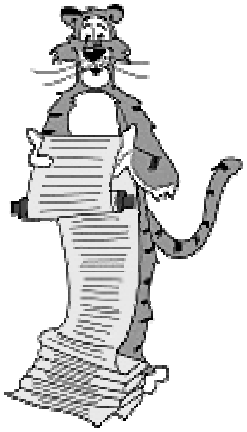
Lecture A3: Boolean Circuits



George Boole
(1815 – 1864)

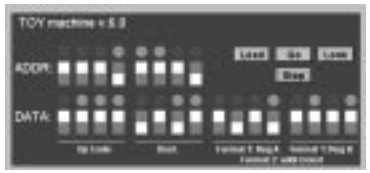
Claude Shannon
(1916 – present)

Architecture

Lecture A1 – A2.
- TOY machine.

Lecture A3 – A4.
- Digital circuits.

Lecture A5.
- Putting it all together.



Digital Circuits

What is a digital system?

Why digital systems?

Basic abstraction.
- On, off.
- Switch that can turn something on or off.

Digital circuits and you.
- Computer microprocessors.
- Antilock brakes.
- VCR.
- Cell phone.

Switches

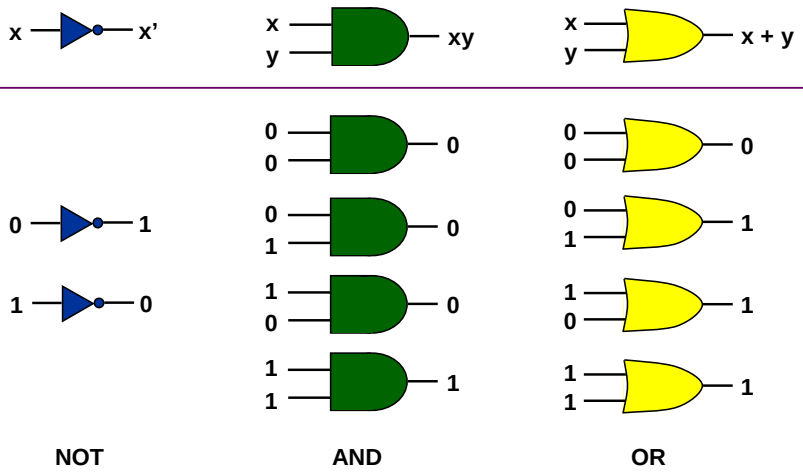
Abstract switch.
- Electrical (transistor).
- Electro-mechanical (relay).
- Hydraulic (water valve).

Hydraulic OR Gate

Logical Gates

Logical gates.

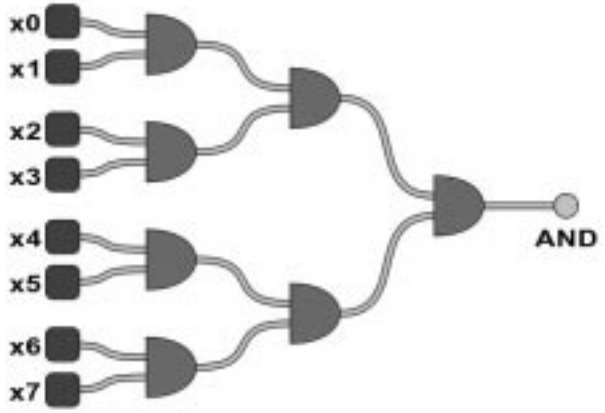
- Fundamental building blocks.



Multiway AND Gates

AND(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7).

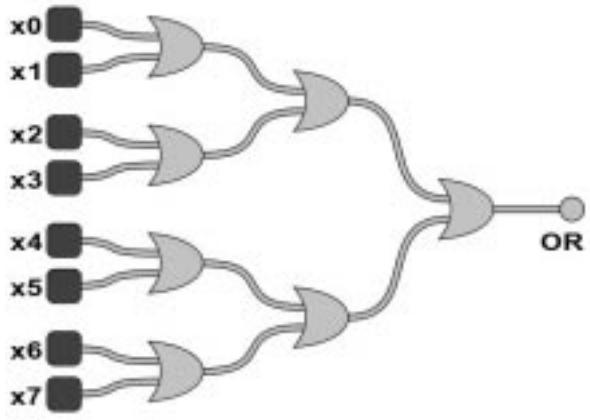
- 1 if all inputs are 1.
- 0 otherwise.



Multiway OR Gates

OR(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7).

- 1 if at least one input is 1.
- 0 otherwise.



Boolean Algebra

History.

- Developed by Boole to solve mathematical logic problems (1847).
- Shannon first applied to digital circuits (1939).

