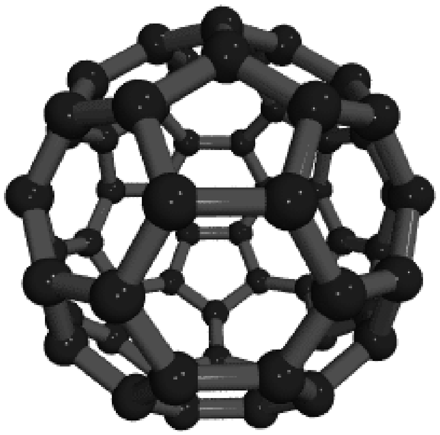
Linear Programming



Linear Programming

What is it?

- Quintessential tool for optimal allocation of scarce resources, among a number of competing activities.
- Powerful and general problem-solving method.
  - shortest path, max flow, min cost flow, multicommodity flow, MST, matching, 2-person zero sum games

Why significant?

- Fast commercial solvers: CPLEX.
- Powerful modeling languages: AMPL, GAMS.
- Ranked among most important scientific advances of 20^{th} century.
- Also a general tool for attacking NP-hard optimization problems.
- Dominates world of industry.
  - ex: Delta claims saving $100 million per year using LP

Applications

Agriculture. Diet problem.
Computer science. Compiler register allocation, data mining.
Electrical engineering. VLSI design, optimal clocking.
Energy. Blending petroleum products.
Economics. Equilibrium theory, two-person zero-sum games.
Environment. Water quality management.
Finance. Portfolio optimization.
Logistics. Supply-chain management, Berlin airlift.
Management. Hotel yield management.
Marketing. Direct mail advertising.
Manufacturing. Production line balancing, cutting stock.
Medicine. Radioactive seed placement in cancer treatment.
Physics. Ground states of 3-D Ising spin glasses.
Telecommunication. Network design, Internet routing.
Transportation. Airline crew assignment, vehicle routing.
Sports. Scheduling ACC basketball, handicapping horse races.

Brewery Problem: A Toy LP Example

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

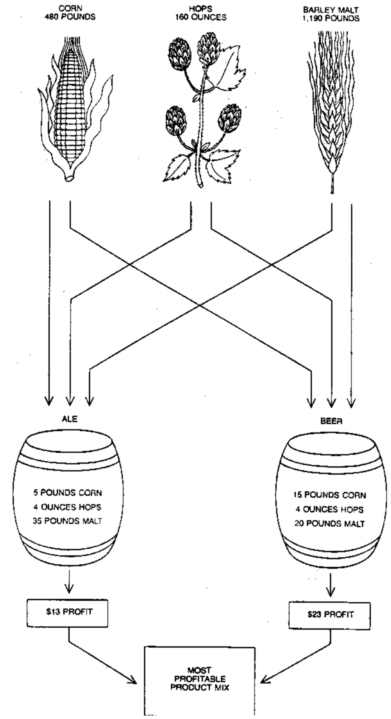
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale	5	4	35	13
Beer	15	4	20	23
Quantity	480	160	1190	

How can brewer maximize profits?

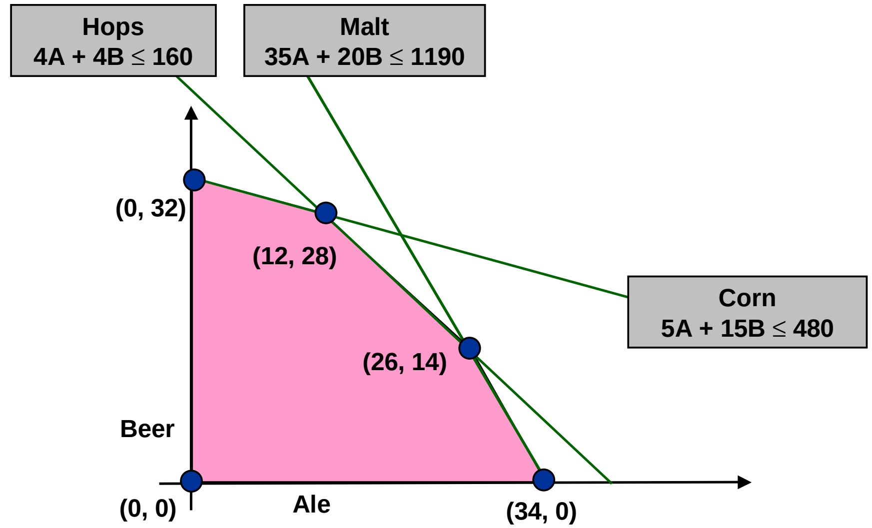
- Devote all resources to ale: 34 barrels of ale ⇒ $442.
- Devote all resources to beer: 32 barrels of beer ⇒ $736.
- 7.5 barrels of ale, 29.5 barrels of beer ⇒ $776.
- 12 barrels of ale, 28 barrels of beer ⇒ $800.

