Nathan Jacobson, Basic Algebra I, p.p. 16-17, 1974.

Proofs by induction are very common in mathematics and are undoubtedly familiar to the reader. One also encounters quite frequently—without being conscious of it—definitions by induction. An example is the definition mentioned above of a^n by $a^0 = 1$, $a^{r+1} = a^r a$. Definition by induction is not as trivial as it may appear at first glance. This can be made precise by the following

RECURSION THEOREM. *Let S be a set, φ a map of S into itself, a an element of S. Then there exists one and only one map f from $\mathbb{N}$ to S such that*

$$1.\ f(0) = a, \qquad 2.\ f(n^+) = \varphi(f(n)),\ n \in \mathbb{N}.$$ [6]

Proof. Consider the product set $\mathbb{N} \times S$. Let Γ be the set of subsets U of $\mathbb{N} \times S$ having the following two properties: (i) $(0, a) \in U$, (ii) if $(n, b) \in U$ then $(n^+, \varphi(b)) \in U$. Since $\mathbb{N} \times S$ has these properties it is clear that $\Gamma \neq \varnothing$. Let f be the intersection of all the subsets U contained in Γ. We proceed to show that f is the desired function from $\mathbb{N}$ to S. In the first place, it follows by induction that if $n \in \mathbb{N}$, there exists a $b \in S$ such that $(n, b) \in f$. To prove that f is a map of $\mathbb{N}$ to S it remains to show that if (n, b) and $(n, b') \in f$ then $b = b'$. This is equivalent to showing that the subset T of $n \in \mathbb{N}$ such that (n, b) and $(n, b') \in f$ imply $b = b'$ is all of $\mathbb{N}$. We prove this by induction. First, $0 \in T$. Otherwise, we have $(0, a)$ and $(0, a') \in f$ but $a \neq a'$. Then let f' be the subset of f obtained by deleting the element $(0, a')$ from f. Then it is immediate that f' satisfies the defining conditions (i) and (ii) for the sets $U \in \Gamma$. Hence $f' \supset f$. But $f' \subsetneq f$ since f' was obtained by dropping $(0, a')$ from f. This contradiction proves that $0 \in T$. Now suppose we have a natural number r such that $r \in T$ but $r^+ \notin T$. Let $(r, b) \in f$. Then $(r^+, \varphi(b)) \in f$ and since $r^+ \notin T$, we have a $c \neq \varphi(b)$ such that $(r^+, c) \in f$. Now consider the subset f' of f obtained by deleting (r^+, c). Since $r^+ \neq 0$ and f contains $(0, a)$, f' contains $(0, a)$. The same argument shows that if $n \in \mathbb{N}$ and $n \neq r$ and $(n, d) \in f'$ then $(n^+, \varphi(d)) \in f'$. Now suppose $(r, b') \in f'$ then $b' = b$ and $(r^+, \varphi(b)) \in f'$ since $(r^+, \varphi(b))$ was not deleted in forming f' from f. Thus we see that $f' \in \Gamma$ and this again leads to the contradiction: $f' \supset f$, $f' \subsetneq f$ We have therefore proved that if $r \in T$ then $r^+ \in T$. Hence $T = \mathbb{N}$ by induction, and so we have proved the existence of a function f satisfying the given conditions. To prove uniqueness, let g be any map satisfying the conditions. Then $g \in \Gamma$ so $g \supset f$. But $g \supset f$ for two maps f and g implies $f = g$, by the definition of a map. Hence f is unique.

[6] One is tempted to say that one can *define* f inductively by conditions 1 and 2. However, this does not make sense since in talking about a function on $\mathbb{N}$ we must have an à priori definition of $f(n)$ for every $n \in \mathbb{N}$. A proof of the existence of f must use *all* of Peano's axioms. An example illustrating this is given in exercise 4, p. 19. For a fuller account of these questions we refer the reader to an article, "On mathematical induction," by Leon Henkin in the *American Mathematical Monthly*, vol. 67 (1960), pp. 323–338. Henkin gives a proof of the recursion theorem based on the concept of "partial" functions on $\mathbb{N}$. The proof we shall give is due independently to P. Lorenzen, and to D. Hilbert and P. Bernays (jointly).