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

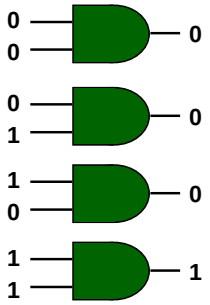
- Boolean variables: signals.
- Boolean functions: circuits.

Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs $\Rightarrow$ 2^N rows.

AND Truth Table		
x	y	AND
0	0	0
0	1	0
1	0	0
1	1	1



Truth Table

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one

Truth Table for All Boolean Functions of 2 Variables									
x	y	ZERO	AND	???	X	???	Y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for All Boolean Functions of 2 Variables									
x	y	NOR	EQ	Y'	???	X'	???	NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one
- 256 Boolean functions of 3 variables.
  - every 8-bit value represents one
- 2^2^N Boolean functions of N variables!

Some Functions of 3 Variables							
x	y	z	AND	OR	MAJ	ODD	MUX
0	0	0	0	0	0	0	0
0	0	1	0	1	0	1	0
0	1	0	0	1	0	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	0
1	1	0	0	1	1	0	1
1	1	1	1	1	1	1	1

Universality of AND, OR, NOT

Any Boolean function can be expressed using AND, OR, NOT.

- "Universal."
- XOR(x,y) = xy' + x'y

Expressing XOR Using AND, OR, NOT							
x	y	x'	y'	x'y	xy'	x'y + xy'	XOR
0	0	1	1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	1	1	1
1	1	0	0	0	0	0	0

Exercise: {AND, NOT}, {OR, NOT}, {NAND} are universal.

Sum-of-Products

Any Boolean function can be expressed using AND, OR, NOT.

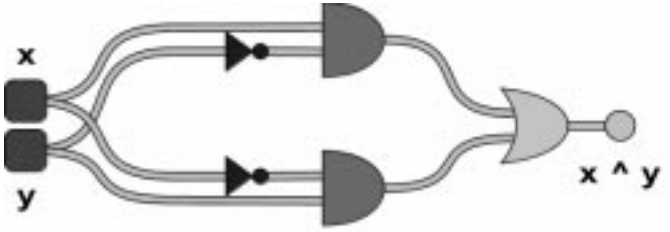
- Sum-of-products is systematic procedure.
  - form AND term for each 1 in Boolean function
  - OR terms together

Expressing MAJ Using Sum-of-Products								
x	y	z	MAJ	x'yz	xy'z	xyz'	xyz	x'yz + xy'z + xyz' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

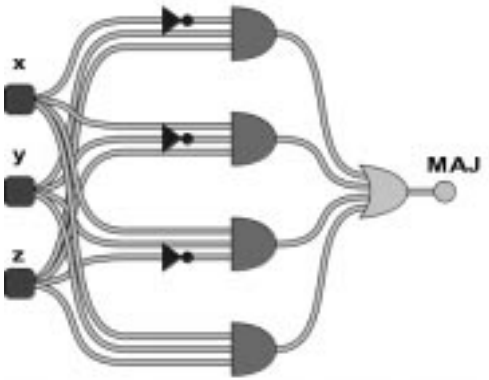
- XOR(x, y) = xy' + x'y.



Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

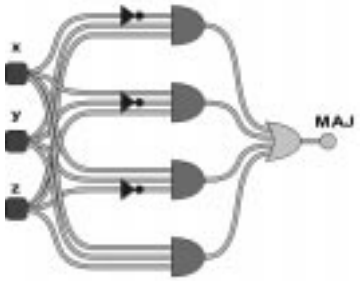
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz.



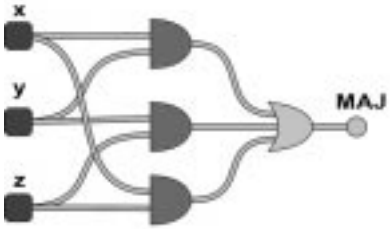
Simplification Using Boolean Algebra

Many possible circuits for each Boolean function.

- Sum-of-products not optimal.
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.



size = 8, depth = 4



size = 4, depth = 3

Recipe for Making Combinational Circuit

Ingredients.

- AND
- OR
- NOT
- wire

White Chocolate Mousse Cake, Balducci's

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

ODD Parity Circuit

ODD(x, y, z).

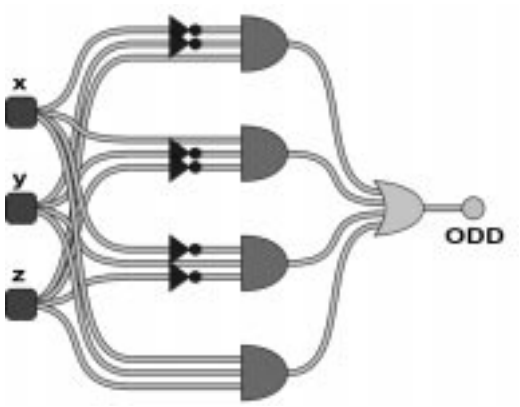
- 1 if odd number of inputs are 1.
- 0 otherwise.

