COS 226 Lecture 20: Shortest Paths

Classic algorithms for natural network problems

SHORTEST PATH
- shortest way to get from u to v

SINGLE-SOURCE SHORTEST PATHS (SPT)
- PFS implementation
- Dijkstra's algorithm

ALL SHORTEST PATHS
- Floyd's algorithm

Negative weights?

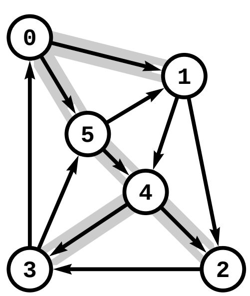
REDUCTION

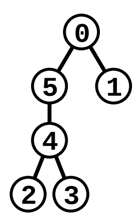
Problem-solving models

Single-source shortest paths

Defines SHORTEST PATHS TREE (SPT) rooted at source

```
0-1 .41
1-2 .51
2-3 .50
4-3 .36
3-5 .38
3-0 .45
0-5 .29
5-4 .21
1-4 .32
4-2 .32
5-1 .29
```





	0	1	2	3	4	5
st	0	0	4	4	5	0
wt	0	.41	.32	.36	.21	.29

SPT algorithm

Another generalized graph-search implementation

RELAXATION
- if wt[w] < wt[v] + wt(v-w) then set wt[w] to that value
  (v-w gives a shorter path to w than the best known)

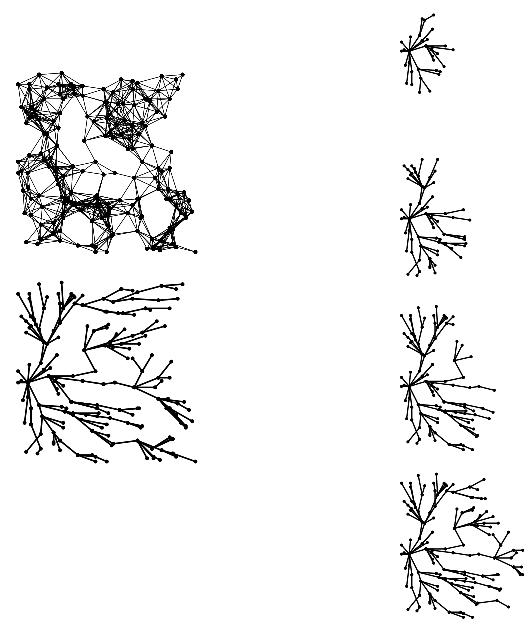
SPT ALGORITHM

```
put s on fringe
while fringe nonempty
     choose node from fringe that is closest to s
     relax along all its edges
```

v on TREE wt[v] is shortest distance from s to v

v on FRINGE: wt[v] is shortest KNOWN distance from s to v
- won't find a shorter path to node with smallest value

larger SPT example



Dijkstra's algorithm

Classical implementation of generic SPT algorithm

SAME CODE as Prim's MST algorithm with

```
#define P wt[v] + t->wt
```

DENSE graphs

- classical Dijkstra's algorithm
- time cost: O(V^3)

SPARSE graphs

- use PQ (heap) implementation
- time cost: O(E lg V)

Better PQs give faster algorithms for sparse graphs

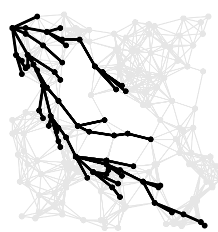
- d-way heap: O(E log_d V)
- F-heap: O(E + V log V)

Shortest paths in Euclidean graphs

Problem: find shortest path from s to d

Algorithm:

- start shortest-path PFS at s
- stop when reaching d



SUBLINEAR algorithm

- need not touch all nodes

better yet: use geometry to limit search

wt[v]:

- TREE: shortest distance from s to v
- FRINGE: shortest POSSIBLE distance from s to d through v
  tree path from s to v PLUS distance from v to d

```
#define P wt[v] + t->wt + dist(t->v, d) - dist(k, d)
```


All shortest paths

Table of shortest paths for each vertex pair

Ex: map of New England

.	P	W	L	N
. Providence	0	53	54	48
. Westerley	53	0	18	101
. newLondon	54	18	0	12
. Norwich	48	101	12	0

Norwich-Westerly: 101 miles??

