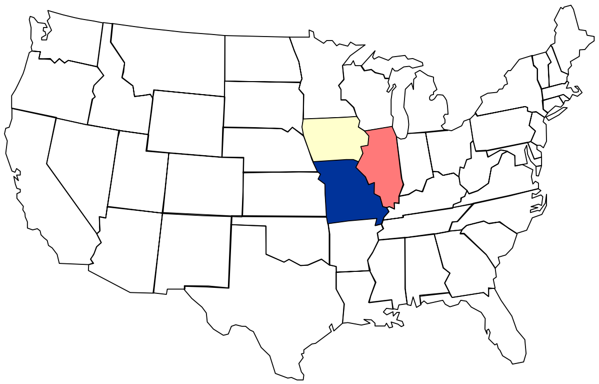
Lecture T6: NP-Completeness



Can you color each of the 48 states red, white, or blue so that no two adjacent states have the same color?

Overview

Lecture T4:

- What is an algorithm?
  - Turing machine
- Which problems can be solved on a computer?
  - not the halting problem

Lecture T5:

- Which algorithms will be useful in practice?
  - polynomial vs. exponential algorithms

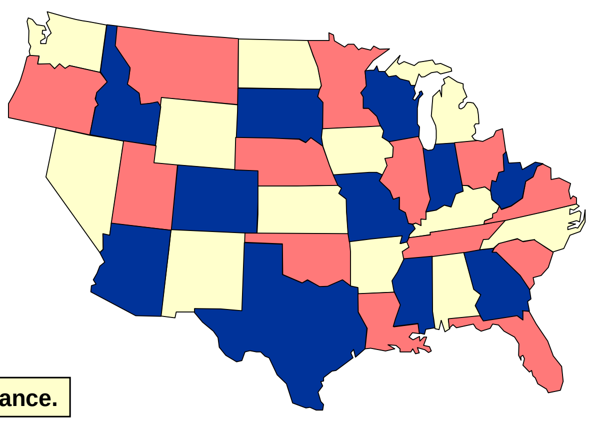
This lecture:

- Which problems can be solved in practice?
  - probably not 3-COLOR or TSP

Some Hard Problems

3-COLOR.

- Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

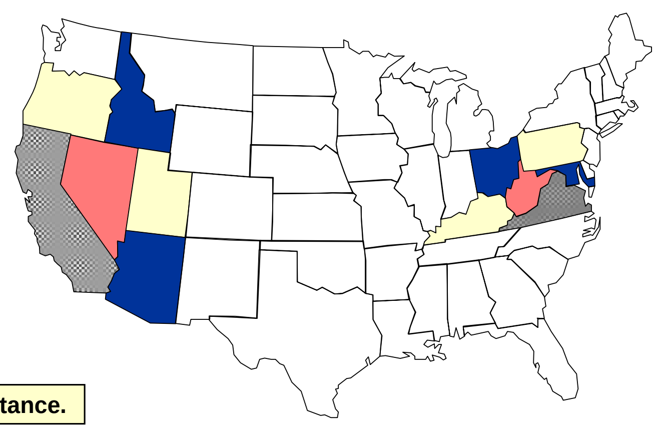


YES instance.

Some Hard Problems

3-COLOR.

- Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

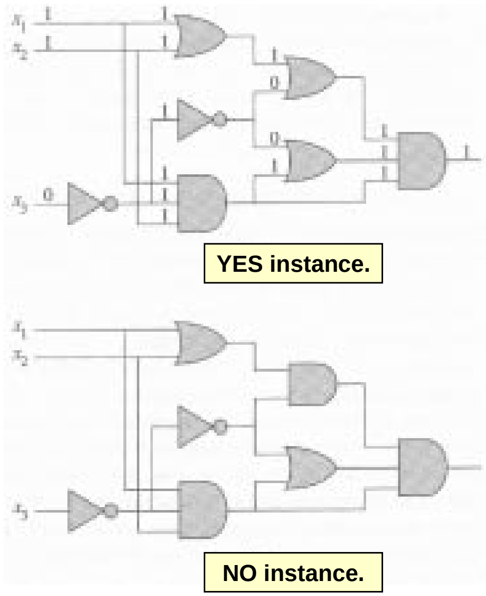


NO instance.

Some Hard Problems

CIRCUIT-SAT.

- Is there a way to assign inputs to a given Boolean (combinational) circuit that makes it true?



YES instance.

NO instance.

Some Hard Problems

FACTOR.

- Given two positive integers x and U, is there a nontrivial factor of x that is less than U?
- Factoring is at the heart of RSA encryption.