Brewery Problem

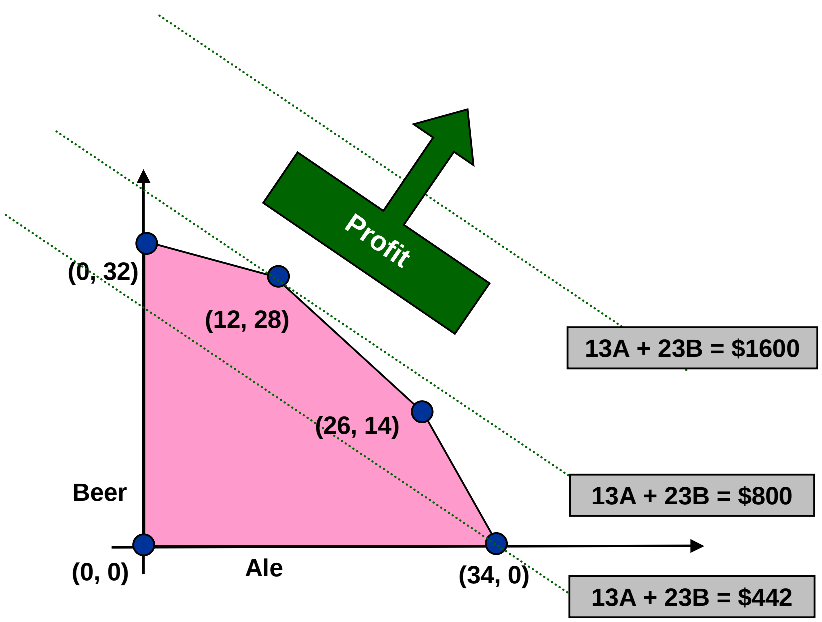
	Ale		Beer			
max	$13A$	$+$	$23B$			Profit
s. t.	$5A$	$+$	$15B$	$\leq$	480	Corn
	$4A$	$+$	$4B$	$\leq$	160	Hops
	$35A$	$+$	$20B$	$\leq$	1190	Malt
	A	,	B	$\geq$	0	



Brewery Problem: Feasible Region

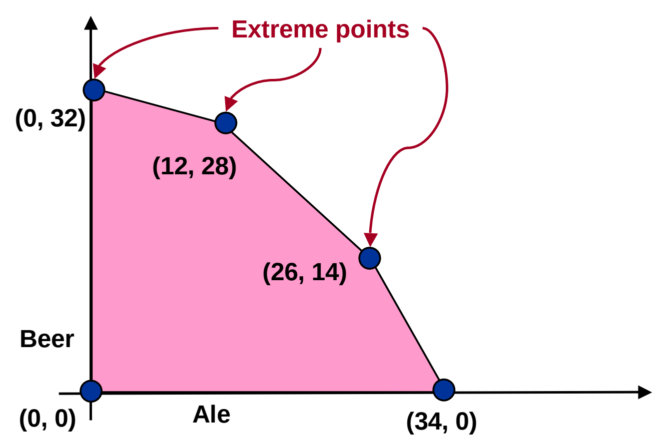


Brewery Problem: Objective Function



Brewery Problem: Geometry

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at an extreme point.



Standard Form LP

"Standard form" LP.

- Input data: rational numbers c_j, b_i, a_{ij} .
- Output: rational numbers x_j.
- n = # nonnegative variables, m = # constraints.
- Maximize linear objective function.
  - subject to linear inequalities

$$
\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\
 & & x_j & \ge & 0 & 1 \le j \le n
\end{array}
$$

$$
\begin{array}{llrcl}
\text{(P)} & \max & c \bullet x & & \\
 & \text{s.t.} & Ax & = & b \\
 & & x & \ge & 0
\end{array}
$$

Linear. No x^2, xy, arccos(x), etc.