Expressing ODD Using Sum-of-Products								
x	y	z	ODD	x'y'z	x'yz'	xy'z'	xyz	x'y'z + x'yz' + xy'z' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	1	0	0	0	1	1

ODD Parity Circuit

ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.



Let's Make an Adder Circuit

Goal: x + y = z.

- We build 4-bit adder: 8 inputs, 4 outputs. (Same idea scales to 64-bit adder.)

	1	1	1	
	2	4	8	7
+	3	5	7	9
	6	1	6	6

Step 1.

- Represent input and output in binary.

	1	1	0	
	0	0	1	0
+	0	1	1	1
	1	0	0	1

	c_3	c_2	c_1	
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Let's Make an Adder Circuit

Goal: x + y = z.

Step 2.

- **Build truth table.**

	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Adder Truth Table

x_0	x_1	x_2	x_3	y_0	y_1	y_2	y_3	z_0	z_1	z_2	z_3
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	1	0	1	0	1	0	1
.	.	.	.	.	.	.	.	.	.	.	.
1	1	1	1	1	1	1	1	1	1	1	0

$2^8 = 256$ rows!

Let's Make an Adder Circuit

Goal: x + y = z.

Step 2.

- **Build truth table for carry bit.**
- **Build truth table for summand bit.**

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Carry Bit

x_i	y_i	c_i	c_{i+1}
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Summand Bit

x_i	y_i	c_i	z_i
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Let's Make an Adder Circuit

Goal: x + y = z.

Step 3.

- **Derive (simplified) Boolean expression.**

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Carry Bit

x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Summand Bit

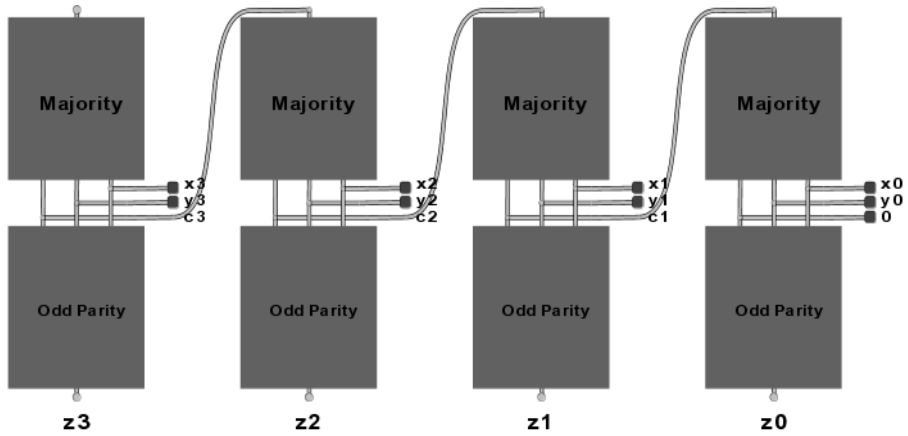
x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Let's Make an Adder Circuit

Goal: x + y = z.

Step 4.

- **Transform Boolean expression into circuit.**

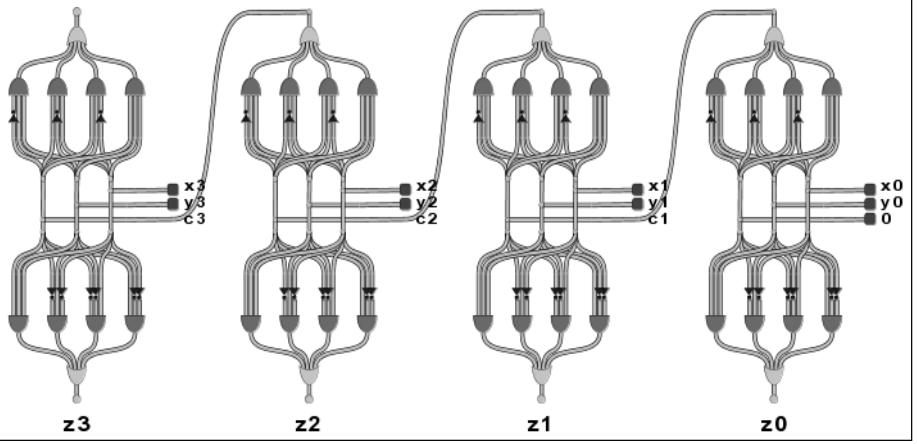


Let's Make an Adder Circuit

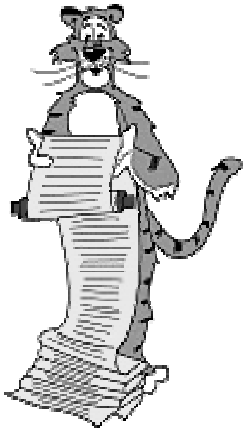
Goal: x + y = z.