Input 1: x = 23,536,481,273, U = 110,000
YES since x = 224,737 × 104,729.

YES instance.

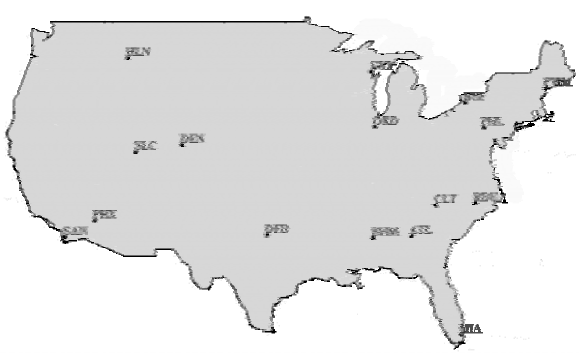
Input 2: x = 23,536,481,277, U = 110,000
NO since x is prime.

NO instance.

Some Hard Problems

TSP.

- A travelling salesperson needs to visit N cities. Is there a route of length at most D?



Is there a tour of length at most 1570?

Properties of Algorithms

A given problem can be solved by many different algorithms (TM's).

- Which ones are useful in practice?

A working definition: (Jack Edmonds, 1962)

- Efficient: polynomial time for ALL inputs.
  - mergesort requires $N \log_2 N$ steps
- Inefficient: "exponential time" for SOME inputs.
  - brute force TSP takes $N! > 2^N$ steps

Robust definition has led to explosion of useful algorithms for wide spectrum of problems.

Exponential Growth

Exponential growth dwarfs technological change.

- Suppose each electron in the universe had power of today's supercomputers.
- And each works for the life of the universe in an effort to solve TSP problem using N! algorithm from Lecture P6.

Some Numbers

quantity	number
Home PC instructions/second	10^9
Supercomputer instructions per second	10^{12}
Seconds per year	10^9
Age of universe in years (estimated)	10^{13}
Electrons in universe (estimated)	10^{79}

- Will not succeed for 1,000 city TSP!

  $1000! >> 10^{1000} >> 10^{79} * 10^{13} * 10^9 * 10^{12}$

Properties of Problems

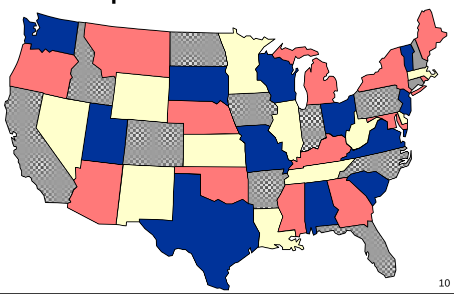
Which ALGORITHMS will be useful in practice?

- Efficient: polynomial-time for ALL inputs.
  - broad and robust definition
  - covers virtually all algorithms running on actual computers
- Inefficient: "exponential-time" for SOME inputs.

Which PROBLEMS will we be able to solve in practice?

- Those with efficient algorithms.
- How can I tell if I am trying to solve such a problem?
  - 2-COLOR: yes
  - 3-COLOR: probably no
  - 4-COLOR: yes

Theorem (Appel-Haken, 1976).
Every planar map is 4 colorable.



P

Definition of P:

- Set of all decision problems solvable in polynomial time on a deterministic Turing machine.

Examples:

- MULTIPLE: Is the integer y a multiple of x?
  - YES: (x, y) = (17, 51).
- RELPRIME: Are the integers x and y relatively prime?
  - YES: (x, y) = (34, 39).
- MEDIAN: Given integers $x_1, \ldots, x_n$, is the median value < M?
  - NO: $(M, x_1, x_2, x_3, x_4, x_5) = (17, 82, 5, 104, 22, 10)$

Definition important because of Strong Church-Turing thesis.

Strong Church-Turing Thesis

Strong Church-Turing thesis:

- P is the set of all decision problems solvable in polynomial time on REAL computers.

Evidence supporting thesis:

- True for all physical computers.
  - can create deterministic TM that efficiently simulates TOY machine (and vice versa)
  - can create deterministic TM that efficiently simulates any real general-purpose machine (and vice versa)
- Possible exception?
  - quantum computers – no conventional gates

NP

Definition of NP:

- Does NOT mean "not polynomial."

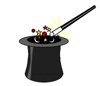
NP

Definition of NP:

- Set of all decision problems solvable in polynomial time on a NONDETERMINISTIC Turing machine.
- Definition important because it links many fundamental problems.

Useful alternate definition:

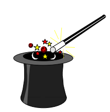
- Set of all decision problems with efficient verification algorithms.
  - efficient = polynomial number of steps on deterministic TM
- Verifier: algorithm for decision problem with extra input.



