E}[p(x)|x \notin P_\varepsilon] \leq 1/3 + \mathrm{o}(1)$$

for every $d = n/\log n - \omega(1/\log n)$ and every ε with $H(\varepsilon) = 1 - \omega(1/n)$. □

6 Further Research

Our paper opens the door to the world of quantum property testing. Several interesting problems remain including

- Can one get the greatest possible separation of quantum and classical property testing, i.e., is there a language that requires $\Omega(n)$ classical queries but only $\mathrm{O}(1)$ quantum queries to test?
- Are there other natural problems that do not have quantum property testers? We conjecture for instance that the language $\{uuvv : u, v \in \Sigma^*\}$ does not have a quantum property tester.
- Beals et al. [5] observed that any k-query quantum algorithm gives rise to a degree-$2k$ polynomial in the input bits, which gives the acceptance probability of the algorithm; thus, a quantum property tester for P gives rise to a polynomial that is on all binary inputs between

0 and 1, that is at least 2/3 on inputs with the property P and at most 1/3 on inputs far from having the property P. Szegedy [27] suggested to algebraically characterize the complexity of classical testing by the minimum degree of such polynomials; as mentioned in the introduction, our results imply that this cannot be the case for classical testers. However, it is an open question whether quantum property testing can be algebraically characterized in this way.

We hope that further research will lead to a greater understanding of what can and cannot be tested with quantum property testers.

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