Programming. Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input.

$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \le & 480 \\
 & 4A & + & 4B & \le & 160 \\
 & 35A & + & 20B & \le & 1190 \\
 & A & , & B & \ge & 0
\end{array}
$$

Standard form.

- Add SLACK variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{lrcrcccccccr}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_H & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_M & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_C & = & 1190 \\
 & A & , & B & , & S_H & , & S_M & , & S_C & \ge & 0
\end{array}
$$

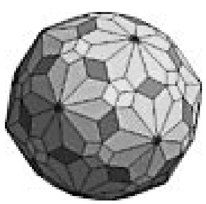
Geometry

2-D geometry.

- Inequalities : halfplanes.
- Bounded feasible region : convex polygon.

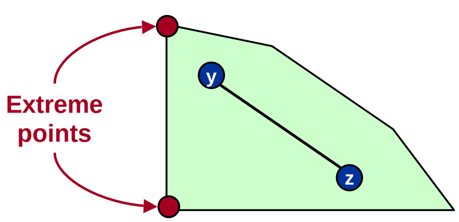
Higher dimensional geometry.

- Inequalities : hyperplanes.
- Bounded feasible region : (convex) polytope.

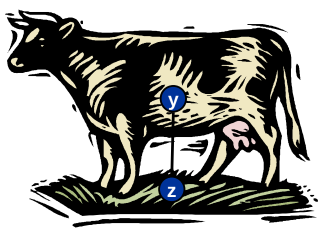


Convex: if y and z are feasible solutions, then so is (y +z) / 2.

Extreme point: feasible solution x that can't be written as (y + z) / 2 for any two distinct feasible solutions y and z.



Convex



Not convex

Geometry

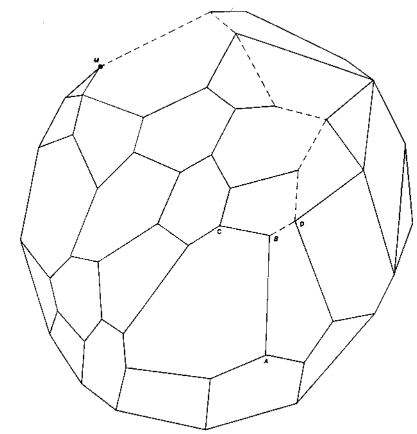
Extreme Point Property. If there exists an optimal solution to (P), then there exists one that is an extreme point.

- Only need to consider finitely many possible solutions.

Challenge. Number of extreme points can be exponential!

- Consider n-dimensional hypercube.

Greed. Local optima are global optima.



Simplex Algorithm

Simplex algorithm. (George Dantzig, 1947)

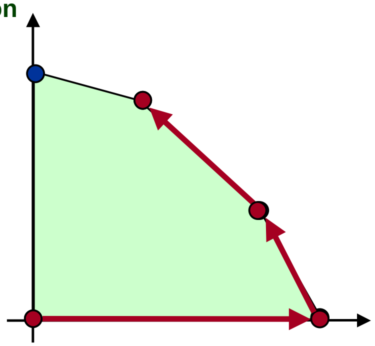
- Developed shortly after WWII in response to logistical problems.
- Used for 1948 Berlin airlift.

Generic algorithm.

- Start at some extreme point.
- Pivot from one extreme point to a neighboring one.
  - never decrease objective function
- Repeat until optimal.

How to implement?

- Use linear algebra.



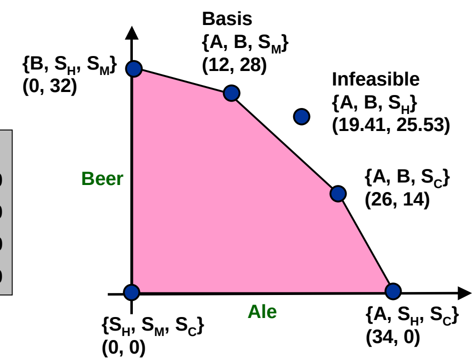
Simplex Algorithm: Basis

Basis. Subset of m of the n variables.

Basic feasible solution (BFS). Set n - m nonbasic variables to 0, solve for remaining m variables.

