Heuristic Search: Let $e(v)$ be an estimate of the distance from v to the goal t.

Use Dijkstra's algorithm with $d(v) + e(v)$ as the selection criterion.

The method works if

$$e(v) \le \ell(v,w) + e(w) \quad \text{for all } v, w$$

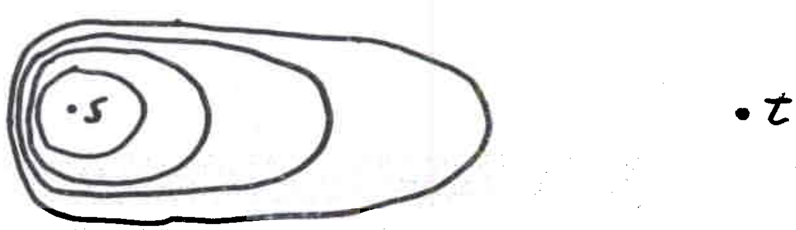
(Estimate e is a consistent lower bound on the actual distance.)

In Euclidean graphs the distance "as the crow flies" works.

Hart, Nilsson, Rafael (1968)

Dijkstra's algorithm

Heuristic search



Bidirectional Search: Search forward from s and backward from t concurrently.

⇒ Getting the stopping rule correct is tricky, especially for bidirectional heuristic search.

The Minimum Spanning Tree Problem

Given a connected graph, find a spanning tree of minimum total edge cost.

