CS 341 — Discrete Math — Fall 1999

Problem Set 4 Answer Key

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1 Section 5.8, Ex. 18

In this problem, we are asked to evaluate: $\sum_{i=0}^{n}(-1)^i \frac{1}{i+1}\binom{n}{i}$. It is easy to verify that $\frac{1}{i+1}\binom{n}{i} = \frac{1}{n+1}\binom{n+1}{i+1}$, so we are going to evaluate $E = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^i\binom{n+1}{i+1}$. It is a well known fact that $\sum_{i=-1}^{n}(-1)^i\binom{n+1}{i+1} = 0$ (you can find this formula in the textbook, section 5.3). Then $E = \frac{1}{n+1}(0-(-1)\binom{n+1}{0}) = \frac{1}{n+1}$.

2 Section 5.8, Ex. 20

To make the equality $m^3 = a\binom{m}{3}+b\binom{m}{2}+c\binom{m}{1}$ holds for all m, the coefficient of m^3 in both sides must be the same. In the left side it is 1, while in the right side it is $\frac{a}{6}$, so we know that $a = 6$. Then we substitute a by 6 in the equality and do some elimination, and it becomes $3m^2 - 2m = b\binom{m}{2} + c\binom{m}{1}$. Then compare the coefficient of m^2 in both sides we know that $3 = \frac{b}{2}$, hence $b = 6$. Then we substitute b by 6 and get $m = cm$, hence $c = 1$.
another way to calculate a, b, c is to let m be 1, 2, 3 in $m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$, and solve three linear equations:

$$\begin{aligned} 1 &= c \\ 8 &= b + 2c \\ 27 &= a + 3b + 3c \end{aligned}$$

From above,

$$\sum_{m=1}^{n} m^3 = 6\sum_{m=1}^{n}\binom{m}{3} + 6\sum_{m=1}^{n}\binom{m}{2} + \sum_{m=1}^{n}\binom{m}{1}$$

It is a formula (you can find it in section 5.3 of the textbook) that for any nonnegative integers k and n, $\sum_{m=1}^{n}\binom{m}{k} = \binom{n+1}{k+1}$. So
$\sum_{m=1}^{n} m^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = n^2(n+1)^2/4$

3 Special Problem 1

The Newton's Binomial Theorem tells us that $(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$
So the derivative of the left side which is $n(1+x)^{(n-1)}$ equals to the derivation of the right side which is $\sum_{k=0}^{n} kx^{k-1}\binom{n}{k}$. Then multiply x on both sides we get $\sum_{k=0}^{n} kx^k\binom{n}{k} = nx(1+x)^{(n-1)}$
notice that in the left side of the above equation, we can start from $k=1$ because when $k=0$, the corresponding item is 0. Then we substitute x by 5 in it then get $\sum_{k=1}^{n} k5^k\binom{n}{k} = 5n6^{n-1}$

4 Special Problem 2

It is a well known fact that (you can find it in section 5.3):

$$\sum_{0\le k\le n} \binom{n}{k}^2 = \binom{2n}{n}$$

we multiply $\binom{2n}{n}$ on both sides and get:

$$\sum_{0\le k\le n} \binom{n}{k}^2 \binom{2n}{n} = \binom{2n}{n}^2$$

just expand the left side according to the definition of combination, then we will get:

$$\sum_{0\le k\le n} \frac{(2n)!}{(k!)^2((n-k)!)^2} = \binom{2n}{n}^2$$

5 Special Problem 3

- First we show a one-to-one correspondence between paying patterns where at some stage the cashier goes at least \$1 in debt and all binary sequences of length $2n$ with exactly $n+1$ 1's.

  1. For each pattern in which we will eventually go to debt, we read it from left to right, and compare the number of 0s we have read with the number of 1s we have read. Since we will go to debt, then there must exist a moment in which we have read more 0s than 1s the first time. So the pattern may be written as $x0y$,

where there are equally number of 0s and 1s in x, and there are at least as many 1s as 0s in any prefix of x. Now we change this pattern to $x'1y$, where x' is the pattern obtained by exchanging 0 with 1 in x. Since we changed exactly one 0 to 1, there are exactly $n+1$ 1s and $n-1$ 0s in the new pattern $x'1y$. It is easy to see that in the above transformation, for different old patterns, we will get different new patterns.

2. Now give us a pattern of length $2n$ in which there are exactly $n+1$ 1s and $n-1$ 0s, we read it from left to right. Since there are more 1s than 0s, there must exist a moment in which we have read more 1s than 0s the first time. So the pattern may be written as $x1y$, where there are equally number of 0s and 1s in x, and there are at least as many 0s as 1s in any prefix of x. Now we change this pattern to $x'0y$, where x' is the pattern obtained by exchanging 0 with 1 in x. Since we changed exactly one 1 to 0, and there are $n+1$ 1s and $n-1$ 0s in the old pattern, we know that there are exactly n 1s and n 0s in the new pattern $x'0y$. What is more, in this new pattern after we read x', we will encounter a 0, which is a debt we will not be able to pay. So this new pattern is a pattern in which we will go to debt. It is easy to see that in the above transformation, for different old patterns we will get different new patterns.

It is easy to see that the two transformations in (1) and (2) are reverse to each other. So now we have constructed a desired one-to-one correspondence between paying patterns where the cashier goes into debt and all binary sequences of length $2n$ with exactly $n+1$ 1's.

- We know that the total number of different paying patterns is exactly $\binom{2n}{n}$, in above we have proved that the number of different paying patterns where the cashier goes into debt is exactly $\binom{2n}{n+1}$, so the number of paying patterns in which the cashier never goes into debt is $\binom{2n}{n} - \binom{2n}{n+1}$.

To help understanding the above proof well, I give a concrete example here. Now suppose $n = 2$, then there are totally $\binom{4}{2} = 6$ paying patterns. There are exactly $\binom{4}{3} = 4$ paying patterns in which the cashier goes into debt, they are listed below in the left side. The corresponding binary sequences for all such paying pattern are listed below in the right side. "$\leftrightarrow$" is the correspondence we used in the proof.

$$\begin{array}{ccc} 0011 & \longleftrightarrow & 1011 \\ 0101 & \longleftrightarrow & 1101 \\ 0110 & \longleftrightarrow & 1110 \\ 1001 & \longleftrightarrow & 0111 \end{array}$$

(*Consider the following problem: you walk from origin O(0,0) to the point P(n,n), in each step you can only go along the positive x direction or the positive y direction, then how many different walking ways are there with the restriction that* **you are not allowed** *to walk through any point (x,y) above the diagonal* $\overline{OP}$ *(that is, the line with equation* $x = y$*). Have you found some relationship between this problem and the paying pattern problem we just proved?*)