- Solve m equations in m unknowns.
- If unique and feasible solution $\Rightarrow$ BFS.
- BFS corresponds to extreme point!
- Simplex only considers BFS.

$$
\begin{array}{lrcrcccccl}
\max & 13A & + & 23B & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_H & & & = & 480 \\
 & 4A & + & 4B & & & + \; S_M & & = & 160 \\
 & 35A & + & 20B & & & & + \; S_C & = & 1190 \\
 & A & , & B & , & S_H & , \; S_M & , \; S_C & \geq & 0
\end{array}
$$



Simplex Algorithm: Pivot 1

max Z subject to

$$
\begin{array}{rcrcccccl}
13A & + & 23B & & & & & - \; Z = & 0 \\
\hline
5A & & \boxed{15B} & + & S_H & & & = & 480 \\
4A & + & 4B & & & + \; S_M & & = & 160 \\
35A & + & 20B & & & & + \; S_C & = & 1190 \\
A & , & B & , & S_H & , \; S_M & , \; S_C & \geq & 0
\end{array}
$$

Basis = $\{S_H, S_M, S_C\}$
A = B = 0
Z = 0
S_H = 480
S_M = 160
S_C = 1190

Substitute: $B = 1/15 \,(480 - 5A - S_H)$

max Z subject to

$$
\begin{array}{rcrcrcccl}
\frac{16}{3}A & & & - & \frac{23}{15}S_H & & & - \; Z = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_H & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_H & + \; S_M & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_H & & + \; S_C & = & 550 \\
A & , & B & , & S_H & , \; S_M & , \; S_C & \geq & 0
\end{array}
$$

Basis = $\{B, S_M, S_C\}$
A = S_H = 0
Z = 736
B = 32
S_M = 32
S_C = 550

Simplex Algorithm: Pivot 1

max Z subject to

$$
\begin{array}{rcrcccccl}
13A & + & 23B & & & & & - \; Z = & 0 \\
\hline
5A & & \boxed{15B} & + & S_H & & & = & 480 \\
4A & + & 4B & & & + \; S_M & & = & 160 \\
35A & + & 20B & & & & + \; S_C & = & 1190 \\
A & , & B & , & S_H & , \; S_M & , \; S_C & \geq & 0
\end{array}
$$

Basis = $\{S_H, S_M, S_C\}$
A = B = 0
Z = 0
S_H = 480
S_M = 160
S_C = 1190

Why pivot on column 2?

- Each unit increase in B increases objective value by \$23.
- Pivoting on column 1 also OK.

Why pivot on row 2?

- Ensures that RHS ≥ 0 (and basic solution remains feasible).
- Minimum ratio rule: min { 480/15, 160/4, 1190/20 }.

Simplex Algorithm: Pivot 2

max Z subject to

$$\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_H & & & & & - & Z & = & -736 \\ \hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_H & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_H & + & S_M & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_H & & & + & S_C & & & = & 550 \\
A & , & B & , & S_H & , & S_M & , & S_C & & & \geq & 0
\end{array}$$

Basis = {B, S_M, S_C}
A = S_H = 0
Z = 736
B = 32
S_M = 32
S_C = 550

Substitute: A = 3/8 (32 + 4/15 S_H – S_M)

max Z subject to

$$\begin{array}{rcrcrcrcrcrcr}
 & & & - & S_H & - & 2\,S_M & & & - & Z & = & -800 \\ \hline
 & & B & + & \frac{1}{10}S_H & + & \frac{1}{8}S_M & & & & & = & 28 \\
A & & & - & \frac{1}{10}S_H & + & \frac{3}{8}S_M & & & & & = & 12 \\
 & & & - & \frac{25}{6}S_H & - & \frac{85}{8}S_M & + & S_C & & & = & 110 \\
A & , & B & , & S_H & , & S_M & , & S_C & & & \geq & 0
\end{array}$$

Basis = {A, B, S_C}
S_H = S_M = 0
Z = 800
B = 28
A = 12
S_C = 110

Simplex Algorithm: Optimality

When to stop pivoting?

- If all coefficients in top row are non-positive.

Why is resulting solution optimal?