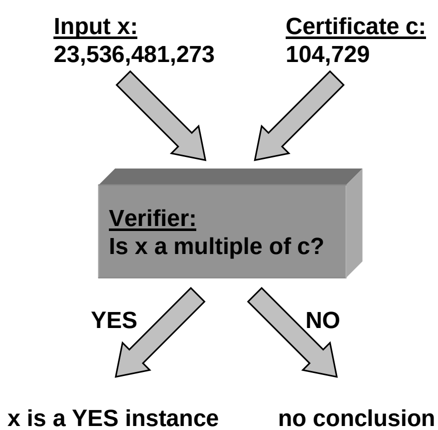
- Intuition: nondeterministic TM can try all possible solutions in parallel.

Verifiers and Certificates

COMPOSITE: Given integer x, is x composite?



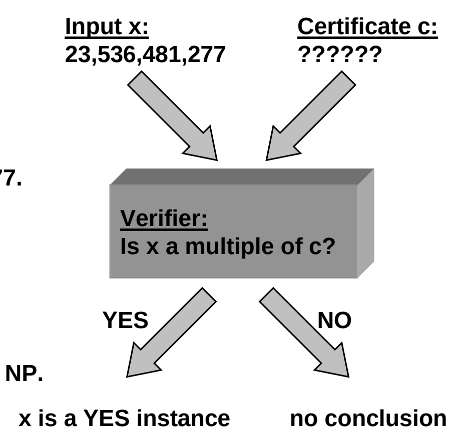
- YES instance: x = 23,536,481,273.
  - a corresponding certificate: c = 104,729 (a factor)
  - every YES instance has such a certificate



Verifiers and Certificates

COMPOSITE: Given integer x, is x composite?

- YES instance: x = 23,536,481,273.
  - a corresponding certificate: c = 104,729 (a factor)
  - every YES instance has such a certificate
- NO instance: x = 23,536,481,277.
  - no NO instance has a valid certificate
  - can never fool verifier into saying YES
- Conclusion: COMPOSITE is in NP.

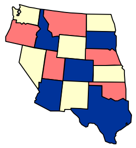


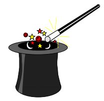
Verifiers and Certificates

3-COLOR: Given planar map, can it be colored with 3 colors?

Input x:

Certificate c:





Verifier:
1. Check that x and c describe same map.
2. Count number of distinct colors in c.
3. Check all pairs of adjacent states.

YES → x is a YES instance

NO → no conclusion

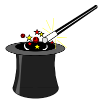
3-COLOR is in NP.

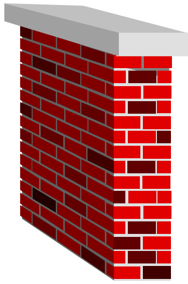
NP

NP = set of decision problems with efficient verification algorithms.

Why doesn't this imply that all problems in NP can be solved efficiently?

- BIG PROBLEM: need to know certificate ahead of time.
  - real computers can simulate by guessing all possible certificates and verifying
  - naïve simulation takes exponential time unless you get "lucky"





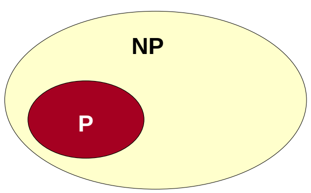
The Main Question

Does P = NP? (Edmonds, 1962)

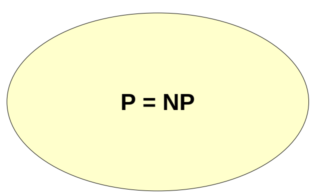
- Is the original DECISION problem as easy as VERIFICATION?
- Does nondeterminism help you solve problems faster?

Most important open problem in computer science.

- If yes, staggering practical significance.
- Even ranked #3 in all of pure mathematics. (Smale, 1999)



If $P \neq NP$



If P = NP

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as VERIFICATION?

If yes, then:

- Efficient algorithms for 3-COLOR, TSP, FACTOR.
- Cryptography is impossible (except for one-time pads) on conventional machines.
- Modern banking system will collapse.
- Harmonial bliss.

If no, then:

- Can't hope to write efficient algorithm for TSP.
  - see NP-completeness
- But maybe efficient algorithm still exists for factoring??

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as VERIFICATION?

Probably no, since:

- Thousands of researchers have spent four decades in search of polynomial algorithms for many fundamental NP problems without success.
- Consensus opinion: $P \neq NP$.

