CS 341 — Discrete Math — Fall 1999

Problem Set 5 Answer Key

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1 Section 7.8, Ex. 2

We note that for every real number r, there is at most one integer i such that $|r - i| < \frac{1}{2}$, and thus i is also the closest integer to r. In our problem, we have $r = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$, and we need to show $|i - r| < \frac{1}{2}$. Now $|i - r| = |(\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n) - \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n| = |\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n|$. We need only show show that $|\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n| < \frac{1}{2}$. And note that $\frac{1}{\sqrt{5}} < \frac{1}{2}$, so we need only show that $\frac{1-\sqrt{5}}{2}^n < 1$, but this is clearly true for all $n > 0$ because $\frac{1-\sqrt{5}}{2} < 1$.

2 Section 7.8, Ex. 12

The characterstic equation is $x^2 - 8x + 16 = 0$, wich has a two-fold root at 4. Thus the general solution is $h_n = c_1 4^n + c_2 n 4^n$. To find c_1 and c_2 so that the intial conditions hold, we need to solve simultaneously the equations $-1 = c_1$ and $0 = c_1 4 + c_2 4$. So clearly $c_1 = -1$ and $c_2 = 1$, so $h_n = -4^n + n4^n$.

3 Section 7.8, Ex. 22

The first thing we do is solve the corresponding homogeneous equation: $h_n = 4h_{n-1} - 4h_{n-2}$, which whose characteristic equation is $x^2 - 4x + 4 = 0$ and has a two-fold root at 2. So, we have the general solution to the homogeneous equation is $h_n = c_1 2^n + c_2 n 2^n$. For the particular solution, we try $h_n = rn + s$. This yields $rn + s = 4(r(n-1) + s) - 4(r(n-2) + s) + 3n + 1 = 3n + (4r + 1)$. Equating coefficients, we get $r = 3$ and $s = 13$. Now we combine our solutions to get the general solution $h_n = c_1 2^n + c_2 n 2^n + 3n + 13$. Simply plugging in $n = 0$, $h_0 = 1$ and $n = 1$, $h_1 = 2$ we get two equations, which we solve simultaneously to get $c_1 = -12$ and $c_2 = 5$. Thus, our final solution is $h_n = (5n - 12)2^n + 3n + 13$.

4 Special Problem 1

Using the same notation as the Math Casino handout, we need to find $\sum_{j=0}^{t} |A_j|$. We still have that $|A_0| = 1$, but now for $j > 0$, $|A_j| = \lfloor \frac{(j+1)^5 - 1 - j^5 + 1}{j} \rfloor = 5j^3 + 10j^2 + 10j + 6$. Now $m = 1 + \sum_{j=1}^{t} |A_j| =$
$1 + 5\sum_{j=1}^{t} j^3 + 10\sum_{j=1}^{t} j^2 + 10\sum_{j=1}^{t} j + \sum_{j=1}^{t} 6 =$
$1 + 5(\frac{t(t+1)}{2})^2 + 10(\frac{t(t+1)(2t+1)}{6}) + 10(\frac{t(t+1)}{2}) + 6(t-1)$. This is closed form invovling t, and $p = \frac{m}{t^5}$. If you go through the algebra to expand and then gather like terms, you get $p = \frac{5}{4t} + \frac{5}{6t^2} + \frac{5}{4t^3} + \frac{8}{3t^4} - \frac{5}{t^5}$.

5 Special Problem 2

I will solve this problem in the general case of a best-of-n series, since the best of 7 series is just a special case of that. Say team A is playing team B. How many ways can the series go the maximum number of games? $2t$, where t is the number of ways the series can be tied after $n-1$ games. How do we calculate t. Both teams must win the same number of games. We intially write down the $(n-1)/2$ a's, standing for games won by team A. Then, for each of $(n-1)/2$ b's we should place it in one of the $((n-1)/2)+1$ spaces between the a's and before and after the word. Thus there are $\binom{(n-1)/2+1+(n-1)/2-1}{(n-1)/2} = \binom{n-1}{(n-1)/2}$ ways the series can be tied, and thus there are $2\binom{n-1}{(n-1)/2}$ ways the series can go the distance. How many total ways can the series play out? Well, for our purposes we should answer the question, what is the weighted total of ways teh series can play out? That is, all outcomes are not equally likely, and if an outcome is twice is likely that an outcome where the series goes the distance, then we should count it twice. With a little thought, you can see that the number we require is simply 2^n. Therefore combining our results, we see then probability of a best-of-n series going the distance is $\frac{\binom{n-1}{(n-1)/2}}{2^{n-1}}$. In the case $n = 7$, this is $\frac{20}{64}$, or $\frac{5}{16}$. For part (b), we have $E(X) = 5.8125$ and $Var(X) = 263/156 = 1.027$.