

Problem Set 7 Answer Key

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1 Section 11.8, Ex. 12

(Ex.9 is easy and straightforward, not worth posting out.)

It is easy to verify that the left graph and the middle one are isomorphic graphs. The right graph is not isomorphic with the left/middle graph because it contains quadrilaterals while the left/middle one does not.

2 Section 11.8, Ex. 20

Suppose by contradiction that there exists a disconnected graph G with n vertices and at least $\frac{(n-1)(n-2)}{2} + 1$ edges, then G can be divided into two disconnect subgraphs G_1 and G_2 . Suppose G_1 has n_1 vertices and G_2 has n_2 vertices, where $n_1 + n_2 = n$, then the number of edges in G satisfies: $|E| \leq \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} = \frac{(n-n_2)(n-n_2-1)}{2} + \frac{n_2(n_2-1)}{2} = \frac{(n-1)(n-2)}{2} + (n_2 - 1)(n_2 + 1 - n) \leq \frac{(n-1)(n-2)}{2}$. The last inequality comes from the fact that $1 \leq n_2 \leq (n-1)$. The above means that $|E| \leq \frac{(n-1)(n-2)}{2}$, which is a contradiction to the condition that G has at least $\frac{(n-1)(n-2)}{2} + 1$ edges. So G must be connected. However, if G contains one fewer edge than given in the problem, then it can be a disconnected graph. An example is a $n-1$ order complete graph plus an extra vertex which is not connected by any edges.

3 Section 11.8, Ex. 40

Using Theorem 11.3.2 in page 430, it is clear that we only need to show this graph satisfies the *Ore property*. For any two non-adjacent vertices u and v of G , the graph $G/\{u, v\}$ has at most $(n-2)(n-3)/2$ edges (it reaches that number when it is a complete graph). So $\deg(u) + \deg(v) \geq (n-1)(n-2)/2 + 2 - (n-2)(n-3)/2 = n$, which means G satisfies the *Ore property*. When G contains one fewer edge, then a counter example is still an order $n-1$ complete graph plus an extra vertex, but this time the

extra vertex is connected to one vertex in the complete graph. It is easy to see that this graph has no Hamilton cycle.

4 Section 13.6, Ex. 5

The left graph has its largest clique size 2, and it is 2-colorable, so its chromatic number is 2; The middle graph has its largest clique size 3, and it is 3-colorable, so its chromatic number is 3. The right graph has its largest clique size 4, and it is 4-colorable, so its chromatic number is 4.

5 Section 13.6, Ex. 13

We prove this result by induction on n :

1. Base case $n = 3$: then C_n is now a triangle whose three vertices must be assigned with three different colors. So its chromatic polynomial $P_3(k) = k(k-1)(k-2) = (k-1)^3 - (k-1)$.
2. Suppose the chromatic polynomial of cycle graph C_{n-1} is $P_{n-1}(k) = (k-1)^{n-1} + (-1)^{n-1}(k-1)$.
3. For cycle graph C_n , using equation (13.1) we know that $P_n(k) = P'_n(k) - P_{n-1}(k)$, where $P'_n(k)$ is the chromatic polynomial of the graph obtained by removing one edge from C_n . By Theorem 13.1.7, it is easy to see that $P'_n(k) = k(k-1)^{n-1}$, so $P_n(k) = k(k-1)^{n-1} - (k-1)^{n-1} + (-1)^n(k-1) = (k-1)^n + (-1)^n(k-1)$. This completes our induction.

6 Section 13.6, Ex. 18

One example is like this: first draw a cycle graph C_5 whose vertices are v_1 through v_5 , then add another vertex u and connect it to every vertex v_i . This graph is obviously a planar graph. Since the chromatic number of C_5 is 3, and since the color of u must be different from every v_i , we conclude that this graph has chromatic number 4. This graph also has no induced subgraph K_4 because otherwise C_5 should contain induced subgraph K_3 , which is impossible.

7 Section 13.6, Ex. 30

First we show that a tree is 2-colorable. In fact, the following procedure colors a tree with 2 colors: first find the root u of this tree, then for every other vertex v there is an unique simple path from u to v . If the length of this path is odd, then color v with color 1; if the length is even, then color v with color 0; Finally color u with color 0. It is easy to verify that this procedure gives a vertex-coloring.

The only thing left is to notice that all vertices with color 0 (1) constitute an independence set of the graph, and since the sum of these two sets is n , one of them must be at least $\lceil n/2 \rceil$.

8 Section 13.6, Ex. 42

By the definition of *perfect* graphs, we only need to show that every induced subgraph H of a bipartite graph G is also perfect, that is, the chromatic number of H equals to the size of its largest clique. This is almost obvious: suppose $G = (X, \Delta, Y)$, if all vertices of H are in X (Y), then the chromatic number and the size of its largest clique are both 1; if the vertices of H come from X and Y but there are no edges connecting between them, then chromatic number and largest clique size are still both 1; if the vertices of H come from X and Y and there exist some edges connecting between these two set of vertices, then the chromatic number and the largest clique size are both 2.

9 Special problem 1

From the spectrum S given in the problem, we can construct G_S as shown in handout number 11 (DNA Sequencing). It is easy to see that G_S has three open Eulerian trails, which correspond to three possible strings for $\overline{\sigma}$ as shown below:

- $\overline{\sigma} = GACTGACGAACTCACGT$ $\sigma = CTGACTGCTTGAGTGCA$
- $\overline{\sigma} = GACTCACGAACTGACGT$ $\sigma = CTGAGTGCTTGACTGCA$
- $\overline{\sigma} = GACGAACTGACTCACGT$ $\sigma = CTGCTTGACTCAGTGCA$

Since in those three possible $\overline{\sigma}$ strings, we can always find the pattern *ACGAACT* and *CTCACG* and we will never find the pattern *TGACCT*, we conclude that this dinosaur is vegetarian and can fly, but may not be a timid one.

10 Special problem 2

- The size of the largest clique is 5, corresponding to the clique composed of all $v_i (1 \leq i \leq 5)$.
- The size of the largest independent set is 4, this is because all v_i is adjacent to each other, so we can only select one from v_i (v_4 or v_5), plus all vertices w_i .
- First we know that $\chi(G) \geq \omega(G) = 5$, second this graph is 5-colorable, so $\chi(G) = 5$.
- G is not a planar graph because it contains K_5 as its subgraph, which is not planar.
- G is not an Eulerian graph because 6 of its vertices, $v_1, v_2, v_3, w_1, w_2, w_3$, have odd degrees.
- G does not contain a Hamiltonian cycle. A short proof is like this: suppose by contradiction that it has a Hamiltonian cycle, then it is easy to see that those part of the cycle concerning with three w_i must has this form: $\dots \square w \square w \square w \square \dots$, where $\square$ must be filled with v_1, v_2 or v_3 . But there are four positions to fill and we only have three candidates, so the required Hamiltonian cycle does not exist.