- 12 miles Norwich-New London
- 18 miles New London-Westerly
- 30 miles total

Need correct algorithm to get correct table

Floyd's algorithm

Another ancient algorithm (1962)
[same as Warshall, in a different context]

Want shorter path from s to d?

- take s to i, then i to d, if shorter (vertex relaxation)

```
for (i = 0; i < G->V; i++)
  for (s = 0; s < G->V; s++)
    if (G->adj[s][i] != maxWT)
      for (t = 0; t < G->V; t++)
        if (G->adj[i][t] != maxWT)
          if (d[s][t] > d[s][i]+d[i][t])
            d[s][t] = d[s][i]+d[i][t];
```

Correctness proof:

- induction on i (same as Warshall)

Shortest paths ADT

Same issues as reachability in digraphs

Classical Floyd-Warshall algorithm gives
- query: O(1)
- preprocessing: O(V^3)
- space: O(V^2)

Easy to reduce preprocessing to O(VE)
- use Dijkstra for each vertex

End of story?

NOT QUITE
- ADT is useful for a variety of disparate problems
- negative weights complicate matters

Reduction

DEF: Problem A REDUCES TO Problem B
if we can use an algorithm that solves B
to develop an algorithm that solves A

Typical reduction:
- given an instance of A
- transform it to an instance of B
- solve that instance of B
- transform the solution to be a solution of A

Uses of reduction
- algorithm for A (programmer using ADT)
- lower bound on B

PROBLEM-SOLVING MODELS
- problems that many other problems reduce to

NP-HARD PROBLEMS
- problems that ANY NP-hard problem reduces to

Reduction example: longest paths

THM: Longest-paths reduces to shortest-paths

Proof:
- given an instance of longest-paths
- transform it to shortest-paths by negating weights
- solve shortest-paths
- negate weights on path to get longest path

CATCH
- SP algs don't work in the presence of negative weights!

Lessons:
- reductions have to be constructed with care
- they may not always give useful information

Reduction example: arbitrage

Currency conversion

	dollars	pounds	1K yen
dollars	1.000	1.631	0.669
pounds	0.613	1.000	0.411
1K yen	1.495	2.436	1.000

- $1000 dollars-pounds-dollars
  $1000*(1.631)*(0.613) = $999
- $1000 dollars-pounds-yen-dollars
  $1000*(1.631)*(0.411)*(1.495) = $1002

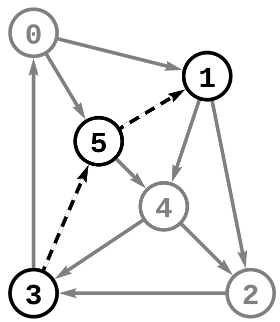
SHORTEST PATH is best arbitrage opportunity
- replace table entry x by -log x
- BUT, weights may be negative!

Need SP algs that work with negative weights

Shortest paths with negative weights

Negative weights
- completely change SPT
- can introduce negative cycles

```
0-1  .41
1-2  .51
2-3  .50
4-3  .36
3-5 -.38
3-0  .45
0-5  .29
5-4  .21
1-4  .32
4-2  .32
5-1 -.29
```



shortest path from 4 to 2: 4-3-5-1-2

Reduction example: SP with negative weights

THM: SP with negative weights is NP-hard

A: Hamilton path
B: SP with negative weights

Hamilton path reduces to SP with negative weights

- given an undirected graph
- transform to network with -1 wt on each edge
- find shortest simple path
- YES to Hamilton path if SP length is -V

Negative weights in SP problems

NP-complete: don't try to solve general problem
- restrict problem to solve it

Versions that we can solve
- no negative weights
- no cycles
- negative-cycle detection
- no negative cycles

Dijkstra's algorithm: doesn't work at all with negative weights
Floyd's algorithm
- detects negative cycles
- solves all-pairs shortest paths if no neg cycles present

Ex: use Floyd's to find SOME arbitrage opportunity
- (much harder to find the BEST one)

Bellman-Ford shortest-paths algorithm

Generic algorithm for single-source problem

- initialize wt[s] to 0, other wts to max
- repeat V times: relax on each edge

Order of processing edges not specified

Running time O(VE)

If no negative cycles present
- can use as preprocessing step for Dijkstra
- VE lg V for all-pairs problem
- improves on V^3 for Floyd

Not much harder to solve all-pairs than single-source (?!)
OPEN: Better alg for single-source?