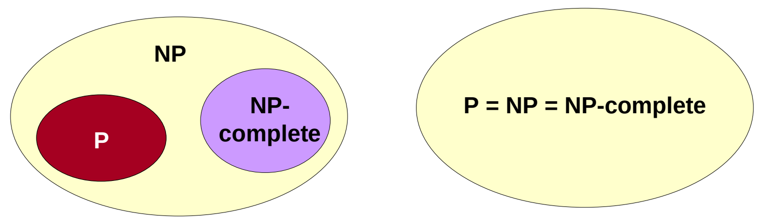
But maybe yes, since:

- No success in proving $P \neq NP$ either.

NP-Complete

Definition of NP-complete:

- A problem with the property that if it can be solved efficiently, then it can be used as a subroutine to solve any other problem in NP efficiently.
- "Hardest computational problems" in NP.



If $P \neq NP$ If $P = NP$

NP-Complete

Definition of NP-complete:

- A problem in NP with the property that if it can be solved efficiently, then it can be used as a subroutine to solve any other problem in NP efficiently.

Links together a huge and diverse number of fundamental problems:

- TSP, 3-COLOR, CIRCUIT-SAT, thousands more.
- Given an efficient algorithm for 3-COLOR, can efficiently solve TSP, CIRCUIT-SAT, FACTOR, etc.
- Can implement any program in 3-COLOR.

Note: FACTOR not known to be NP-complete.

Notorious complexity class.

- Only exponential algorithms known for these problems.
- Called intractable - unlikely that they can be solved given limited computing resources.

Reduction

Reduction is a general technique for showing that one problem is harder (easier) than another.

- For problems A and B, we can often show: if A can be solved efficiently, then so can B.
- In this case, we say B reduces to A. (B is "easier" than A).

Intuition: Finding median of n items reduces to sorting.

- Given an algorithm for sorting, want to design algorithm for finding the median.
  - Step 1: Sort $x_1, x_2, x_3, \ldots, x_N$
  - Step 2: Compute $m = N / 2$
  - Step 3: Return x_m

Reduction

Reduction is a general technique for showing that one problem is harder (easier) than another.

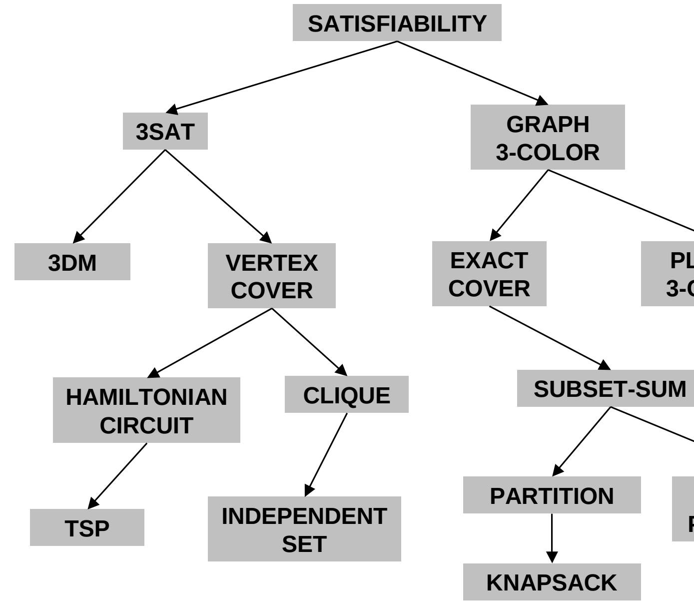
- For problems A and B, we can often show: if A can be solved efficiently, then so can B.
- In this case, we say B reduces to A. (B is "easier" than A).

Warmup: PRIMALITY reduces to FACTOR.

- Given an efficient algorithm for FACTOR(X, L), want to design an efficient algorithm for PRIMALITY(p).
  - Step 1: Compute FACTOR(p, p).
  - Step 2: If answer = YES, return NO.
    Else return YES.

  - original problem: Is p = 23,536,481,273 prime?
  - transformed instance: Does X = 23,536,481,273 have a nontrivial factor less than L = 23,536,481,273?

Reduction



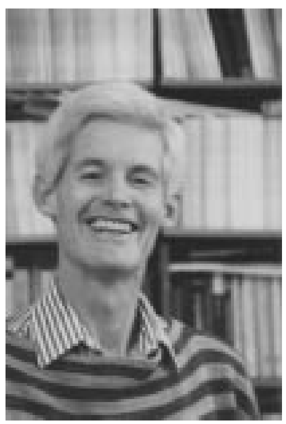
Dick Karp (1972)

The "World's First" NP-Complete Problem

SAT is NP-complete. (Cook-Levin, 1960's)

Idea of proof:

- By definition, nondeterministic TM can solve problem in NP in polynomial time.
- Polynomial-size Boolean formula can describe (nondeterministic) TM.
- Given any problem in NP, establish a correspondence with some instance of SAT.
- SAT solution gives simulation of TM solving the corresponding problem.
- IF SAT can be solved in polynomial time, then so can any problem in NP (e.g., TSP).



