CS 341 — Discrete Math — Fall 1999

Problem Set 6 Answer Key

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1 Section 7.8, Ex. 39

If we use h_n to denote the number of possible colorings for a 1 by n chessboard, then the exponential generating function for h_n is:
$(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)^2(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots)^2 = \left(\frac{e^x+e^{-x}}{2}\right)^2(e^x)^2 = \left(\frac{e^{2x}+1}{2}\right)^2 = \frac{e^{4x}}{4}+\frac{e^{2x}}{2}+\frac{1}{4}$
The coefficient of $x^n(n \geq 1)$ in this generating function is $\frac{4^{n-1}}{n!}+\frac{2^{n-1}}{n!}$, so $h_n = 4^{n-1}+2^{n-1}$.

2 Section 9.5, Ex. 6

(Ex. 2 is very easy, everyone did it well, so we omit its answer here)
For the given m-by-n chessboard, we associate it with a bipartite graph $G = (X, \Delta, Y)$ as shown in the textbook (page 296-297). Here $X = w_1, w_2, ...w_k$ is the set of white squares and $Y = b_1, b_2, ...b_k$ is the set of black squares, $k = (mn)/2$. Since mn is even, we know that there exists a perfect cover with dominoes, which implies that G has a perfect matching M. Now suppose w_i is the forbidden white square and b_j is the forbidden black one. To show there is a perfect cover when w_i and b_j are forbidden, we only need to show that there exists a perfect matching in $G - \{w_i, b_j\}$. There are two cases:

- $< w_i, b_j >$ is an edge of M, then we simply ignore it, and the rest of M constitutes a desired perfect matching.
- $< w_i, b_j >$ is not an edge of M, then we try to find an alternative chain connecting w_i and b_j by the following process:
    1. Initially vertex set W only has one element w_i which is marked as "unused", vertex set B is empty. If all vertices in W are marked as "used", then quit; otherwise for every "unused" vertex w_i in W, we find the edge in M that meets w_i, suppose the other endpoint is b_{i1}, add b_{i1} to B and mark it as "unused", mark w_i as "used".

2. Now for every "unused" vertex b_k in B, find all edges starting from it that do not in M, mark b_k as "used", for each newly found edge, look at the other endpoint (the one other than b_k). If it has been marked as "used", then ignore this edge and turn to the next edge connecting b_k. If that endpoint has not been marked as "used" sofar, put it into W and mark it as "unused". If in this step we do not add any new vertices to W, then quit this process, else go back to step 1.

Now we want to show that the above process must end up with W equals to X (that is, W contains all white square vertices) and B equals to Y. Suppose to the contrary this is not the case, then we know that when we quit the process, because of the action of step 1, there is a perfect matching between W and B and so $|W| = |B|$. What's more, for every vertex in B, it only connects with vertices in W. This corresponds to a "strange" partition of the chessboard into two parts: $A_1 = W \cup B$ and $A_2 = X/W \cup Y/B$, where every black square in A_1 does not "touch" any white squares in A_2, in other words, all white squares of A_1 are "enclosed" by black squares of A_1, and yet the number of white squares equals to the number of black squares. One way to show that this kind of "strange" partition cannot actually happen is to consider the total number of sides of white squares and black squares in A_1. Notice that each square has four sides, so if we sum the total number of sides over all white squares of A_1 and over all black squares, these two sums should be equal since $|W| = |B|$. However for an "internal" side (a side between a white square in W and a black square in B), it "contributes" one to both two sums, so all "internal" sides as a whole contribute equally to those two sums. Then we consider those "border" sides (those sides which are part of the border of the chessboard), it is easy to see that such "border" sides contribute more (or at least the same) to the sum over W than to the sum over B. What's more, since all sides between A_1 and A_2 are those that contribute only to W, we know that the sum over W should be greater than that of B, which is a contradiction with $|W| = |B|$. So in conclusion we know that such "strange" situation cannot occur, thus W becomes X after the above process. Suppose the edge in perfect matching M that meets forbidden square b_j is $< w_l, b_j >$, then we know that that there exists a sequence of vertices $x_1, y_1, x_2, y_2, ...x_t$ such that all vertices x_m are in X and all vertices y_n are in Y, $x_1 = w_i$ and $x_t = w_l$. So we find a cycle $w_i, y_1, x_2, ...w_l, b_j$ in which the edges are alternately ”in” and ”not in” perfect matching M. Now we exchange the edges in this cycle that are in M with those that are not in, and obtain another perfect matching M' in which $< w_i, b_j >$ is an edge of this matching, then we actually come to the case already discussed. So we can simply ignore $< w_i, b_j >$ from M', and the rest constitutes a perfect matching, which means the resulting board has a perfect cover with dominoes.

(note: another method (much shorter when being expressed) of attacking this problem is to ignore all stuff about perfect matching, and say everything directly about the chessboard. We can divide the chessboard into several parts, one of which contains those two forbidden squares as oppositing corners. Then we show that each part has a perfect cover. Basically there are 3 cases to discuss, but they are similar to each other)

3 Section 9.5, Ex. 8

Proof by induction as follows:

- $p = 1$, by Theorem 9.2.5 we know that there exists a perfect matching in G, which is just a partition of G into ”1” perfect matching.
- Suppose this is true for $p - 1$.
- Now consider the case for p, by Theorem 9.2.5 we know that there exists a perfect matching M_1 of G. Now we remove all edges in M_1 from G and obtain graph G'. It is easy to see that G' is a regular bipartite graph of degree $p - 1$, by induction hypothesis we know that G' can be partitioned into $p - 1$ perfect matchings $M_2, M_3, ...M_p$, so we know that $G = G' \cup M_1$ can be partitioned into p perfect matchings $M_1, M_2, ...M_p$.

4 Section 9.5, Ex. 20

Suppose by contradiction that A is not paired with a in a stable marriage, then she must pair with another man b. Similarly a must pair with another woman C. Then since A ranks a first, she perfers a than her partner b; It is also true that a prefers A than his partner C. So by definition this is not a stable marriage, a contradiction. So A must pair with a in any stable marriage.

5 Section 9.5, Ex. 26

Below is a stable complete marriage obtained by applying the deferred acceptance algorithm:
$A \leftrightarrow c$; $B \leftrightarrow d$; $C \leftrightarrow a$; $D \leftrightarrow b$;

6 Special Problem 1

$f(x) = (1 + x^2 + x^4 + ...)(\sum_{i=0}^{\infty} \binom{\frac{1}{3}}{i} x^i)$, so the coefficient of x^4 in $f(x)$ $a_4 = \binom{\frac{1}{3}}{0} + \binom{\frac{1}{3}}{2} + \binom{\frac{1}{3}}{4} = \frac{206}{243}$. Similarly $a_5 = \binom{\frac{1}{3}}{1} + \binom{\frac{1}{3}}{3} + \binom{\frac{1}{3}}{5} = \frac{310}{729}$