- Any feasible solution satisfies system of equations in tableaux.
  - in particular: Z = 800 – S_H – 2 S_M
- Thus, optimal objective value Z* ≤ 800 since S_H, S_M ≥ 0.
- Current BFS has value 800 ⇒ optimal.

max Z subject to

$$\begin{array}{rcrcrcrcrcrcr}
 & & & - & S_H & - & 2\,S_M & & & - & Z & = & -800 \\ \hline
 & & B & + & \frac{1}{10}S_H & + & \frac{1}{8}S_M & & & & & = & 28 \\
A & & & - & \frac{1}{10}S_H & + & \frac{3}{8}S_M & & & & & = & 12 \\
 & & & - & \frac{25}{6}S_H & - & \frac{85}{8}S_M & + & S_C & & & = & 110 \\
A & , & B & , & S_H & , & S_M & , & S_C & & & \geq & 0
\end{array}$$

Basis = {A, B, S_C}
S_H = S_M = 0
Z = 800
B = 28
A = 12
S_C = 110

Simplex Algorithm

Remarkable property. Simplex algorithm typically requires less than 2(m+n) pivots to attain optimality.

- No polynomial pivot rule known.
- Most pivot rules known to be exponential in worst-case.

Issues.

- Which neighboring extreme point?
- Cycling.
  - get stuck by cycling through different bases that all correspond to same extreme point
  - doesn't occur in the wild
  - Bland's least index rule ⇒ finite # of pivots
- Degeneracy.
  - new basis, same extreme point
  - "stalling" is common in practice

LP Duality: Economic Interpretation

Brewer's problem: find optimal mix of beer and ale to maximize profits.

$$\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}$$

A* = 12
B* = 28
OPT = 800

Entrepreneur's problem: Buy individual resources from brewer at minimum cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if 5C + 4H + 35M < 13.

$$\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & & 15C & + & 4H & + & 20M & \geq & 23 \\
 & & C & , & H & , & M & \geq & 0
\end{array}$$

C* = 1
H* = 2
M* = 0
OPT = 800

LP Duality

Primal and dual LPs. Given rational numbers a_{ij}, b_i, c_j, find rational numbers x_i, y_j that optimize (P) and (D).

$$(P)\quad \max \sum_{j=1}^{n} c_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m, \qquad x_j \ge 0 \quad 1 \le j \le n$$

$$(D)\quad \min \sum_{i=1}^{m} b_i y_i \quad \text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad 1 \le j \le n, \qquad y_i \ge 0 \quad 1 \le i \le m$$

Duality Theorem (Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947).
If (P) and (D) have feasible solutions, then max = min.

- Special case: max-flow min-cut theorem.
- Sensitivity analysis.

LP Duality: Economic Interpretation

Sensitivity analysis.

- How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
  - corn \$1, hops \$2, malt \$0.
- Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
  - Breakeven: 2 (\$1) + 5 (\$2) + 24 (0\$) = \$12 / barrel.

How do I compute marginal prices (dual variables)?

- Simplex solves primal and dual simultaneously.
- Top row of final simplex tableaux provides optimal dual solution!

History

1939. Production, planning. (Kantorovich)
1947. Simplex algorithm. (Dantzig)
1950. Applications in many fields.
1979. Ellipsoid algorithm. (Khachian)
1984. Projective scaling algorithm. (Karmarkar)
1990. Interior point methods.

Current research.

- Approximation algorithms.
- Software for large scale optimization.
- Interior point variants.

Ultimate Problem Solving Model

Ultimate problem-solving model?

- Shortest path.
- Min cost flow.
- Linear programming.
- Semidefinite programming.
- . . .
- TSP??? (or any NP-complete problem)

Does P = NP?

- No universal problem-solving model exists unless P = NP.

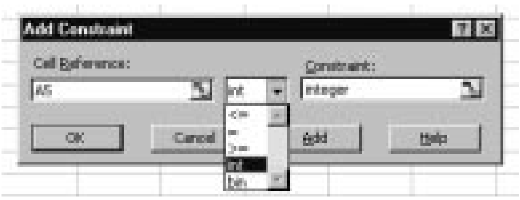
Perspective

LP is near the deep waters of NP-completeness.

- Solvable in polynomial time.
- Known for less than 25 years.

Integer linear programming.

- LP with integrality requirement.
- NP-hard.



An unsuspecting MBA student transitions from tractable LP to intractable ILP in a single mouse click.