Stephen Cook

Coping With NP-Completeness

Hope that worst case doesn't occur.

- Complexity theory deals with worst case behavior. The instance(s) you want to solve may be "easy."
  - TSP where all points are on a line or circle
  - 13,509 US city TSP problem solved

(Cook et. al., 1998)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.
- Develop a heuristic, and hope it produces a good solution.
  - TSP assignment.
- Design an approximation algorithm: algorithm that is guaranteed to find a high-quality solution in polynomial time.
  - active area of research, but not always possible!
  - Euclidean TSP tour within 1% of optimal

Sanjeev Arora (1997)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

Exploit intractability.

Keep trying to prove P = NP.

Summary

Many fundamental problems are NP-complete.
- TSP, CIRCUIT-SAT, 3-COLOR.

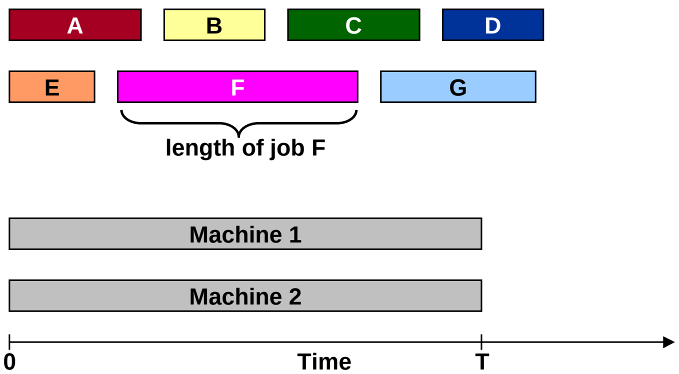
Theory says we probably won't be able to design efficient algorithms for NP-complete problems.
- You will likely run into these problems in your scientific life.
- If you know about NP-completeness, you can identify them and avoid wasting time.

Lecture T6: Extra Slides

Some Hard Problems

SCHEDULE

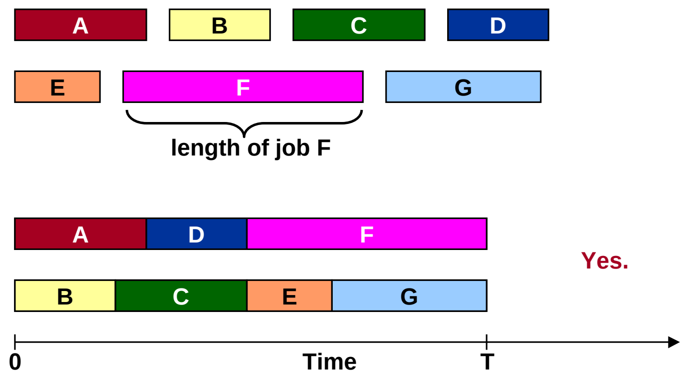
- A set of jobs of varying length need to be processed on two identical machines before a certain deadline T. Can the jobs be arranged so that the deadline is met?



Some Hard Problems

SCHEDULE

- A set of jobs of varying length need to be processed on two identical machines before a certain deadline T. Can the jobs be arranged so that the deadline is met?



Yes.

Some Hard Problems

CLIQUE

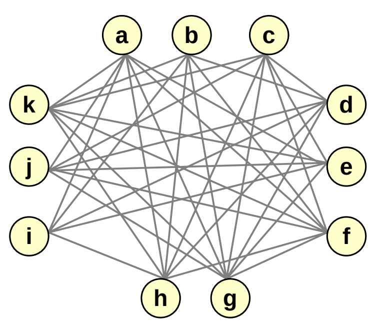
- Given N people and their pairwise relationships. Is there a group of S people such that every pair in the group knows each other.

Friendship Graph

People: a, b, c, d, e, . . ., k

Friendships: (a, e), (a, f), (a, g), . . ., (h, k)

Clique size: S = 4?



Some Hard Problems

CLIQUE

- Given N people and their pairwise relationships. Is there a group of S people such that every pair in the group knows each other.

Friendship Graph

People: a, b, c, d, e, . . ., k

Friendships: (a, e), (a, f), (a, g), . . ., (h, k)

Clique size: S = 4?

Yes - {b, d, i, h} is a witness.