where,

n = the number of vertices

m = the number of edges

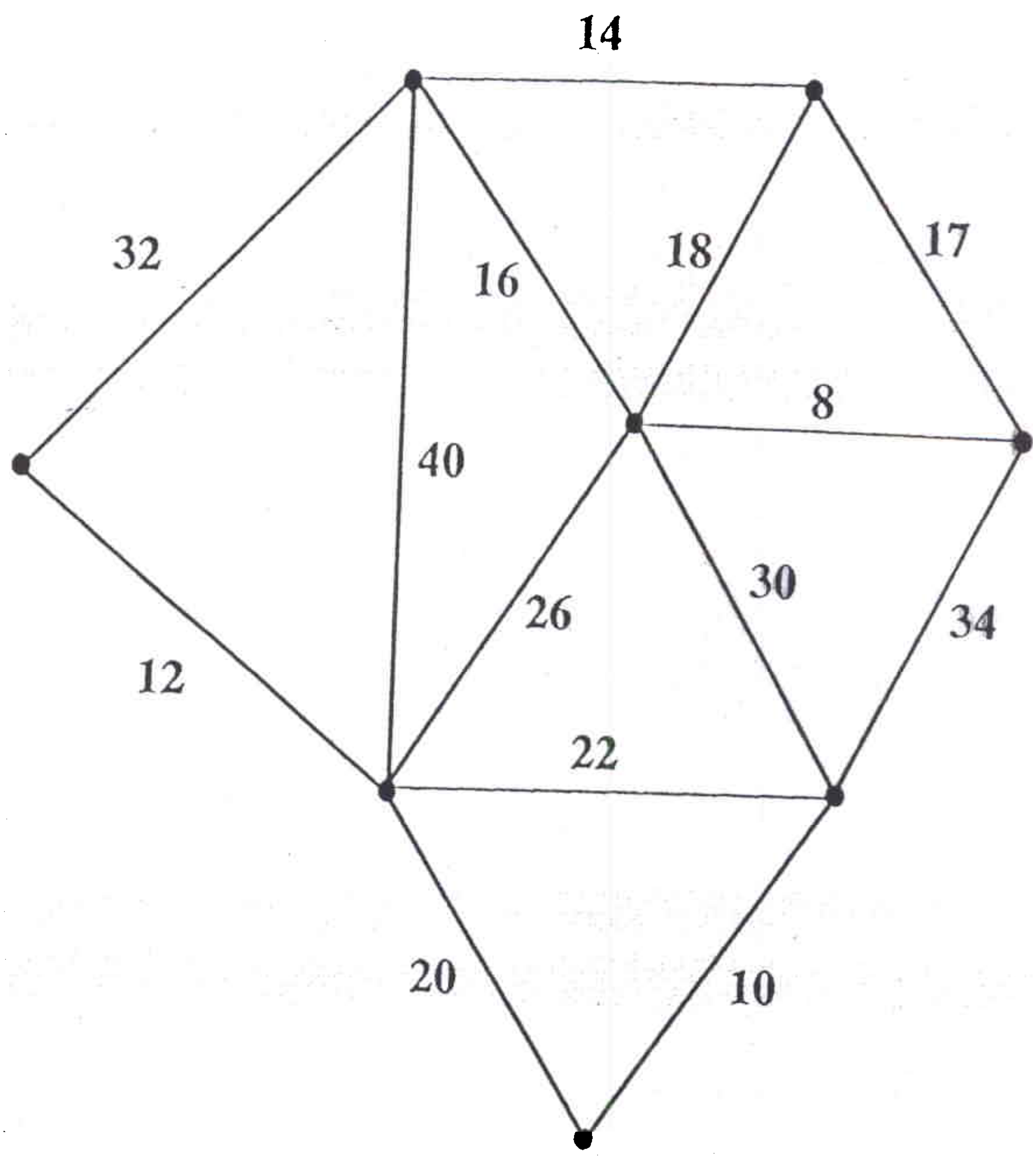
$$n-1 \le m \le \binom{n}{2}$$

Applications

Network Construction

Clustering

Minimum Tour Relaxation (Held-Karp 1-trees)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A Simple Solution From the 80's

(with apologies to Oliver Stone)

Gorden Gecko: "Greed is Good"

Repeatedly select the cheapest unselected edge and add it to the tree under construction if it connects two previously disconnected pieces.

Kruskal, 1956

The greedy method generalizes to matroids.

We shall generalize the method rather than the domain of application.

Generalized Greedy Method

Beginning with all edges uncolored,

sequentially color edges

blue (accepted) or red (rejected).

Blue Rule:

Color blue any minimum-cost uncolored edge crossing a cut with no blue edges crossing.

Red Rule:

Color red any maximum-cost uncolored edge on a cycle with no red edges.

"Classical" Algorithms

(before algorithm analysis)

Kruskal's algorithm, 1956

$O(m \log n)$ **time**

Jarnik's algorithm, 1930

$O(n^2)$ **time**

also Prim, Dijkstra

Boruvka's algorithm, 1926

$O(\min\{m \log n, n^2\})$ **time**

and many others

Jarnik's Algorithm

Grow a tree from a single start vertex.

At each step add a cheapest edge with

exactly one end in the tree.

Boruvka's Algorithm

Repeat the following step until all vertices are connected:

> **For each blue component, select a cheapest edge connecting to another component; color all selected edges blue.**

For correctness, a tie-breaking rule is needed.

Henceforth, assume all edge costs are distinct.

Then there is a unique spanning tree.

Selected History

Boruvka, 1926	$O(\min\{m\log n, n^2\})$
Jarnik, 1930 **Prim, 1957** **Dijkstra, 1959**	$O(n^2)$
Kruskal, 1956	$O(m\log n)$
Williams, Floyd, 1964 **heaps**	$O(m\log n)$
Yao, 1975 **packets in Boruvka's algorithm**	$O(m\log\log n)$
Fredman, Tarjan, 1984 **F-heaps in:**	
Jarnik's algorithm	$O(n\log n + m)$
a hybrid Jarnik-Boruvka algorithm	$O(m\log^* n)$
Gabow, Galil, Spencer, 1984 **Packets in F-T algorithm**	$O(m\log\log^* n)$

$$\log^* n = \min\{i \mid \log\log\log \ldots \log n \leq 1\}$$

where the logarithm is iterated i times

Models of Computation

We assume comparison of the two edge costs takes unit time, and no other manipulation of edge costs is allowed.

Another model:

bit manipulation of the binary representations of edge costs is allowed.

In this model,

Fredman-Willard, 1990, achieved O(m) time.

(fast small heaps by bit manipulation)

Goal: An O(m)-time algorithm

without bit manipulation of edge weights

Boruvka's algorithm with contraction:

If G contains at least two vertices:

select cheapest edge incident to each vertex;

Contract all selected edges;

Recur on contracted graph.

If contraction preserves sparsity ($m = O(n)$),

this algorithm runs in $O(n) = O(m)$ time

on sparse graphs.

E.g. planar graphs

How to handle non-sparse graphs?

Thinning: remove all but O(n) edges by finding edges that can't be in the minimum spanning tree.

How to thin?