CS 341 — Discrete Math — Fall 1999

Problem Set 3 Answer Key

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1 Section 3.6, Ex. 34

Let us assume the first child receives the orange. How many ways can we distribute the apples. The answer is the same as the number of solutions to the equation $x + y + z = 12$, where $x \geq 0, y \geq 1, z \geq 1$. Let $y\prime = y - 1$ and $z\prime = z - 1$. Then we should solve $x + y\prime + z\prime = 10$. From Section 2.5, we know there are $\binom{10+3-1}{10}$ ways to solve this. Now we remember that we assumed the first child received the orange, but each child is equally likely to get oragne, so we multiply by 3. Thus the answer is $3 * \binom{12}{10}$.

2 Section 3.6, Ex. 36

The set of r-combinations can be paritioned into the following two groups: those that contain a member of type a_1 and those that don't. The number of ways to create to create an r-combination with a member of type a_1 is the same as the number of $r-1$ combinations of the set $\{\infty*a_2, \infty*a_3, ..., \infty*a_k\}$, or $\binom{r-1+k-1-1}{r-1}$. The number of ways to create to create an r-combination without a member of type a_1 is the same as the number of r combinations of the set $\{\infty * a_2, \infty * a_3, ..., \infty * a_k\}$, or $\binom{r+k-1-1}{r}$. Thus, by the addition principle, the total is $\binom{r+k-3}{r-1} + \binom{r+k-2}{r}$.

3 Section 5.8, Ex. 21

For any real number r, if $k > 0$ then we have, by definition, that $\binom{-r}{k} = \frac{-r(-r-1)...(-r-k+1)}{k!}$. Simply factoring we obatin the desired form, $(-1^k)(\frac{r(r+1)...(r+k-1)}{k!}) = \frac{(r+k-1)!(-1)^k}{(r-1)!k!} = \binom{r+k-1}{k}(-1)^k$. We must also consider the cases of $k = 0$ and $k < 0$. The reader may check the if $k = 0$, then both sides of the equation always evaluate to the constant 1. And if $k < 0$, both sides always evaluate to 0. Thus the equality is proved for all cases.

4 Section 5.8, Ex. 21

Suppose we partition some set M into two sets M_1 and M_2 with $M_1 \cap M_2 = \emptyset$. Let $m_1 = |M_1|$ and $m_2 = |M_2|$. Then $\binom{m_1+m_2}{n}$ is the number of n-combinations. Now, for each n-combination, there is some number $0 \leq k \leq n$ such that k elements of the combination are also elements of M_1. And thus, the remaining $n-k$ elements of the combination are elements of M_2. Note that we could partion the set of n-combinations into n sets by the the number of elements k of each combination that are in M_1. How many n-combinations are there with k elements from M_1? There are $\binom{m_1}{k}$ ways to choose the k elements from M_1, and $\binom{m_2}{n-k}$ ways to choose the $n-k$ elements from M_2, for a total of $\binom{m_1}{k}\binom{m_2}{n-k}$ by the multiplication principle. Thus summing over all values for k, we obtain the total which proves our equality, $\sum_{k=0}^{n}\binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}$.

5 Section 5.8, Ex. 27

This problem is a special case of the next problem. There are several ways to solve this, but the logic from the following proof would work.

6 Section 5.8, Ex. 29

Let P be any symetric chain parition of S. Let $P_1, P_2, ..., P_k$ be the chains in P, where $k = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Note that any clutter can contain at most one element from any chain in P. We will first prove the case where n is even. Let us first show that there can be no clutter of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ that contains a 1-combination if $\binom{n}{\lfloor \frac{n}{2} \rfloor} > 1$. Assume that this is false. Let C_1 be a 1-combination in the clutter, and P_1 be the chain from P containing C_1. We have that $n > 1$, so certainly there is at least one 2-combinatiion C_2' from some other chain P_2 that is a superset of C_1. But we must choose a combination from P_2, and it certainly can't be C_2' or any "larger" combination. Therfore we must choose the 1-combination from P_2 (assuming for the moment that there is one). And now there is at least one 2-combinatiion C_3' from some other chain P_3 that is a superset of C_2. And we will be forced to choose the 1-Combination again from that chain. This will continue on, but note that we will run out of 1-combinations before we fill our clutter with $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ sets. Now using the fact that we have already shown our clutter could not contain a 1-combination, we could follow the same logic to show the our clutter could not contain a

2-combination. And continuing inductively, 3-combinations, and on up to $\lfloor \frac{n}{2} \rfloor - 1$-combinations. Then, we would start from n combinations start the same inductive logic coming back down to $\lfloor \frac{n}{2} \rfloor + 1$-combinations. Thus, we have ruled out everything but $\lfloor \frac{n}{2} \rfloor + 1$-combinations for membership in our clutter. The odd case is very similar, save a few modifications to consider that there are the same number of $\frac{n-1}{2}$-combinations and $\frac{n+1}{2}$-combinations. I'll leave it to the student to work out the odd case.

7 Section 5.8, Ex. 30

$\{\{1,2,3,4,5\},\{1,2,3,4\},\{1,2,3\},\{1,2\},\{1\},\emptyset\}$
$\{\{2,3,4,5\},\{2,3,4\},\{2,3\},\{2,3\},\{2\}\}$
$\{\{1,3,4,5\},\{1,3,4\},\{1,3\},\{3\}\}$
$\{\{1,2,4,5\},\{1,4,5\},\{1,4\},\{4\}\}$
$\{\{1,2,3,5\},\{2,3,5\},\{2,5\},\{5\}\}$
$\{\{1,2,5\},\{1,5\}\}$
$\{\{1,2,4\},\{2,4\}\}$
$\{\{3,4,5\},\{3,4\}\}$
$\{\{1,3,5\},\{3,5\}\}$
$\{\{2,4,5\},\{4,5\}\}$

8 Section 5.8, Ex. 37

$\binom{9}{3312}(1)^3(-1)^3(2)^1(-2)^2$