Step 4.

- **Transform Boolean expression into circuit.**



Lecture A1: Extra Slides



Two Important Combinational Circuits

Multiplexer.

- **Selects an addressed input line.**

Decoder.

- **Converts from binary to unary.**

Translates directly to circuit without sum-of-products.

Useful to implement key computer components.

Multiplexer

N-bit multiplexer.

- **N address inputs, 2^N data inputs.**
- **1 output.**
- **Copies one data input to output.**
- **Which one?**

Ex.

- **2-bit multiplexer.**
- **Address inputs: s_0, s_1.**
- **Data inputs: x_0, x_1, x_2, x_3.**
- **Output: Q.**

s_1	s_0
1	1

x_3	x_2	x_1	x_0
1	0	0	1

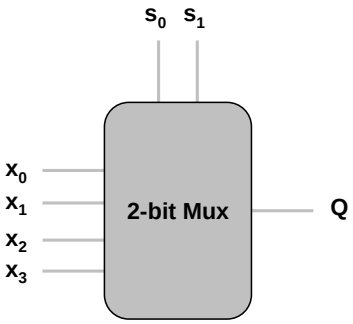
$11_2 = 3_{10}$

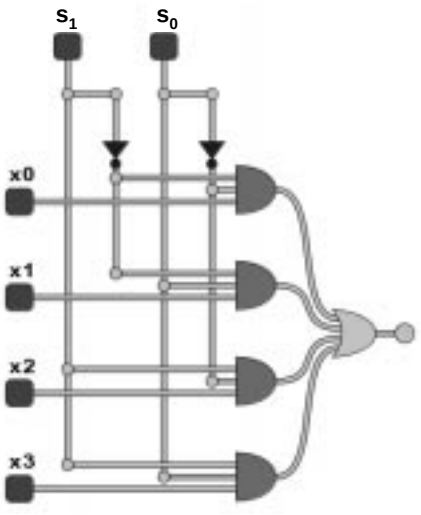
Q
1

2-Bit Multiplexer Circuit

MUX(x_0, x_1, x_2, x_3, s_0, s_1).

- x_0 if $s_1 = 0$, $s_0 = 0$.
- x_1 if $s_1 = 0$, $s_0 = 1$.
- x_2 if $s_1 = 1$, $s_0 = 0$.
- x_3 if $s_1 = 1$, $s_0 = 1$.





Decoder

N-bit decoder.

- N inputs.
- 2^N outputs.
- Exactly one of outputs is 1; rest are 0.
- Which one?

Ex.

- 3-bit decoder.
- Inputs: x_0, x_1, x_2.
- Outputs: Q_0, Q_1, Q_2, …, Q_7.

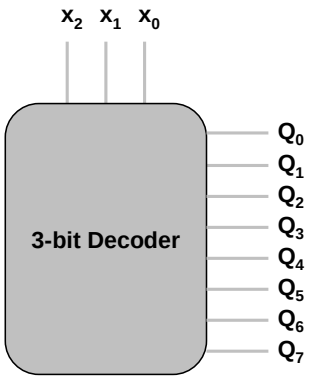
x_2	x_1	x_0
0	1	1

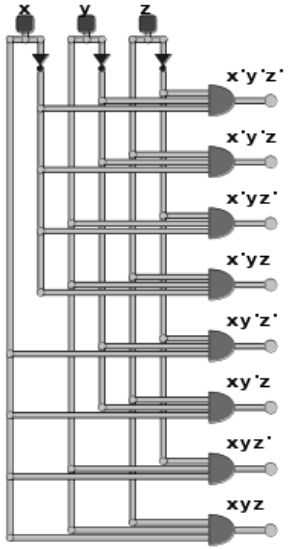
$011_2 = 3_{10}$

Q_7	Q_6	Q_5	Q_4	Q_3	Q_2	Q_1	Q_0
0	0	0	0	1	0	0	0

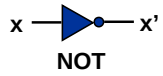
3-Bit Decoder

DECODE(x, y, z).

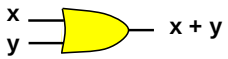




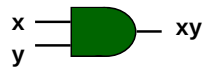
Cheat Sheet



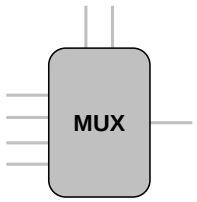
NOT



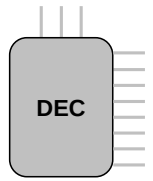
OR



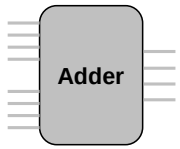
AND



2-bt Multiplexer



3-bit Decoder



4-bit Adder